Investigation of continuous and discontinuous contact cases in the contact mechanics of graded materials using analytical method and FEM

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Abstract. The aim of this paper was to examine the continuous and discontinuous contact problems between the functionally graded (FG) layer pressed with a uniformly distributed load and homogeneous half plane using an analytical method and FEM. The FG layer is made of non-homogeneous material with an isotropic stress–strain law with exponentially varying properties. It is assumed that the contact at the FG layer-half plane interface is frictionless, and only the normal tractions can be transmitted along the contacted regions. The body force of the FG layer is considered in the study. The FG layer was positioned on the homogeneous half plane without any bonds. Thus, if the external load was smaller than a certain critical value, the contact between the FG layer and half plane would be continuous. However, when the external load exceeded the critical value, there was a separation between the FG layer and half plane on the finite region, as discontinuous contact. Therefore, there have been some steps taken in this study. Firstly, an analytical solution for continuous and discontinuous contact cases of the problem has been realized using the theory of elasticity and Fourier integral transform techniques. Then, the problem modeled and two-dimensional analysis was carried out by using ANSYS package program based on FEM. Numerical results for initial separation distance and contact stress distributions between the FG layer and homogeneous half plane for continuous contact case; the start and end points of separation and contact stress distributions between the FG layer and homogeneous half plane for discontinuous contact case were provided for various dimensionless quantities including material inhomogeneity, distributed load width, the shear module ratio and load factor for both methods. The results obtained using FEM were compared with the results found using analytical formulation. It was found that the results obtained from analytical formulation were in perfect agreement with the FEM study.

Keywords: contact mechanics; functionally graded layer; finite element method; contact area; contact stress

1. Introduction

Materials science is one of the most important pillars of modern technology. Advances and research in materials science led to the discovery of materials forming the basis of the high technology in the 21st century. However, the material requirement of special the character increases rapidly with these developments. The lack of a homogeneous material, especially in space vehicles providing high strength and thermal resistance has led researchers to new searches. As at result of these studies, functionally graded materials (FGMs) have been found. FGMs are advanced composite materials with a sharp interface found in the traditional composite materials being replaced with the gradually changing interface that helps the material to survive in extreme working environments. FGMs are widely used in engineering applications such as sensor technology, civil engineering, nanotechnology, biomechanics, tribology and thermal barrier coatings for turbine blades due to their superior thermal, mechanical, optical and electrical performances. In recent years, FGMs have started to be used in the field of contact mechanics, and studies where layers are considered functionally graded are frequently encountered in the solution of contact problems in layered mediums. The studies carried out by various methods such as analytical method, FEM, BEM etc. in the literature on the contact mechanics of graded materials and they were summarized as follows:

Birinci and Erdol (2001) solved the frictionless contact problem for a layered composite which consists of two elastic layers having different elastic constants and heights resting on two simple supports. Cakiroglu et al. (2005) presented the possibilities of adapting artificial neural networks to predict the dimensionless parameters related to the maximum contact pressures of an elasticity problem. Mohamed et al. (2006) developed a new incremental finite element model to simulate the frictional contact of elastic bodies. The contact problem for an elastic layer resting on an elastic half plane loaded by means of two dissimilar rigid punches is solved according to the theory of elasticity by Ozsahin (2007). Rhimi et al. (2009) considered an axisymmetric problem of a frictionless receding contact between an elastic functionally graded layer and a homogeneous half-space, when the two bodies are pressed together. They investigated the effect of the material nonhomogeneity parameter and the thickness of the graded layer. The FG layer is made of non-homogeneous material with an isotropic stress–strain law with exponentially varying properties.
layer on the contact pressure and on the length of the receding contact. Choi (2009) examined the contact mechanics of a functionally graded layer loaded by a frictional sliding flat punch and presented the effects of several parameters, on the distributions of the contact pressure and the in-plane surface stress component, such as the material nonhomogeneity, the friction coefficient, the punch width, and Poisson’s ratio. Liu et al. (2010) investigated two-dimensional elastic contact problems, including normal, tangential, and rolling contacts with the finite element method. Elloumi et al. (2010) discussed the effect of the graded medium non-homogeneity parameter and the friction coefficient on the size of the stick zone and the contact stresses for the cases of flat and circular stamp profiles. Aizikovich et al. (2011) developed an approximate analytical method allowing one to efficiently solve, to a preassigned accuracy, contact problems for materials with properties arbitrarily varying in depth. The two-dimensional thermoelastic sliding frictional contact of functionally graded material (FGM) coated half-plane under the plane strain deformation is addressed by Liu et al. (2011). Trubchik et al. (2011) investigated the contact problem of the layer for the case when elastic properties of the medium are arbitrary continuously differentiable functions of its thickness. Rhimi et al. (2011) extended the previous study conducted by Rhimi et al. (2009) in the sense that the axisymmetric double receding contact problem is considered instead of the single receding contact problem. Guler et al. (2012) studied the rolling contact problem of two elastically similar cylinders coated with functionally graded materials. An exact three-dimensional axisymmetric elastic solution for a functionally graded coating submitted to a point-force load is examined by Sburlati (2012). Liu et al. (2012) considered the axisymmetric partial slip contact problem of a graded coating. Guler et al. (2012) investigated the contact mechanics of thin films bonded to graded coatings both analytically and numerically. Volkov et al. (2013) presented an analytical solution of the axisymmetric contact problem about indentation of a circular indenter into a soft functionally graded elastic layer. Frictionless elastic contact of a functionally graded vitreous enameled low carbon steel plate and a rigid spherical punch is studied by Nikbakht et al. (2013). Yaylaci and Birinci (2013) solved the receding contact problem of two elastic layers supported by two elastic quarter planes using the theory of elasticity and integral transform techniques. Chidlow et al. (2013) examined the two-dimensional solution of both adhesive and non-adhesive contact problems involving functionally graded materials. Abhilash and Murthy (2014) realized finite element analysis of two dimensional elastic contacts involving FGMs. Gun and Gao (2014) presented analysis of frictional contact problems for FGMs using BEM. Elastic contact of a functionally graded plate of finite dimensions with continuous variation of material properties and a rigid spherical indenter is considered by Nikbakht et al. (2014). Contact problem for an FG layer indented by a moving punch is solved by Çömez (2015). Sarfarazi et al. (2015) investigated a new approach for measurement of anisotropic tensile strength of concrete. Yan and Li (2015) examined a double receding contact plane problem between an FG layer and an elastic layer. Öner et al. (2015) presented a comparative study of analytical method and FEM for analysis of a continuous contact problem. Frictional contact analysis of functionally graded materials with Lagrange finite block method is studied by Li et al. (2015). Another study was conducted by Alinia et al. (2016). They considered the fully coupled contact problem between a rigid cylinder and an FG coating bonded to a homogeneous substrate system under plane strain and generalized plane stress sliding conditions. Frictional contact problem between an FG magneto-electroelastic layer and a rigid conducting flat punch with frictional heat generation is investigated by Ma et al. (2016). Elloumi et al. (2016) considered the contact problem of a rigid punch with friction on an FG magneto-electro-elastic half-plane. Karabulut et al. (2017) solved a receding contact problem for an elastic layer pressed by two rectangular stamps placed symmetrically and resting on a half plane. The contact problem with finite friction for a graded piezoelectric coating under an insulating spherical indenter is examined by Liu et al. (2017). El-Borgi and Çömez (2017) solved a receding frictional contact problem between a graded layer and a homogeneous substrate pressed by a rigid stamp. Güler et al. (2017) studied the plane frictional contact problem of a cylindrical punch on a functionally graded orthotropic medium using both analytical and computational methods. Patra et al. (2018) investigated frictionless contact between a rigid punch and a transversely isotropic an FG layer. Yilmaz et al. (2018) considered frictional receding contact problem for a graded bilayer system indented by a rigid punch. Plane receding contact problem for an FG layer supported by two quarter-planes is examined by Çömez et al. (2018). The frictionless contact plane problem of an infinitely graded layer supported by two rigid cylindrical punches and subjected to a concentrated normal force by means of a rigid cylindrical punch is studied by Çömez and El-Borgi (2018). Chen and Yue (2019) investigated the normal frictionless point contact between two dissimilar elastic spheres reinforced by FGM coatings with arbitrarily varied shear modulus and Poisson’s ratio. Another study was conducted by Balci and Dag (2019). They developed an analytical method to investigate the dynamic frictional contact mechanics between a FG coating and a rigid moving cylindrical punch. A semi-analytical approach for sliding frictional contact problem between a rigid insulating sphere and a transversely isotropic FG magneto-electro-elastic film and half-space based on frequency response functions is proposed by Zhang et al. (2019). Arslan (2020) presented a solution for the plane contact problem between a rigid punch and a bidirectional FG medium. Frictional contact mechanics analysis for a rigid moving punch of an arbitrary profile and an FG coating/homogeneous substrate system is examined by Balci and Dag (2020). The frictional contact problem of a FG monoclinic layer is solved according to the linear elasticity theory by Çömez (2020). Zhang et al. (2020) used the smoothed finite element method based on linear triangular elements to solve 2D solid contact problems for FGMs. Yaylaci et al. (2019, 2020) used the artificial neural network to predict the dimensionless
parameters for the maximum contact pressures and contact areas of a contact problem. Taherifar et al. (2021) studied an application of differential quadrature and Newmark methods for dynamic response in pad concrete foundation covered by the piezoelectric layer.

As a result of a large literature review, it has been revealed that the solution of the continuous and discontinuous contact problems of the FG layer, which rests on the elastic half plane and is pressed with uniformly distributed load, have not been examined by two different methods such as analytical and finite element methods in the literature. In order to fill this gap in the literature, the problem in question has been analytically solved using the ANSYS package program based on FEM. In conclusion, it was found that the results obtained by the analytical method and FEM were consistent.

2. Analytical solution

2.1 The definition of the problem

As shown in Fig. 1, consider the (a) continuous and (b) discontinuous plane strain contact problem of an infinitely long FG layer of thickness $h$ resting on a homogeneous half plane. Subscripts 0 and 1 are used to represent the terms related to the homogeneous half plane and FG layer, respectively. Poisson ratios of the layer and the half plane, $v_1$ and $v_0$, respectively, and the shear modulus of the half plane, $\mu_0$, are taken as constant, whereas shear modulus, $\mu_1$, and mass density, $\rho_1$, of the FG layer change exponentially thorough thickness as given below, respectively

$$\mu(y) = \mu_0 \exp(\beta y), \rho(y) = \rho_0 \exp(\alpha y), (0 \leq y \leq h)$$

layer (i.e., $y=0$), $\beta$ and $\alpha$ are the non-homogeneity parameters controlling the variation of the shear modules and the density in the graded layer, namely stiffness and density parameters, respectively. The FG layer is pressed into the half plane by means of a distributed load which subjects over a finite segment, $|x| \leq a$. The body force of the homogeneous half plane is ignored (i.e., $\rho_0 = 0$), whereas body force only in the direction of the gravity is included for the FG layer. In addition, it is assumed that the contact surface is frictionless, and the loading and the geometry of the problem are symmetric with respect to $x=0$ axis, for simplicity. Clearly, it is sufficient to consider only one half (i.e., $x \geq 0$) of the system.

The problem can be separated into two sub-problems, namely continuous and discontinuous contact problem, depending on the applied force. If the effect of the applied force is relatively small compared to the effect of the gravitational force, the contact between the FG layer and the half plane becomes continuous and no separation occurs along the contact surface (Fig. 1(a)). If the applied load is increased, the first separation occurs between the FG layer and the half plane at a critical load. The critical load causing the first separation, the point in which the first separation occurs and stresses on the contact surface are the quantities of interest in the case of continuous contact problem. If the applied load increases further from the critical load, the FG layer is separated from the half plane, and the discontinuous contact case is formed. However, the layer bends at some point depending on the magnitude of the applied load because of gravity and the FG layer comes into contact with the half plane at some point. Therefore, only a finite separation distance occurs such as $b \leq x \leq c$ (Fig. 1(b)). The points in which the separation starts and ends, and contact stresses are the quantities of interest in the case of a discontinuous contact problem.

In the case of continuous contact problem (Fig. 1(a)), since there is no separation, following boundary conditions can be written

$$\sigma_{y1}(x, \delta) = -q(x)H(a - x), \tau_{xy1}(x, \delta) = 0, \tau_{xy1}(x, 0) = 0, (0 \leq x < \infty)$$

(3-5)

$$\tau_{xy0}(x, 0) = 0, \sigma_{y1}(x, 0) = \sigma_{y0}(x, 0), \frac{dv_1(x, 0)}{dx} = \frac{dv_0(x, 0)}{dx}, (0 \leq x < \infty)$$

(6-8)

in which, subscript 1 and 0 represent FG layer and half plane related terms, respectively; $v$ is the y component of the displacement field; $\sigma_y$ and $\tau_{xy}$ are the components of the stress field in the same coordinate system, and $H$ is the Heaviside function. In the case of discontinuous contact problem (Fig. 1(b)), the boundary conditions can be revised as follows

$$\sigma_{y1}(x, \delta) = -q(x)H(a - x), \tau_{xy1}(x, \delta) = 0, \tau_{xy1}(x, 0) = 0, (0 \leq x < \infty)$$

(9-11)
\[ \tau_{xy0}(x, 0) = 0, \quad \sigma_{yy}(x, 0) = \sigma_{yy}(x, 0), \quad (0 < x < \infty) \] (12-13)
\[ \frac{d\sigma_{iy}(x, 0)}{dx} - \frac{d\sigma_{iy}(x, 0)}{dx} = f(x), \quad \sigma_{yy}(x, 0) = 0 \]
where, \( f(x) \) is an unknown function which equals to the derivative of separation distance between the FG layer and the half plane with respect to \( x \), and single-valuedness for vertical displacements requires that \( f(x) \) satisfies the following condition
\[ \int_0^c f(x)dx = \int_b^c f(t)dt = 0 \] (16)

### 2.2 The solution of the problem

Assuming that the materials used in FG layer and half plane are isotropic, equilibrium equations, the strain-displacement relationships and the linear elastic stress-strain law, respectively, can be written as follows
\[ \frac{\partial \sigma_{ix}}{\partial x} + \frac{\partial \tau_{ix}}{\partial y} = 0, \quad \frac{\partial \tau_{ix}}{\partial x} + \frac{\partial \sigma_{iy}}{\partial y} - \rho_i g = 0 \] (17-18)
\[ \epsilon_{ix} = \frac{\partial u_i}{\partial x}, \quad \epsilon_{iy} = \frac{\partial v_i}{\partial y}, \quad \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} \right) \] (19-21)
\[ \sigma_{ii} = \frac{\mu_i}{\kappa_i-1} \left[ (1+\kappa_i)\epsilon_{ii} + (3-\kappa_i)\epsilon_{ij} \right] \] (22)
\[ \sigma_{ij} = \frac{\mu_i}{\kappa_i-1} \left[ (3-\kappa_i)\epsilon_{ii} + (1+\kappa_i)\epsilon_{ij} \right] \] (23)
\[ \tau_{iy} = 2\mu_i (y) \epsilon_{yi} \] (24)
in which, subscript \( i \) equals to 1 for the half plane and equals to 0 for the FG layer. \( u_i \) is the \( x \) component of the displacement field, respectively. \( \sigma_{ix} \) is the \( x \) component of the stress field. \( \epsilon_{ix} \) and \( \epsilon_{iy} \) are the components of the strain field in the same coordinate system. \( \kappa_i \) is a material property defined as \( \kappa_i = 3 - 4v_i \) for plane strain problems, and \( g \) is gravitational acceleration. Equilibrium equations in terms of displacements (Navier equations) can be obtained simply combining Eqs. (1), (2), (17-24).

\[ (\kappa_i + 1) \frac{\partial^2 u_i}{\partial x^2} + (\kappa_i - 1) \frac{\partial^2 u_i}{\partial y^2} + 2 \frac{\partial u_i}{\partial y} + \beta (\kappa_i - 1) \frac{\partial u_i}{\partial x} = 0 \] (25)
\[ (\kappa_i + 1) \frac{\partial^2 v_i}{\partial x^2} + (\kappa_i - 1) \frac{\partial^2 v_i}{\partial y^2} + 2 \frac{\partial v_i}{\partial y} + \beta (3 - \kappa_i) \frac{\partial v_i}{\partial x} + \beta (\kappa_i + 1) \frac{\partial v_i}{\partial y} = \rho_m \delta (\beta-\alpha i) \] (26)

Eqs. (25) and (26) are second order partial differential equations and valid for both homogeneous half plane and FG layer provided that the \( \beta_i = 0, \alpha_i = 0 \) and \( \rho_m = 0 \) for the homogeneous half plane and \( \rho_m = 0 \) is described as follows for the FG layer
\[ \rho_m = \frac{\kappa_i - 1}{\mu_{10}} \rho_{100} \] (27)

For the FG layer, Eqs. (25) and (26) form a non-homogeneous second order partial differential equation system. The solution can be obtained combining particular and homogeneous solutions as follows
\[ \zeta_i = \zeta_{i1p} + \zeta_{i1h} \] (28)
in which, \( \zeta \) is the representation of any of the stress and displacement components and subscripts \( p \) and \( h \) represent particular and homogeneous solution, respectively. Particular part of stress component \( \sigma_{iy} \) can obtained as follows assuming displacement components \( u_{1p} = u_{1p}(x) \) and \( v_{1p} = v_{1p}(y) \)
\[ \sigma_{iy}(y) = -\frac{p_a g}{\alpha} \left( e^{\alpha y} - e^{\alpha y} \right) \] (29)

In the homogeneous solution of the FG layer, using symmetry considerations and Fourier transforms, the displacement components for FG layer may be written as follows
\[ u_{1h}(x, y) = \frac{2}{\pi} \int_0^\infty \phi(\xi, y) \sin(\xi x) d\xi, \quad v_{1h}(x, y) = \frac{2}{\pi} \int_0^\infty \psi(\xi, y) \cos(\xi x) d\xi \] (30-31)
where \( \phi(\xi, y) \) and \( \psi(\xi, y) \) are the Fourier sine and Fourier cosine transforms of \( u_{1h} \) and \( v_{1h} \) with respect to \( x \), respectively. After substituting Eqs. (30) and (31) into Eqs. (25) and (26), the solution can be obtained for homogeneous solution of stress and displacement fields of interest for graded layer can be expressed as follows
\[ \sigma_{inh} = \frac{2\mu_i p}{\pi} \exp(\beta y) \left( \sum_{n=1}^{\infty} F_n \exp(n y) \cos(\xi x) d\xi \right) \] (32)
\[ \tau_{inh} = \frac{2\mu_i p}{\pi} \exp(\beta y) \left( \sum_{n=1}^{\infty} T_n \exp(n y) \sin(\xi x) d\xi \right) \] (33)
\[ v_{lh} = \frac{2}{\pi} \int_0^\infty \sum_{n=1}^{4} F_n m \exp(n y) \cos(\xi x) d\xi \] (34)

\( S_j \) and \( T_j \) \((j = 1, \ldots, 4)\) are known functions whereas \( F_j \) \((j = 1, \ldots, 4)\) are an unknown functions, which will be determined from the boundary conditions, obtained from the solution of the partial differential Eqs. (25) and (26). Eqs. (25) and (26) turn into homogeneous second order partial differential equation system for the homogeneous half plane as follows
\[ (\kappa_0 - 1) \frac{\partial^2 u_0}{\partial x^2} + (\kappa_0 - 1) \frac{\partial^2 u_0}{\partial y^2} + 2 \frac{\partial^2 u_0}{\partial x \partial y} = 0 \] (35)
\[ (\kappa_0 - 1) \frac{\partial^2 v_0}{\partial x^2} + (\kappa_0 - 1) \frac{\partial^2 v_0}{\partial y^2} + 2 \frac{\partial^2 v_0}{\partial x \partial y} = 0 \] (36)

Similar to the process in the FG layer, using symmetry considerations and Fourier transforms, following interested stress and displacement components can be obtained from the solution of Eqs. (35) and (36)
\[
\sigma_{y_0} = \frac{4\mu e}{\pi} \int_0^\infty [\xi (B_1 + B_2 y) + \left(1 + \frac{\kappa_0}{2}\right) B_2] \frac{e^{i\tau}}{\xi} \cos (\xi x) d\xi
\] (38)

\[
\tau_{\phi_0} = \frac{4\mu e}{\pi} \int_0^\infty [\xi (B_1 + B_2 y) - \left(\frac{\kappa_0 - 1}{2}\right) B_2] \frac{e^{i\tau}}{\xi} \sin (\xi x) d\xi
\] (39)

in which, \(B_i\) \((i=1,2)\) are unknown functions and can be found using boundary conditions. Obtained stress and displacement expressions for the FG layer and homogeneous half plane (Eqs. (32)-(34) and (37)-(39)) have a total of six unknown functions, namely \(B_i\) \((i=1,2)\) and \(F_j\) \((i=1,2,3,4)\).

**Continuous contact case**

The unknown functions in the stress and displacement components in case of continuous contact case can be easily obtained using six boundary conditions given in Eqs. (3) - (8). The critical load can be obtained from the following equation which can be obtained by equalizing any vertical stress component (i.e., \(\sigma_{y_0}\) or \(\sigma_{y_1}\)) to zero at \(y=0\).

\[
\frac{1}{\lambda} = \frac{2h}{\mu} \left(2\mu_0 \frac{K_1}{\mu}\right) \sin \frac{\xi a}{\mu_0} \cos \xi x d\xi
\] (40)

In Eq. (40), \(K_1\) is a known kernel function (given in App. A) and \(\lambda\) is a dimensionless parameter which is named load factor defined as follows

\[
\lambda = \frac{P}{\rho_0 \delta^2 k^2}
\] (41)

in which \(P\) is the total resultant force of the distributed load acting on the FG layer and can be expressed as \(P = \int_a^b q(x) dx = 2q_0 a\) assuming \(q(x)\) is a uniformly distributed load with a magnitude of \(q_0\). For \(\lambda = \lambda_{cr}\), the applied load becomes critical load, and the first separation occurs at a point \(x=x_{cr}\), which can be found by the help of equalizing the derivative of any vertical stress component with respect to \(x\) (i.e., \(\frac{\partial \sigma_{y_0}}{\partial x}\) or \(\frac{\partial \sigma_{y_1}}{\partial x}\)) to zero at \(y=0\).

**Discontinuous contact case**

In discontinuous contact case, using corresponding boundary conditions (9-14), the unknown functions in the stress and displacement components can be found in terms of \(F\) which is the Fourier sine transform of \(f(x)\) and defined as follows

\[
F = \int_0^\infty f(x) \sin \xi x dx = \int_b^c f(t) \sin \xi t dt
\] (42)

The problem can be converted into the solution of a singular integral equation given in Eq. (43) by the help of unused boundary condition (15).

\[
\frac{4\mu e}{\rho_0 \delta k^2} \left[\int_b^c f(t) K_5 \left(\frac{1}{\lambda e^{-\xi t}} + \frac{1}{\lambda e^{\xi t}}\right) dt + \frac{1}{\pi} \int_b^c f(t) K_2 dt\right]
\]

\[
- \left[\frac{2\mu e}{\pi} \frac{\lambda}{\mu} K_2 + \frac{e^{\alpha^2-1}}{a^2}\right] = 0
\] (43)

In Eq. (43), \(K_5\) is a known value, \(K_2\) and \(K_2\) are known kernel functions (given in App. A). The singular integral Eq. (43) can be solved using corresponding Gauss-Jacobi integration formulas similar to given in (Adiyaman et al. 2017).

**3. Finite element analysis**

For FG materials in engineering applications, several analysis techniques have been proposed, as well. Most of the analyses of structural elements with FGM are based on theory of elasticity solution techniques. An alternative to such analysis would be the development of plate, shell, layer or solid finite elements that will enable analysis of problems with FGM. The available software package like ANSYS does not automatically provide such capabilities and requires the development of user defined subroutines and functions. First of all, a function code was developed to define the functionally graded layers by changing the material properties of the layers along the layer and added to the program's log files with the extension of “.log”. According to the change in the properties of FGM material in line with the layer thickness, the function for the application of matter specifications is used to define matter. Of course, simultaneous changes of thickness should be taken into consideration in this program. Modeling functionally graded layer using the finite element method is quite complicated. Therefore, many parameters are involved in this numerical method.

Our goal in this article is to study contact problem between the FG layer and homogeneous half plane and compare the analytical and FEM results on the value of initial separation distance, contact area, contact stress, the start point of separation and the end point of separation. This job is done by the simulation of FG layer with the help of the finite element method. To simulate the FG layer, the mechanical analysis is employed by using the ANSYS software. The solid element was used for mechanical analysis.

Verification of the developed analytical method is provided by utilizing the computational results for contacts and FG layer generated through the general purpose finite
element analysis ANSYS software Mechanical APDL (2013). Fig. 2 illustrates the computational model constructed to calculate initial separation distance, contact areas and contact stresses for the problem. The problem is assumed as a two-dimensional contact problem. The material properties of the functionally graded layer of the model are taken to be elastic and isotropic. The system is physically symmetrical in terms of being geometrical, material properties and loading. Therefore, half of the geometric problems can be modeled because the system is symmetrical. In Fig. 2, the dimensions, $L$, $h$ and $q$ respectively denote the FG layer length, the FG layer thickness, and load. In the analyses, geometric properties are taken as $L = 50$ m (length of the FG layer in $x$ direction), $h = 1$ m (thickness of the FG layer in $y$ direction), the load $q = 120$ N/m and the Poisson’s ratio is given as constant for layer $\nu = 0.315$. The shear modulus and density of the element used in layer are calculated according to the grade function $\mu(y) = \mu_{\text{ref}}e^{(\beta y)}$ and $\rho(y) = \rho_{\text{ref}}e^{(\alpha y)}$ respectively. ANSYS does not include any specified module to incorporate functional grading of material properties. For this reason, a function code was developed to define the functionally graded layers by changing the material properties of the layers along the $y$ axis and added to the log files of the program.

The increment of mesh size gives more close results to theoretical ones. However, the increment of the mesh size after a certain point does not affect the results or even adversely affect it. Furthermore, the increment of the mesh size has a negative effect in terms of analysis time. Therefore, at first the most appropriate mesh size has been determined and then solutions are found according to the determined mesh size. The finite element mesh involves a total of 97536 quadrilateral solid 8 node 183 finite elements. The PLANE 183 element has 8 nodes and 2 degrees of freedom and has 8 nodes-4 corner and 4 mid nodes with 2 translational degrees of freedom in the nodal $x$, $y$ directions. Fig. 3 displays the quadrilateral finite elements used in analyses.

In this problem, there is a contact between the FG layer and the homogeneous half plane. A total of 300-line contact elements and 597 contact nodes exist in the finite element model. The contact between the FG layer and the half plane is defined by CONTA169 and TARGE172 elements available in ANSYS Mechanical APDL. Each CONTA169 element has three nodes. Hence, a total of 597 contact nodes are created. The density of the finite element mesh around the contact zone is refined in order to capture the sharp variations in the contact stresses.

Special elements are used to ensure contact between the parts in the model. The selection of the elements depends on the dimension of the problem and the type of used element. Namely, in two-dimensional problems the contact is linear, while in three-dimensional ones the contact is superficial. Various contact application methods are available in contact analysis;

- Node-Node
- Node-Surface contact
- Surface-to-surface contact.

The interaction between contact surfaces is modeled with surface to surface. The surface-to-surface contact model has been utilized for modelling the contact pair. The surface-to-surface contact pattern also allows for a solution if the joints do not overlap. In the problem, a contact pair is formed in the contact area. Contact pairs consist of two element types. These are the TARGET and CONTACT. The Software packages offer different contact algorithms for behavior on contact surfaces. These algorithms can be listed as follows:

- Penalty method
- Augmented Lagrangian Method
- MPC algorithm
- Lagrange Multiplier

The contact regions are modeled using the Augmented Lagrange Method.

Fig. 5 shows meshed model of the contact problem consisting of FGM material in the ANSYS software. The deformed shape that occurs after the analysis of these models is shown in Fig. 6.

The steps of the contact analysis can be summarized as:

1. Modeling of 2D geometry model;
2. Identification of
material properties; (3) Meshing; (4) Identification of contact pairs, identification of target surface and contact surface; (5) Application of boundary conditions and load steps; (6) Identification of solution options; (7) solution of contact problem; (8) obtaining and interpreting of results (Fig. 2).

4. Numerical results

By using the finite element simulation, the results of initial separation distance, contact areas, and contact stresses are investigated. Finite element (FE) simulation is performed with the ANSYS Mechanical software package. By comparing the simulation with analytical formulas, it is observed that the FEM results are in good agreement with the analytical results.

In this study, the continuous and discontinuous contact problems of the FG layer resting on homogeneous half space was analyzed using the ANSYS software. As a result of the analysis, the variations of the separation distance, the starting and ending points of the separation for various dimensionless quantities were examined. All results are shown as dimensionless. In the problem, when the amplitude of the uniformly distributed load \((\frac{a}{h}=0.01)\) applied to the layer is taken, the load is assumed to be singular and the values are taken as \(v_1 = v_0 = 0.315, h = 1\) and when calculating.

The stiffness and density at the top of the FG layer, namely \(\mu_{1h}\) and \(\rho_{1h}\) respectively, can be defined as follows.

\[
\mu_{1h}(y) = \mu_{0h} \exp(\beta y), \quad \rho_{1h}(y) = \rho_{0h} \exp(\alpha y) \quad (44.45)
\]

In the numerical results, it is assumed that \(\mu_{1h}\) and \(\rho_{1h}\) remain constant whereas the stiffness and the density at the bottom of the layer, \(\mu_{10}\) and \(\rho_{10}\), respectively, change when \(\beta\) and \(\alpha\) vary, respectively. In other words, if the stiffness parameter \(\beta\) increases, the stiffness at the bottom of the layer decreases. Similarly, the density at the bottom of the layer decreases if the density parameter \(\alpha\) increases.

In Tables 1-4, the variation of the initial separation distances for continuous contact is given for different \(\mu_{1h}/\mu_0\), \(\beta\), \(\alpha\) and \(a/h\) values. When the tables are analyzed, it is seen that the initial separation distances increase with the shear module ratio \((\mu_{1h}/\mu_0)\), the stiffness parameter \((\beta)\) and distributed load width \((a/h)\) increasing. However, the change in the density parameter \((\alpha)\) does not have a significant effect on the initial separation distances.

| Table 1 Comparison of the analytical and FEM results of the initial separation distance based on \(\mu_{1h}/\mu_0\) value \((\beta=0.001, \alpha=0.001, a/h=0.01)\) |
|----------------|-------------|--------------|-------------|
| \(\mu_{1h}/\mu_0\) | Analytical  | FEM          | Difference Rate % |
| 0.000           | 1.770790    | 1.77         | 0.04        |
| 0.001           | 1.771842    | 1.75         | 1.23        |
| 0.010           | 1.782388    | 1.80         | 0.99        |
| 0.100           | 1.890620    | 1.85         | 2.15        |
| 1.000           | 2.903093    | 3.00         | 3.34        |
| 10.000          | 5.960430    | 6.00         | 0.66        |

| Table 2 Comparison of the analytical and FEM results of the initial separation distance based on \(\beta\) value \((\mu_{1h}/\mu_0=1, \alpha=0.001, a/h=0.01)\) |
|----------------|-------------|--------------|-------------|
| \(\beta\)     | Analytical  | FEM          | Difference Rate % |
| -1.000         | 2.677192    | 2.70         | 0.85        |
| -0.500         | 2.774720    | 2.80         | 0.91        |
| 0.001          | 2.903093    | 2.90         | 0.11        |
| 0.500          | 3.059464    | 3.00         | 1.94        |
| 1.000          | 3.243962    | 3.20         | 1.36        |

| Table 3 Comparison of the analytical and FEM results of the initial separation distance based on \(\alpha\) value \((\mu_{1h}/\mu_0=1, \beta=0.001, a/h=0.01)\) |
|----------------|-------------|--------------|-------------|
| \(\alpha\)    | Analytical  | FEM          | Difference Rate % |
| -1.000         | 2.903093    | 2.90         | 0.11        |
| -0.500         | 2.903093    | 2.90         | 0.11        |
| 0.001          | 2.903093    | 2.95         | 1.62        |
| 0.500          | 2.903093    | 3.00         | 3.34        |
| 1.000          | 2.903093    | 3.00         | 3.34        |

| Table 4 Comparison of the analytical and FEM results of the initial separation distance based on \(a/h\) value \((\mu_{1h}/\mu_0=1, \beta=0.001, \gamma=0.001)\) |
|----------------|-------------|--------------|-------------|
| \(a/h\)       | Analytical  | FEM          | Difference Rate % |
| 0.01           | 3.243962    | 3.20         | 1.36        |
| 0.10           | 3.247909    | 3.30         | 1.60        |
| 1.00           | 3.616058    | 3.60         | 0.44        |
| 2.00           | 4.478384    | 4.50         | 0.48        |
| 5.00           | 7.46262     | 7.50         | 0.50        |

In addition, it is seen that the results obtained from the analytical solution and the results obtained from the numerical solution are very close.

Contact stresses occurring between the FG layer and the homogeneous half plane for different \(\lambda\) and \(\beta\) values in the case of continuous contact are given in Figs. 7 and 8, respectively. When the figures are examined, it is seen that the greatest contact stress occurs in the axis of symmetry. As the \(\lambda\) value increases, the greatest contact stress value increases. On the other hand, it is observed that the greatest contact stress decreases with the increase in the stiffness.
Fig. 7 Variation of contact stresses for continuous contact depending on $\lambda$, $\beta = 0.01, \alpha = 0.001, \mu_s/\mu_q = 1$.

Table 5 Comparison of the start and end points of separation ($b/h$ and $c/h$) for $\beta = -1$ based on $\lambda$ value ($a/h = 0.01, \alpha = 0.001, \mu_s/\mu_q = 1$)

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Analytical</th>
<th>FEM</th>
<th>Difference Rate %</th>
<th>Analytical</th>
<th>FEM</th>
<th>Difference Rate %</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>3.243962</td>
<td>3.30</td>
<td>1.73</td>
<td>3.243962</td>
<td>3.20</td>
<td>1.36</td>
</tr>
<tr>
<td>140</td>
<td>2.922158</td>
<td>3.00</td>
<td>2.66</td>
<td>2.922158</td>
<td>2.90</td>
<td>1.05</td>
</tr>
<tr>
<td>160</td>
<td>2.712827</td>
<td>2.70</td>
<td>0.47</td>
<td>2.712827</td>
<td>2.70</td>
<td>0.27</td>
</tr>
<tr>
<td>200</td>
<td>2.521988</td>
<td>2.50</td>
<td>0.87</td>
<td>2.521988</td>
<td>2.50</td>
<td>0.52</td>
</tr>
<tr>
<td>250</td>
<td>2.402738</td>
<td>2.40</td>
<td>0.11</td>
<td>2.402738</td>
<td>2.40</td>
<td>0.07</td>
</tr>
<tr>
<td>300</td>
<td>2.332376</td>
<td>2.30</td>
<td>1.39</td>
<td>2.332376</td>
<td>2.30</td>
<td>0.89</td>
</tr>
</tbody>
</table>

Fig. 8 Variation of contact stresses for continuous contact depending on $\gamma$, $a/h = 0.01, \alpha = 0.001, \mu_s/\mu_q = 1$.

In Tables 5-9, the change of the start and end points of the separation for discontinuous contact condition is given for different load factors, stiffness parameter and density parameter. When the tables are analyzed, as the value of $\lambda$ increases, the starting point of the separation moves to the left, the end point of the separation moves to the right, and the interval in which the separation occurs grows. Similarly, as the $\alpha$ value increases, the starting point of the separation moves to the left, the end point of the separation moves to the right, and the separation distance increases. In addition, as the value of $\beta$ increases, the starting and ending point of the separation moves to the left, and the separation distance decreases.

In case of discontinuous contact, the variation of the contact stresses occurring across the contact surface between the FG layer and the half plane for different $\alpha$ and $\beta$ values are given in Figs. 9 and 10, respectively. When the graphs are examined, it is seen that greatest contact stress value occurs in the axis of symmetry, similar to the state of continuous contact. In the separation zone, the stress values are zero and the boundary condition is provided. It is seen that the graphics obtained from the analytical solution and the graphics from the FEM solution are compatible with each other.
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Table 7 Comparison of the start and end points of separation (b/h and c/h) for β = 1 based on λ value (a/h = 0.01, α = 0.001, μ₁/μ₀ = 1)

<table>
<thead>
<tr>
<th>λ</th>
<th>Analytical</th>
<th>FEM</th>
<th>Difference Rate %</th>
<th>Analytical</th>
<th>FEM</th>
<th>Difference Rate %</th>
</tr>
</thead>
<tbody>
<tr>
<td>181</td>
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<td>2.60</td>
<td>2.88</td>
<td>2.677192</td>
<td>2.60</td>
<td>2.88</td>
</tr>
<tr>
<td>200</td>
<td>2.159275</td>
<td>2.10</td>
<td>2.75</td>
<td>3.337445</td>
<td>3.30</td>
<td>1.12</td>
</tr>
<tr>
<td>225</td>
<td>2.074099</td>
<td>2.00</td>
<td>3.57</td>
<td>3.582432</td>
<td>3.50</td>
<td>2.30</td>
</tr>
<tr>
<td>250</td>
<td>2.016984</td>
<td>2.00</td>
<td>0.84</td>
<td>3.787932</td>
<td>3.80</td>
<td>0.32</td>
</tr>
<tr>
<td>300</td>
<td>1.942637</td>
<td>1.90</td>
<td>2.19</td>
<td>4.129826</td>
<td>4.00</td>
<td>3.14</td>
</tr>
<tr>
<td>350</td>
<td>1.895222</td>
<td>1.80</td>
<td>5.02</td>
<td>4.414829</td>
<td>4.50</td>
<td>1.93</td>
</tr>
</tbody>
</table>

Table 8 Comparison of the start and end points of separation (b/h and c/h) for α = −1 based on λ value (a/h = 0.01, β = 0.001, μ₁/μ₀ = 1)

<table>
<thead>
<tr>
<th>λ</th>
<th>Analytical</th>
<th>FEM</th>
<th>Difference Rate %</th>
<th>Analytical</th>
<th>FEM</th>
<th>Difference Rate %</th>
</tr>
</thead>
<tbody>
<tr>
<td>238</td>
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<td>3.00</td>
<td>3.34</td>
<td>2.903093</td>
<td>3.00</td>
<td>3.34</td>
</tr>
<tr>
<td>260</td>
<td>2.556762</td>
<td>2.50</td>
<td>2.22</td>
<td>3.360077</td>
<td>3.50</td>
<td>4.16</td>
</tr>
<tr>
<td>300</td>
<td>2.376338</td>
<td>2.30</td>
<td>3.21</td>
<td>3.741255</td>
<td>3.75</td>
<td>0.23</td>
</tr>
<tr>
<td>350</td>
<td>2.259769</td>
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<td>2.64</td>
<td>4.088704</td>
<td>4.00</td>
<td>2.17</td>
</tr>
<tr>
<td>400</td>
<td>2.187159</td>
<td>2.10</td>
<td>3.99</td>
<td>4.371151</td>
<td>4.25</td>
<td>2.77</td>
</tr>
<tr>
<td>500</td>
<td>2.098380</td>
<td>2.00</td>
<td>4.69</td>
<td>4.829169</td>
<td>5.00</td>
<td>3.54</td>
</tr>
</tbody>
</table>

Table 9 Comparison of the start and end points of separation (b/h and c/h) for α = 1 based on λ value (a/h = 0.01, β = 0.001, μ₁/μ₀ = 1)

<table>
<thead>
<tr>
<th>λ</th>
<th>Analytical</th>
<th>FEM</th>
<th>Difference Rate %</th>
<th>Analytical</th>
<th>FEM</th>
<th>Difference Rate %</th>
</tr>
</thead>
<tbody>
<tr>
<td>87</td>
<td>2.903093</td>
<td>3.00</td>
<td>3.34</td>
<td>2.903093</td>
<td>3.00</td>
<td>3.34</td>
</tr>
<tr>
<td>100</td>
<td>2.487013</td>
<td>2.50</td>
<td>0.52</td>
<td>3.490572</td>
<td>3.50</td>
<td>0.27</td>
</tr>
<tr>
<td>125</td>
<td>2.278807</td>
<td>2.25</td>
<td>1.26</td>
<td>4.024303</td>
<td>4.00</td>
<td>0.60</td>
</tr>
<tr>
<td>150</td>
<td>2.178172</td>
<td>2.00</td>
<td>8.18</td>
<td>4.410954</td>
<td>4.50</td>
<td>2.02</td>
</tr>
<tr>
<td>200</td>
<td>2.072156</td>
<td>2.00</td>
<td>3.48</td>
<td>5.000052</td>
<td>5.00</td>
<td>0.00</td>
</tr>
<tr>
<td>250</td>
<td>2.015127</td>
<td>1.90</td>
<td>5.71</td>
<td>5.461196</td>
<td>5.50</td>
<td>0.71</td>
</tr>
<tr>
<td>300</td>
<td>1.978900</td>
<td>1.90</td>
<td>3.99</td>
<td>5.851343</td>
<td>6.00</td>
<td>2.54</td>
</tr>
</tbody>
</table>

Fig. 9 Variation of contact stresses for discontinuous contact depending on α (a/h = 0.01, β = 0.001, μ₁/μ₀ = 1)

Fig. 10 Variation of contact stresses for discontinuous contact depending on β (a/h = 0.01, α = 0.001, μ₁/μ₀ = 1)

5. Conclusions

Functionally graded materials (FGMs) are a kind of advanced composites involving two or more constituent phases with a gradual and functionally variable composition of microstructure and material properties. They are used as
protective coatings and interfacial zones in engineering applications. In this paper, the continuous and discontinuous contact problems of an FG layer resting on a homogeneous half plane is investigated using analytical method and FEM. ANSYS software was used to obtain finite element results.

The paper successfully highlights the effect of material composition parameter in FG layers. The following inferences can be drawn from the numerical simulation solutions:

- Results obtained from FEM solution are in perfect agreement with analytical results in the literature. It can be seen that results of the present analytical study are in excellent agreement with those of finite element analysis.
- The initial separation distances increase with the shear module ratio ($\mu_1/\mu_0$), the stiffness parameter ($\beta$) and distributed load width ($\alpha/h$) increasing. However, the change in the density parameter ($\alpha$) does not have a significant effect on the initial separation distances.
- Stiffness parameter ($\beta$) and load parameter ($\lambda$) have a significant effect on the starting and ending points of separation.
- The effect of the stiffness and density parameters ($\beta$ and $\alpha$) on the separation distance is investigated using a parametric study. As the $\alpha$ value increases, the separation distance increases. On the contrary the value of $\beta$ increases, the separation distance decreases.

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CC
\[ K_i = \frac{1}{\Delta K} \frac{2 \xi_{10}}{\mu_0} \left[ e^{\text{inh}} \left( S_1 T_2 T_m - S_1 T_2 T_m - S_1 T_2 T_m - S_1 T_2 T_m + S_1 T_2 T_m + S_1 T_2 T_m - S_1 T_2 T_m \right) \right. \\
+ e^{\text{inh}} \left( -S_1 T_2 T_m + S_1 T_2 T_m + S_1 T_2 T_m - S_1 T_2 T_m - S_1 T_2 T_m + S_1 T_2 T_m + S_1 T_2 T_m - S_1 T_2 T_m \right) \\
+ e^{\text{inh}} \left( +S_1 T_2 T_m - S_1 T_2 T_m - S_1 T_2 T_m + S_1 T_2 T_m + S_1 T_2 T_m - S_1 T_2 T_m + S_1 T_2 T_m - S_1 T_2 T_m \right) \\
\left. + e^{\text{inh}} \left( -S_1 T_2 T_m + S_1 T_2 T_m - S_1 T_2 T_m + S_1 T_2 T_m + S_1 T_2 T_m - S_1 T_2 T_m + S_1 T_2 T_m - S_1 T_2 T_m \right) \right] \] (A1)

\[ K_{21} = \int_{0}^{\frac{1}{\Delta K} \frac{2 \mu_0}{\mu_i}} \left[ e^{\text{inh} \xi} + e^{\text{inh} \xi} \right] \left[ \frac{S_1 T_2 T_m}{S_1 T_2 T_m} + \frac{S_1 T_2 T_m}{S_1 T_2 T_m} \right] - K \sin \xi \cos \xi \, d\xi 
\sin \xi \cos \xi \, d\xi 
\] (A2)

\[ K_{22} = \lim_{\xi \to \infty} \left[ \frac{2 \mu_0}{\Delta K} \frac{1 - \frac{1}{\mu_i}}{e^{\beta \xi}} \right] \sin \xi \cos \xi \, d\xi \] (A3)

\[ K_5 = \lim_{\xi \to \infty} \left[ \frac{2 \mu_0}{\Delta K} \frac{1 - \frac{1}{\mu_i}}{e^{\beta \xi}} \right] \sin \xi \cos \xi \, d\xi \] (A4)

\[ \Delta K = \mu_0 \left( 1 + \kappa_0 \right) + \left( e^{\text{inh} \xi} + e^{\text{inh} \xi} \right) \left( S_1 T_2 T_m - S_1 T_2 T_m - S_1 T_2 T_m \right) \\
+ e^{\text{inh} \xi} \left( -S_1 T_2 T_m + S_1 T_2 T_m + S_1 T_2 T_m \right) \\
+ e^{\text{inh} \xi} \left( -S_1 T_2 T_m + S_1 T_2 T_m + S_1 T_2 T_m \right) \\
+ e^{\text{inh} \xi} \left( -S_1 T_2 T_m + S_1 T_2 T_m + S_1 T_2 T_m \right) \\
\left. + e^{\text{inh} \xi} \left( -S_1 T_2 T_m + S_1 T_2 T_m + S_1 T_2 T_m \right) \right] \] (A5)