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Dimensional synthesis problem of 3R for three precision poses using dual quaternions

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Abstract. In this paper, three precision poses geometric synthesis problem of 3R serial manipulators is solved using a polynomial continuation method. Denavit-Hartenberg parameters and a new formulation using dual quaternions are used to formulate the problem and obtain the design equations. Upon choosing six of the design parameters arbitrarily, a system of thirteen polynomials in thirteen unknowns is derived. Four new types for selecting the free choices are introduced and their design equations are solved using polynomial homotopy continuation method. Numerical example is included in which the multi-homogeneous bounds for the solutions paths for the first type is 1152 and for the others are 9216.

Keywords: geometric synthesis problem; 3R serial manipulators; homotopy continuation; dual quaternions

1. Introduction

Calculation of the geometric parameters of a mechanism so that it guides a rigid body in a number of prescribed spatial locations or precision poses is known as the Rigid Body Guidance synthesis Problem. A precision pose is given by six parameters; three for position and another three for orientation. This problem has been studied extensively for 4-bar linkages and has recently drawn much attention to researchers for spatial manipulators (Mavroidis *et al*. 2001, Huang and Lai 2012). Solution techniques for this problem can be classified into two categories; namely, exact and approximate synthesis methods. Exact synthesis method result in mechanisms or manipulators, which guides a rigid body exactly through the precision poses. If this is the case, solutions exist only if the number of independent design equations obtained by the precision poses does not exceed the number of the design parameters. For each mechanism only for a limited number of precision poses the design problem leads to a finite number of exact solutions (Tsai 1972, Roth 1986). While approximate synthesis is mainly used in over-determined systems where more precision poses are given than the design parameters, no exact solution exists in general.

The geometric design problem usually leads to a set of highly nonlinear multivariate polynomial equations (Dhingra *et al*. 1994, Lee and Mavroidis 2003). Numerical continuation and

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algebraic methods can be used to solve these systems of polynomial equations (Raghavan and Roth 1995). The latter method solves the polynomial system by eliminating all but one variable that leads to a univariate polynomial. All the solutions are then calculated by finding the roots of this polynomial. However finding the univariate polynomial is either tedious and/or mathematically complex. Therefore, one has to resort to numerical techniques to solve such a system. Polynomial continuation method computes the solutions of a system of polynomials by tracing continuously solution paths from a polynomial start system to the final one (Varedi *et al.* 2009, Wu 2006).

The objective of the present work is to design a 3R serial manipulator so that it guides a rigid body through three precision poses. For three prescribed poses the problem leads to 18 closure equations, while we have 24 parameters including Denavit-Hartenberg (DH) parameters of the manipulator (Lee and Mavroidis 2001). Therefore, to solve the exact synthesis problem, six of the design parameters are set as free choices and their values are selected arbitrarily.

Throughout the development of kinematics, numerous mathematic theories and tools have been introduced. Dual quaternion is a powerful mathematical tool for spatial analysis and synthesis of rigid body motions (Yang and Freudenstein 1964, Bottema and Roth 1979, McCarthy 1990). It is perhaps the most compact and efficient tool to express the general rigid body motion known as screw motion (Funda and Paul 1990, Funda *et al*. 1990, Gouasmi *et al*. 2012, Mohammadi Daniali *et al*. 1995, Gan *et al*. 2008). In this paper, we use dual quaternions to formulate the problem. Using this powerful tool, we derived different formulations, in terms of DH parameters and the pose of the end-effector, than those reported by Lee and Mavroidis (Lee and Mavroidis 2001). Upon selecting the values of six design parameters, we reduce the closure equations to five equations in eleven unknowns for one precision pose. It is noteworthy that each additional precision pose adds four equations, while only introduces one new variable to the system. Therefore, the three precision poses leads to thirteen design equations in thirteen unknowns. Four different cases for selecting the free choices are considered and their design equations are solved using Bertini software (Sommese *et al*. 2006). It is noteworthy that the types introduced here are different than those reported by Lee and Mavroidis (Lee and Mavroidis 2001). Moreover, we include a numerical example in which the multi-homogeneous bounds for the solutions paths for the first type is 1152 and the bounds for the rest is 9216, while only 60 (384; 416; 296) paths converge to the finite solutions for type 1 (type2; type3; type4) formulation.

2. Polynomial continuation

Polynomial systems are found in many scientific fields. When dealing with any numerical problem, e.g., the Newton-Raphson method, there are two troublesome questions. One is that good initial guesses are not easy to detect and another is related to whether the method will converge to useful solutions. Homotopy continuation method can eliminate these shortcomings (Wu 2006). This method involves following paths from the solutions of a simpler system to the solutions of the target system (Morgan *et al*. 1990).

To find the solution in a system of nonlinear equations, a new simple start system, called auxiliary homotopy function is chosen, as

$$\boldsymbol{G}(\boldsymbol{X})=0 \tag{3}$$

$\boldsymbol{G}(\boldsymbol{X})$ must be known or controllable and easy to solve. Then, the homotopy continuation function is as follows:

$$\boldsymbol{H}(\boldsymbol{X}, t) \equiv t\boldsymbol{F}(\boldsymbol{X}) + (1-t)\boldsymbol{G}(\boldsymbol{X}) = 0 \tag{4}$$

In which t is an arbitrary parameter and changes from 0 to 1, i.e., $t \in [0, 1]$. Thus, we have the following two boundary conditions

$$\boldsymbol{H}(\boldsymbol{X}, 0) = \boldsymbol{G}(\boldsymbol{X}), \qquad \boldsymbol{H}(\boldsymbol{X}, 1) = \boldsymbol{F}(\boldsymbol{X}) \tag{5}$$

The purpose is to solve the $\boldsymbol{H}(\boldsymbol{X}, t)=0$ instead of $\boldsymbol{F}(\boldsymbol{X})=0$ by varying parameter t from 0 to 1 and avoid divergence. To this end, (Wu 2006) suggested some useful auxiliary homotopy functions. They are polynomial, harmonic, exponential or any combinations of them. By appropriate choosing/adjusting the auxiliary homotopy function, we can obtain the solutions of the system.

Moreover, a comprehensive reference on the numerical solution of systems of polynomials and homotopy continuation can be found in (Sommese and Wampler 2005).

3. Dual quaternions

Here, we give an overview on dual quaternions and how they can be used to represent screw motion.

Unit Quaternions: Unit quaternion q is a rotation operator. The word quaternion is derived from Latin word quaterni and means a set of four. It is a linear combination of four quaternion units, 1, **i**, **j**, and **k**, namely,

$$q = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

With the definitions,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\mathbf{ij} = \mathbf{k},\ \mathbf{jk} = \mathbf{i},\ \mathbf{ki} = \mathbf{j}$$

Moreover, d, a, b, and c are all real numbers. The three quaternion units **i**, **j**, and **k** can be considered as orthogonal unit vectors with respect to the scalar product. A unit quaternion is a quaternion whose norm is unit takes on the general form

$$\text{Norm of a quaternion} = \sqrt{d^2 + a^2 + b^2 + c^2} \tag{1}$$

$$q = cos\left(\frac{\theta}{2}\right) + \mathbf{q}\, sin\left(\frac{\theta}{2}\right) \tag{2}$$

in which, $\mathbf{q}$ is the direction of rotation and θ is the angle of rotation. This operator is able to rotate a unit vector as depicted in Fig. 1, i.e.,

$$\mathbf{w} = q\ \mathbf{v}\ k(q) \tag{3}$$

in which $k(q)$ is the conjugate of q and is defined as

$$k(q) = cos\left(\frac{\theta}{2}\right) - \mathbf{q}\, sin\left(\frac{\theta}{2}\right) \tag{4}$$

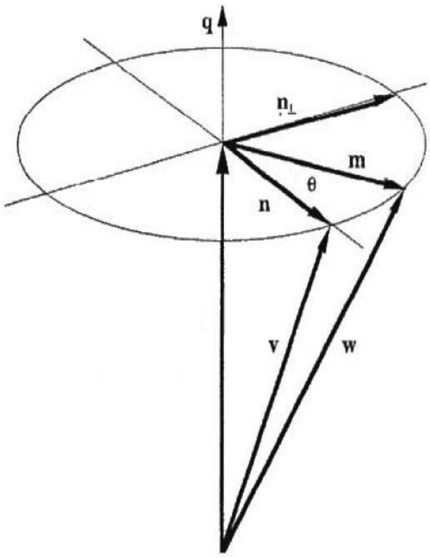

Fig. 1 Unit quaternion as a rotation operator

Lines: A dual vector $\hat{\mathbf{a}}$ is defined as the sum of a real part and a dual part. Moreover, a line can be specified by a unit dual vector; namely

$$\hat{\mathbf{a}} = \mathbf{a} + \epsilon \mathbf{a_0} \qquad \mathbf{a.a} = 1 \qquad \mathbf{a_0} = \mathbf{p} \times \mathbf{a} \tag{5}$$

in which the six real coefficients in **a** and $\mathbf{a_0}$ are the Pluker coordinates of the line and **p** is the position vector of a point on the line. Note that ϵ denotes the dual unit, which is a quasi-imaginary unit with two properties, namely

$$\epsilon \neq 0\,, \qquad \epsilon^2 = 0 \tag{6}$$

Unit Dual Quaternions: A unit dual quaternion is a unit quaternion with dual quantities, i.e.

$$\hat{p} = cos(\hat{\theta}/2) + sin(\hat{\theta}/2)\hat{\mathbf{s}} \tag{7}$$

$$\hat{\theta} = \theta + \epsilon\, s \tag{8}$$

in which $\hat{\mathbf{s}}$ is the line of the dual quaternion. It can be used as an operator to transform any line in 3-dimensional space in a similar way to how quaternions operate vectors, i.e.

$$\hat{\boldsymbol{w}} = \hat{q}\ \hat{\boldsymbol{v}}\ k(\hat{q}) \tag{9}$$

4. Problem formulation

In this work, the relative positions of links and joints are described using DH parameters. This convention allows one to write the kinematic equations with dual quaternions.

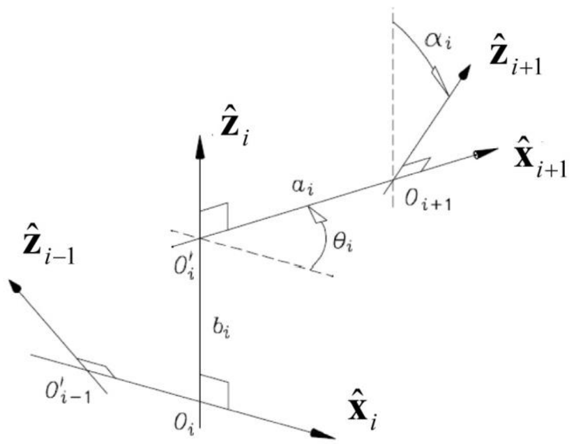

Fig. 2 DH parameters in three successive coordinate frames

This mathematical tool is perhaps the best which offers an elegant geometric insight of the screw motions involved in the DH parameters. As depicted in Fig. 2, by two successive screw motions, frame i can be transformed to frame i+1. In the first motion, the axis of the screw is $\hat{\mathbf{z}}_{\boldsymbol{i}}$ and the associated dual angle is $\hat{\theta}_i$, i.e.

$$\hat{q}_i = cos(\hat{\theta}_i/2) + sin(\hat{\theta}_i/2)\,\hat{z}_i \tag{10}$$

$$\hat{\theta}_i = \theta_i + \epsilon\, b_i \;\;,\;\; \hat{z}_i = \begin{bmatrix}0\\0\\1\end{bmatrix} + \epsilon\,\begin{bmatrix}0\\0\\0\end{bmatrix} \tag{11}$$

where b_i and θ_i are the twist angle and the distance between $\hat{\mathbf{x}}_{\boldsymbol{i}}$ and $\hat{\mathbf{x}}_{\boldsymbol{i+1}}$, respectively.

The axis of the second screw is $\hat{\mathbf{x}}_{\boldsymbol{i+1}}$ and the dual angle is $\hat{\alpha}_i$

$$\hat{p}_i = cos(\hat{\alpha}_i/2) + sin(\hat{\alpha}_i/2)\,\hat{x}_{i+1} \tag{12}$$

$$\hat{\alpha}_i = \alpha_i + \epsilon\, a_i \;\;,\;\; \hat{x}_{i+1} = \begin{bmatrix}1\\0\\0\end{bmatrix} + \epsilon\,\begin{bmatrix}0\\0\\0\end{bmatrix} \tag{13}$$

in which a_i and α_i are the twist angle and the distance between $\hat{\mathbf{Z}}_{\boldsymbol{i}}$ and $\hat{\mathbf{Z}}_{\boldsymbol{i+1}}$, respectively.

Here we consider 3R spatial manipulator studied by Lee and Mavroidis [10], as its kinematic model is depicted in Fig. 3. A reference frame $\{o\}$ and a moving frame $\{e\}$ attached to the base and the end-effector, respectively. The latter frame is given in three distinct spatial poses.

Therefore, the kinematic equations using unit dual quaternions can be written as

$$\hat{q}_0\,\hat{p}_0\,\hat{q}_1\,\hat{p}_1\hat{q}_2\hat{p}_2\,\hat{q}_3\,\hat{p}_3\,\hat{q}_e = \hat{r} \tag{14}$$

in which, $\hat{q}_e$ is a unit dual quaternion to transform frame {4} to the frame {e}

$$\hat{q}_e = cos(\hat{\emptyset}/2) + sin(\hat{\emptyset}/2)\,\hat{z}_4 \tag{15}$$

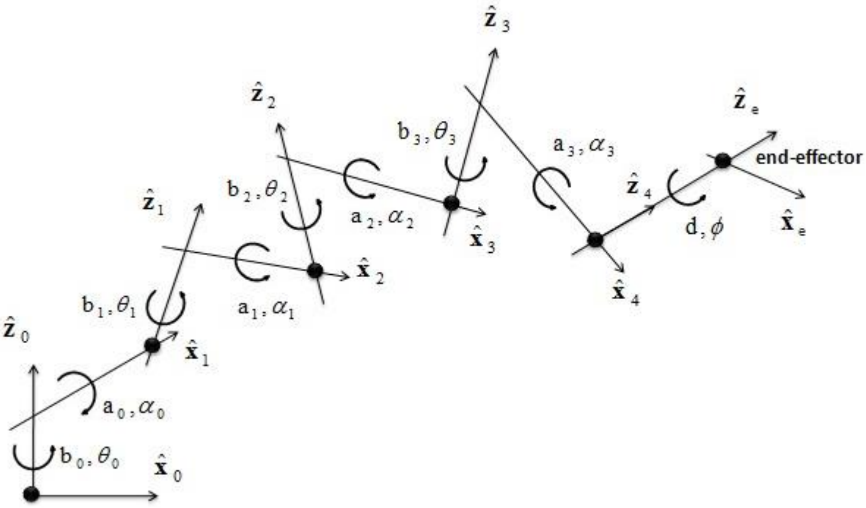

Fig. 3 The kinematic model of 3R spatial manipulator

$$\hat{\emptyset} = \emptyset + \epsilon\, d \;\;,\;\; \hat{z}_4 = \begin{bmatrix}1\\0\\0\end{bmatrix} + \epsilon \begin{bmatrix}0\\0\\0\end{bmatrix} \tag{16}$$

where d and $\emptyset$ are the twist angle and the distance between $\hat{\mathbf{x}}_4$ and $\hat{\mathbf{x}}_e$, respectively. Moreover, $\hat{r}$ is a unit dual quaternion which relates frame $\{o\}$ to frame $\{e\}$ and is known for each precision pose.

5. Design equations at each precision pose

The loop closure Eq. (13), leads to eight scalar design equations in the unknowns, namely; the structural parameters of the manipulator and the joint variables θ_1, θ_2 and θ_3. To simplify the solution process, we eliminate two joint variables θ_1, θ_2 from the design equations. To this end, we rewrite Eq. (13) as

$$\hat{p}_0\, \hat{q}_1\, \hat{p}_1\, \hat{q}_2\, \hat{p}_2 \;=\; k(\hat{q}_0)\, \hat{r}\;\; k(\hat{q}_e)\, k(\hat{p}_3)\, k(\hat{q}_3) \tag{17}$$

which can be expressed as

$$\left(L_{r_1} + \begin{bmatrix}L_{r_2}\\L_{r_3}\\L_{r_4}\end{bmatrix}\right) + \epsilon\left(L_{d_1} + \begin{bmatrix}L_{d_2}\\L_{d_3}\\L_{d_4}\end{bmatrix}\right) = \left(R_{r_1} + \begin{bmatrix}R_{r_2}\\R_{r_3}\\R_{r_4}\end{bmatrix}\right) + \epsilon\left(R_{d_1} + \begin{bmatrix}R_{d_2}\\R_{d_3}\\R_{d_4}\end{bmatrix}\right) \tag{18}$$

where those terms with subscripts r and d are related to the real part and the dual part, respectively. Moreover, capital letters L and R stand for the left and the right hand sides of the equation, respectively.

Four scalar equations of the real part only include rotations, while those four scalar equations related to the dual part include both rotations and translations. Let consider the trigonometric expression containing θ_1 and θ_2 in the left hand side of Eq. (17) i.e., L_{ri} and L_{di} for i=1,2,3,4 as

$$V_1 = c\theta_1\, c\theta_2, \quad V_2 = s\theta_1\, s\theta_2$$
$$V_3 = s\theta_1\, c\theta_2, \quad V_4 = c\theta_1\, s\theta_2 \tag{19}$$

Then, substituting V_i, for i=1,2,3,4 from real part L_{ri} into the dual part L_{di} and simplifying them, leads to the following equations free from θ_1 and θ_2, i.e.,

$$\begin{aligned}
&R_{r_1}\frac{a_1\,c\alpha_1 - a_1\,c(\alpha_0+\alpha_2)}{2\,s\alpha_1} - R_{r_2}\frac{a_0\,s\alpha_1 + a_2\,s\alpha_1 + a_1\,s(\alpha_0+\alpha_2)}{2\,s\alpha_1}\\
&\quad + R_{r_3}\frac{b_2\,c(\alpha_1+\alpha_2) - b_2\,c(\alpha_1-\alpha_2) - b_1\,c(\alpha_0+\alpha_1) + b_1\,c(\alpha_1-\alpha_0)}{4\,s\alpha_1}\\
&\quad - R_{r_4}\frac{b_2\,s(\alpha_1+\alpha_2) + b_2\,s(\alpha_1-\alpha_2) + b_1\,s(\alpha_0+\alpha_1) + b_1\,s(\alpha_1-\alpha_0)}{4\,s\alpha_1}\\
&\quad = R_{d_1}
\end{aligned}$$

$$\begin{aligned}
&R_{r_2}\frac{a_1\,c\alpha_1 + a_1\,c(\alpha_0+\alpha_2)}{2\,s\alpha_1} + R_{r_1}\frac{a_0\,s\alpha_1 + a_2\,s\alpha_1 - a_1\,s(\alpha_0+\alpha_2)}{2\,s\alpha_1}\\
&\quad + R_{r_4}\frac{b_2\,c(\alpha_1+\alpha_2) - b_2\,c(\alpha_1-\alpha_2) + b_1\,c(\alpha_0+\alpha_1) - b_1\,c(\alpha_1-\alpha_0)}{4\,s\alpha_1}\\
&\quad + R_{r_3}\frac{b_2\,s(\alpha_1+\alpha_2) + b_2\,s(\alpha_1-\alpha_2) - b_1\,s(\alpha_0+\alpha_1) - b_1\,s(\alpha_1-\alpha_0)}{4\,s\alpha_1}\\
&\quad = R_{d_2}
\end{aligned}$$

$$\begin{aligned}
&R_{r_3}\frac{a_1\,c\alpha_1 + a_1\,c(\alpha_2-\alpha_0)}{2\,s\alpha_1} + R_{r_4}\frac{a_2\,s\alpha_1 - a_0\,s\alpha_1 - a_1\,s(\alpha_2-\alpha_0)}{2\,s\alpha_1}\\
&\quad + R_{r_1}\frac{b_2\,c(\alpha_1-\alpha_2) - b_2\,c(\alpha_1+\alpha_2) + b_1\,c(\alpha_0+\alpha_1) - b_1\,c(\alpha_1-\alpha_0)}{4\,s\alpha_1}\\
&\quad + R_{r_2}\frac{b_1\,s(\alpha_0+\alpha_1) + b_1\,s(\alpha_1-\alpha_0) - b_2\,s(\alpha_1+\alpha_2) - b_2\,s(\alpha_1-\alpha_2)}{4\,s\alpha_1}\\
&\quad = R_{d_3}
\end{aligned} \tag{20}$$

$$\begin{aligned}
&R_{r_4}\frac{a_1\,c\alpha_1 - a_1\,c(\alpha_2-\alpha_0)}{2\,s\alpha_1} + R_{r_3}\frac{a_0\,s\alpha_1 - a_2\,s\alpha_1 - a_1\,s(\alpha_2-\alpha_0)}{2\,s\alpha_1}\\
&\quad + R_{r_2}\frac{b_2\,c(\alpha_1-\alpha_2) - b_2\,c(\alpha_1+\alpha_2) + b_1\,c(\alpha_1-\alpha_0) - b_1\,c(\alpha_1+\alpha_0)}{4\,s\alpha_1}\\
&\quad + R_{r_1}\frac{b_1\,s(\alpha_0+\alpha_1) + b_1\,s(\alpha_1-\alpha_0) + b_2\,s(\alpha_1+\alpha_2) + b_2\,s(\alpha_1-\alpha_2)}{4\,s\alpha_1}\\
&\quad = R_{d_4}
\end{aligned}$$

As mentioned earlier, for three precision poses, six of the design parameters are set as free choices and their values are selected arbitrarily. We choose θ_0, Ø as two of these six free choices. Then, R_{ri} and R_{di} for i=1,2,3,4 can be expressed as,

$$R_{r_i} = A_i\, c\theta_3\, c\alpha_3 + B_i\, s\theta_3\, s\alpha_3 + D_i\, c\theta_3\, s\alpha_3 + E_i\, s\theta_3\, c\alpha_3 \tag{21}$$

$$R_{d_i} = L_i\, c\theta_3\, c\alpha_3 + M_i\, s\theta_3\, s\alpha_3 + N_i\, c\theta_3\, s\alpha_3 + H_i\, s\theta_3\, c\alpha_3 \tag{22}$$

where $c\alpha_3$=cos(α_3/2), $s\alpha_3$=sin(α_3/2), $c\theta_3$=cos(θ_3/2) and $s\theta_3$=sin(θ_3/2). Moreover, A_i, B_i, D_i, and E_i depend only on the free selected parameters and the EE pose, while L_i, M_i, N_i, H_i are given as

$$\begin{aligned} L_i &= l_{i1}\, a_3 + l_{i2}\, b_0 + l_{i3}\, b_3 + l_{i4}\, d + l_{i5} \\ M_i &= m_{i1}\, a_3 + m_{i2}\, b_0 + m_{i3}\, b_3 + m_{i4}\, d + m_{i5} \\ N_i &= n_{i1}\, a_3 + n_{i2}\, b_0 + n_{i3}\, b_3 + n_{i4}\, d + n_{i5} \\ H_i &= h_{i1}\, a_3 + h_{i2}\, b_0 + h_{i3}\, b_3 + h_{i4}\, d + h_{i5} \end{aligned} \tag{23}$$

where the coefficients l_{ij}, m_{ij}, n_{ij}, h_{ij}, for i=1,2,3,4 and j=1,…,5 depend only on the data. Here, we choose a_1, α_1, α_2 and α_3 as four other free choices. Then, dividing Eqs. (19) by ($c\theta_3$) leads to

$$[M]_{4\times16}[v]_{16\times1} = [N]_{4\times2}[w]_{2\times1} \tag{24}$$

where matrix M depends on the data, the elements of matrix N depends linearly on the translational variables of the right hand sides of Eq. 0(19); v and w can be expressed as

$$v = [a_0 \quad a_2 \quad b_2 \quad s\alpha_0 \quad c\alpha_0 \quad b_1 s\alpha_0 \quad b_1 c\alpha_0 \quad T_{\theta_3} \quad T_{\theta_3}a_0 \quad T_{\theta_3}a_2 \quad T_{\theta_3}b_2 \quad T_{\theta_3}s\alpha_0 \quad T_{\theta_3}c\alpha_0 \quad T_{\theta_3}b_1 s\alpha_0 \quad T_{\theta_3}b_1 c\alpha_0 \quad 1]^T$$

$$w = [T_{\theta_3} \quad 1]^T \tag{25}$$

In which $T_{\theta3}$=tan (θ_3/2). It is noteworthy that each additional precision pose adds four equations to the system, while only introduces one new variable. Therefore, the three precision poses lead to 12 equations in 13 unknowns. Moreover, since $c\alpha_0$ and $s\alpha_0$ are regarded as independent variables there is a constraint equation $s\alpha_0^2+c\alpha_0^2-1=0$. Therefore, the new system is a multivariate polynomial system with 13 equations in 13 unknowns. This type is summarized in Table 1.

Alternatively, one can choose θ_0, Ø α_1, α_2, a_1 as five of the six free choices. Then, dividing Eqs. (19) by ($c\theta_3 c\alpha_3$) leads to

$$[M]_{4\times16}[v]_{16\times1} = [D]_{4\times4}[u]_{4\times1} \tag{26}$$

where the elements of D depend linearly on the translational variables of the right hand sides of Eq. (19) and u can be expressed as

$$u = [T_{\theta_3} \quad T_{\alpha_3} \quad T_{\theta_3}T_{\alpha_3} \quad 1]^T \tag{27}$$

In which $T_{\alpha3}$=tan(α_3/2). Similarly, the problem leads to 12 equations in 14 unknowns. Again, there is an additional constraint equation, namely; $s\alpha_0^2+c\alpha_0^2-1=0$. Thus, by incorporating this constraint equation the number of equation in the system will be 13. Finally, choosing any of three translational variables {a_0, a_2, b_0} as the free choices, reduces the number of variables to 13. Therefore, the new system will be a multivariate polynomial system with 13 equations in 13 unknowns. These types are summarized in Table 1.

Table 1 4 types of selecting free choices

type	free choices
type 1	$\alpha_1, \alpha_2, \theta_0, \emptyset, a_1, \alpha_3$
type 2	$\alpha_1, \alpha_2, \theta_0, \emptyset, a_1, a_2$
type 3	$\alpha_1, \alpha_2, \theta_0, \emptyset, a_1, b_0$
type 4	$\alpha_1, \alpha_2, \theta_0, \emptyset, a_1, a_0$

6. Solution procedure using polynomial continuation

In this section we design a 3R serial manipulator so that it guides a rigid body through three precision poses. Therefore, the foregoing types for selecting the free choices have been considered for three prescribed precision poses. We include a numerical example in which the system of each type is solved by the homotopy continuation method. The computation is carried out using Bertini software. This software is a general-purpose solver, written in C language by Sommese *et al*. (2006) and is publicly available (Sommese *et al*. 2006). The three poses are given as

$$\begin{gathered}
\hat{r}(pose1) = \left(0.439711915724543 + \begin{bmatrix} -0.688882686479297 \\ -0.524932267602671 \\ 0.237781811475222 \end{bmatrix}\right) \\
+\epsilon\left(1.69137689287664 + \begin{bmatrix} 1.16951408702047 \\ -0.0657019883633454 \\ 0.115443390960603 \end{bmatrix}\right) \\[2ex]
\hat{r}(pose2) = \left(0.24712940823383 + \begin{bmatrix} -0.181838058062701 \\ 0.893649179757841 \\ 0.327495220948523 \end{bmatrix}\right) \\
+\epsilon\left(-1.03107518555242 + \begin{bmatrix} 0.167458074920762 \\ -0.181477473236855 \\ 1.36623809473432 \end{bmatrix}\right) \\[2ex]
\hat{r}(pose3) = \left(0.903660574694749 + \begin{bmatrix} -0.321264889057329 \\ 0.277881763353437 \\ -0.0544808443117538 \end{bmatrix}\right) \\
+\epsilon\left(0.312856119053275 + \begin{bmatrix} 0.793200496856447 \\ 0.153665765812824 \\ 1.29566980065216 \end{bmatrix}\right)
\end{gathered} \tag{28}$$

It is noteworthy that, in the present study the angular parameters are given in degrees while the lengths can be considered in any system.

Types 1: We solve a numerical example for the data given in Table 2. Using a 5-partitions $G_1=\{T_{\theta 3}(pose1)\}$, $G_2=\{T_{\theta 3}(pose2)\}$, $G_3=\{T_{\theta 3}(pose3)\}$, $G_4=\{s\alpha_0, c\alpha_0\}$ and $G5=\{a_0,a_2,a_3,b_0,b_1,b_3,d\}$, the multi-homogeneous bound is found to be 1152.

A homotopy continuation method based on this 5-homogeneous number is employed and the numerical values of the variables are computed. Finally, the DH parameters of the design solutions are computed using a back-substitution procedure, which is outlined in the Appendix.

Table 2 The data for the free choices of type 1

	type 1
α_1	166.1577605879387
α_2	80.21409131831524
θ_0	22.91831180523293
$\emptyset$	45.83662361046586
a_1	1.3
α_3	126.0507149287811

Table 3 DH parameters of 3R manipulators for type 1

	#1	#2	#3	#4	#5	#6
$\theta_1(pose1)$	111.779	104.849	64.876	63.5257	101.591	68.7549
$\theta_2(pose1)$	210.365	203.044	162.336	161.007	199.612	166.158
$\theta_3(pose1)$	155.05	151.096	157.412	158.445	149.714	154.699
$\theta_1(pose2)$	171.201	-32.4075	156.756	157.896	158.455	-22.9183
$\theta_2(pose2)$	105.247	265.056	89.1582	90.4499	91.0823	-85.9437
$\theta_3(pose2)$	-159.912	177.795	-151.229	-151.917	-152.254	-177.617
$\theta_1(pose3)$	222.663	119.009	104.959	263.741	109.538	97.4028
$\theta_2(pose3)$	34.2935	-64.095	-78.2367	75.1275	-73.5987	-85.9437
$\theta_3(pose3)$	-96.8133	-118.646	-119.955	-92.2768	-119.616	-120.321
α_0	63.8072	69.3796	72.6403	71.9054	71.547	74.4845
a_0	3.11736	2.73325	6.79542	2.1685	10.0882	2.4
a_2	-0.03322	1.91247	3.60981	-1.3113	2.97305	1.7
a_3	-0.19502	0.80705	0.052769	-0.17737	0.079982	0.9
b_0	-0.4141	1.60622	3.54975	0.705258	7.36523	1.5
b_1	10.731	2.54292	-20.1835	12.638	-37.9413	1.1
b_2	9.46586	1.97369	-23.0594	11.7259	-40.2759	0.75
b_3	0.77594	2.4106	5.72664	0.242614	7.40174	2.1
d	2.58963	1.95359	6.06777	2.8846	5.74016	1.8

Table 4 The data for the free choices of types 2, 3 and 4

type 2		type 3		type 4	
α_1	166.1577605879387	α_1	166.1577605879387	α_1	166.1577605879387
α_2	80.21409131831524	α_2	80.21409131831524	α_2	80.21409131831524
θ_0	22.91831180523293	θ_0	22.91831180523293	θ_0	22.91831180523293
$\emptyset$	45.83662361046586	$\emptyset$	45.83662361046586	$\emptyset$	45.83662361046586
a_1	1.3	a_1	1.3	a_1	1.3
a_2	1.7	b_0	1.5	a_0	2.4

It is found that out of 1152 paths, only 60 paths converge to the true solutions, which only 6 of them are real. These solutions are all numerically distinct. The real distinct solutions for this type are given in Table 3.

Types 2 to 4: In these types the multi-homogeneous bound is found to be 9216. In the formulations we used a 6-partitions $G_1=T_{\theta 3}(pose1)\}$, $G_2=\{T_{\theta 3}(pose2)\}$, $G_3=\{T_{\theta 3}(pose3)\}$, $G_4=\{T_{\alpha 3}\}$, $G_5=\{s\alpha_0,c\alpha_0\}$ and $G_6=\{a_0,a_3,d,b_0,b_1,b_2,b_3\}$ for type 2; another 6-partitions $G_1=\{T_{\theta 3}(pose1)\}$, $G_2=\{T_{\theta 3}(pose2)\}$, $G_3=\{T_{\theta 3}(pose3)\}$, $G_4=\{T_{\alpha 3}\}$, $G_5=\{s\alpha_0,c\alpha_0\}$ and $G_6=\{a_0,a_2,a_3,b_1,b_2,b_3,d\}$ for type 3; and finally 6-partitions $G_1=\{T_{\theta 3}(pose1)\}$, $G_2=\{T_{\theta 3}(pose2)\}$, $G_3=\{T_{\theta 3}(pose3)\}$, $G_4=\{T_{\alpha 3}\}$, $G_5=\{s\alpha_0,c\alpha_0\}$ and $G_6=\{a_2,a_3,b_0,b_1,b_2,b_3,d\}$ for type 4. Now, we solve the numerical example for the data given in Table 4.

A homotopy continuation method based on these 6-homogeneous numbers is employed and the numerical values of the variables are computed. The DH parameters of the design solutions are computed using a back-substitution procedure, which are outlined in the Appendix. It is found that out of 9216 paths, only 384 paths for type2; 416 paths for type 3; and 296 paths for type 5 converge to the true solutions. These solutions are all numerically distinct, which only 12 in type 2, 24 in type 3, and 20 in type 4 are real. Although these solutions are all numerically different, they are not geometrically distinct. It is noteworthy that each geometrically distinct solution has two equivalent representations in terms of the DH parameters. Thus, despite of existing 12 numerically distinct solutions in type 2, there are only 6 distinct manipulators for the three precision poses problem. This is true for types 3 and 4, as well. Consequently, there are 12 distinct manipulators in type 3, and 10 distinct manipulators in type 4. These real geometrically distinct solutions for types 2 to 4 are given in Tables 5, 6 and 7, respectively.

To give a better view toward the solutions, solution 2 of type 1 is modeled in Fig. 4. SolidWorks is used to model the manipulator which is designed to be able to pass through three

Table 5 DH parameters of 3R manipulators for type 2

	#1	#2	#3	#4	#5	#6
$\theta_1(pose1)$	68.7549	-295.807	-2.05139	-227.982	-35.9564	-332.187
$\theta_2(pose1)$	166.158	28.0318	256.786	-127.226	223.653	-7.45762
$\theta_3(pose1)$	154.699	101.349	-164.753	145.898	-146.781	94.1494
$\theta_1(pose2)$	-22.9183	142.143	257.35	145.123	315.644	180.867
$\theta_2(pose2)$	-85.9437	-191.64	-31.5563	76.8086	26.2611	-151.506
$\theta_3(pose2)$	-177.617	-115.075	154.802	-154.637	155.237	-125.289
$\theta_1(pose3)$	97.4028	-252.112	239.247	130.994	-45.7989	-265.418
$\theta_2(pose3)$	-85.9437	39.5345	73.6227	-56.7481	-213.953	26.4164
$\theta_3(pose3)$	-120.321	-7.81622	117.596	-116.47	115.74	-8.65379
α_0	74.4845	156.377	-105.434	68.9967	-91.0788	158.436
α_3	126.051	-111.658	-65.4076	121.476	-60.9219	-110.65
a_0	2.4	0.89083	4.47878	25.1098	20.694	1.79041
a_3	0.9	-0.40151	-0.12831	0.164309	0.169408	-0.60097
b_0	1.5	-5.68828	0.838976	26.6801	-19.5853	-5.6515
b_1	1.1	0.599541	11.339	-136.402	113.988	-1.10064
b_2	0.75	7.44528	13.3312	-133.746	119.068	5.29345
b_3	2.1	0.421248	-0.99876	6.26557	-1.4425	1.28482
d	1.8	0.297599	3.94855	4.11246	3.69679	0.316494

Table 6 DH parameters of 3R manipulators for type 3

	#1	#2	#3	#4	#5	#6
$\theta_1(pose1)$	-212.914	-208.116	-211.122	-15.6233	-77.512	-50.6785
$\theta_2(pose1)$	-109.076	54.5779	-108.014	244.594	108.847	132.798
$\theta_3(pose1)$	139.176	54.4674	143.789	-165.77	110.904	108.517
$\theta_1(pose2)$	20.5826	18.3546	5.21997	234.232	-80.1357	-73.1612
$\theta_2(pose2)$	-44.8955	-222.831	301.602	-55.6475	-271.949	-261.218
$\theta_3(pose2)$	-174.064	-17.253	179.288	152.867	-47.5489	-48.8913
$\theta_1(pose3)$	121.457	95.6099	141.651	238.262	233.874	-44.3937
$\theta_2(pose3)$	-71.1871	-256.169	-50.5811	69.1749	-30.4718	-309.411
$\theta_3(pose3)$	-117.562	-60.0691	-114.226	117.61	-144.546	-131.269
α_0	73.5332	86.4019	68.7823	-107.389	131.666	135.988
α_3	116.363	-37.4148	117.169	-61.8702	28.3607	32.0699
a_0	0.84387	0.689601	2.37977	4.79421	6.72114	1.27025
a_2	1.3	1.3	1.3	1.3	1.3	1.3
a_3	4.3692	-3.86288	2.56661	2.46818	0.215365	-1.59165
b_1	20.5323	16.3617	10.1625	11.5926	13.2506	-0.06897
b_2	19.1837	15.7255	9.11092	13.7162	14.1788	-0.12519
b_3	3.91331	-4.38383	3.20651	-1.89061	-0.00134	0.691464
d	2.02653	3.60838	1.87008	4.43541	-0.86229	-0.59959
	#7	#8	#9	#10	#11	#12
$\theta_1(pose1)$	26.0964	40.7761	-300.611	-33.6182	68.7549	-7.93319
$\theta_2(pose1)$	298.488	139.293	-36.0618	-303.852	166.158	252.374
$\theta_3(pose1)$	20.766	159.423	35.0703	-25.8681	154.699	-169.989
$\theta_1(pose2)$	8.33404	139.626	121.072	-111.095	-22.9183	-6.49426
$\theta_2(pose2)$	-234.163	70.0688	-115.879	128.648	-85.9437	57.231
$\theta_3(pose2)$	-8.02836	-148.96	-39.1229	33.5235	-177.617	-179.676
$\theta_1(pose3)$	147.514	-82.0838	244.99	-117.129	97.4028	219.294
$\theta_2(pose3)$	-204.97	-273.749	-98.4694	234.682	-85.9437	51.6393
$\theta_3(pose3)$	-66.3928	-91.7531	-87.9112	61.5212	-120.321	114.403
α_0	77.9459	75.5247	86.4011	-98.0289	74.4845	-110.971
α_3	-39.7117	122.584	-32.9262	141.808	126.051	-62.9789
a_0	2.16062	1.18039	1.481	6.01159	2.4	2.16381
a_2	1.3	1.3	1.3	1.3	1.3	1.3
a_3	-0.5113	-2.83392	1.43826	-2.95442	1.7	0.495113
b_1	0.294728	17.4061	12.8214	17.4979	1.1	0.147766
b_2	-0.23541	16.3153	11.7047	20.0895	0.75	0.676289
b_3	-2.57513	0.694187	-2.56114	3.74643	2.1	-1.71378
d	3.13918	3.18777	4.83479	6.8081	1.8	1.92263

given poses. All solutions can be verified by using the well-known software in 3D modeling. However, because of lack of enough space, we showed just one of the solutions as an example.

Table 7 DH parameters of 3R manipulators for type 4

	#1	#2	#3	#4	#5	#6	#7	#8	#9	#10
$\theta_1(1)$	-218.135	86.1542	-248.121	68.7549	-241.539	-202.736	-18.7113	69.874	-211.425	283.901
$\theta_2(1)$	-116.927	264.948	14.9137	166.158	21.9291	63.9756	172.326	167.253	-108.378	20.0054
$\theta_3(1)$	150.826	-111.861	39.7108	154.699	40.2173	47.5082	101.173	158.246	143.894	-34.9567
$\theta_1(2)$	150.923	109.993	145.197	-22.9183	144.509	14.8694	-27.0203	162.253	4.55055	20.8275
$\theta_2(2)$	83.822	-61.9234	-90.9055	-85.9437	-92.0305	-227.618	-219.721	95.2904	300.978	260.817
$\theta_3(2)$	-160.307	49.3681	-34.1096	-177.617	-33.2384	-10.1357	-54.3785	-153.207	179.138	9.62009
$\theta_1(3)$	189.709	147.961	211.521	97.4028	205.592	139.265	277.746	258.517	141.799	-98.202
$\theta_2(3)$	-2.2346	49.5155	-131.515	-85.9437	-138.406	-214.019	12.3362	70.4314	-50.3372	247.216
$\theta_3(3)$	-103.257	145.225	-82.4943	-120.321	-81.0616	-64.6897	-143.451	-92.5853	-114.203	60.4675
α_0	62.6146	-52.1906	78.6247	74.4845	77.6885	79.2002	141.292	70.637	68.7097	-97.5205
α_3	120.439	-146.922	-31.1381	126.051	-31.9406	-40.3472	25.2583	126.684	117.274	148.937
a_2	0.043791	-0.86479	0.22436	1.7	0.146155	-1.78986	-0.41	-0.97782	2.54221	-1.33475
a_3	0.071447	-0.14862	-0.03273	0.9	-0.00945	0.449856	-0.03856	-0.19036	0.549558	0.689869
b_0	-2.48144	2.71076	-0.23085	1.5	-0.47227	1.27758	4.56967	0.565449	1.50532	1.24966
b_1	18.2559	0.895178	11.5886	1.1	12.0859	7.30547	8.11345	11.6441	9.97634	0.32946
b_2	16.0763	1.61053	10.5948	0.75	11.0717	6.80806	6.02756	10.7518	8.93482	0.684882
b_3	1.67737	-1.42557	-2.43606	2.1	-2.5268	-3.86784	2.6746	0.24352	3.1851	3.55405
d	2.45989	-0.59943	4.25738	1.8	4.17526	3.03341	-1.54456	2.80296	1.87011	3.52611

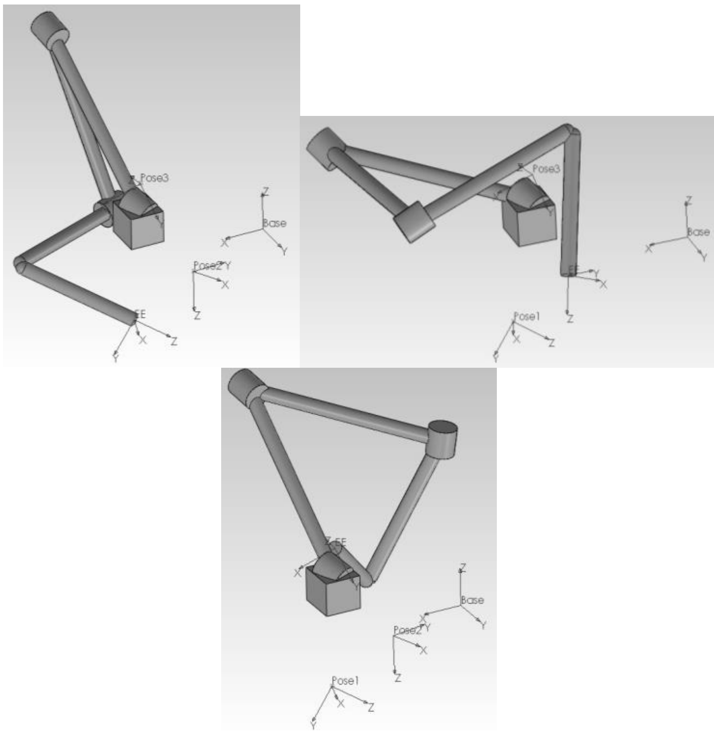

Fig. 4 Type 1, solution 2 at three precision poses

7. Conclusions

In this paper, the three precision poses geometric synthesis problem of serial manipulators with three revolute joints has been solved using a polynomial continuation method. We used dual quaternion to formulate the problem. This powerful mathematical tool and our novel elimination technique have led to some new types of 3R manipulators for the problem. Upon choosing six of the design parameters arbitrarily as the free choices, the problem led to a system of thirteen polynomials in thirteen unknowns. We introduced four new types of formulations according to the set of the free choices and solved the system using polynomial homotopy continuation. Numerical example is included in which the multi-homogeneous bounds for the solutions paths for the first type is 1152 and for the remaining three types are 9216, while only 60 (384; 416; 296) paths converge to the finite solutions for type 1 (type2; type3; type4) formulation.

References

Mavroidis, C., Lee, E. and Alam, M. (2001), “A new polynomial solution to the geometric design problem of spatial R-R robot manipulators using the Denavit and Hartenberg parameters”, *J. Mech. Des*., ASME, **123(**1), 58-67.

Huang, C. and Lai, C. (2012), “Spatial generalizations of planar point-angle and path generation problems”, *J. Mech. Robot*., ASME, **4**(3), 031010.

Tsai, L.W. (1972), “Design of open loop chains for rigid body guidance”, Ph.D. Thesis, *Department of Mechanical Engineering*, Stanford University.

Roth, B. (1986), “Analytic design of open chains”, *Proceedings of the Third International Symposium of Robotic Research*, Eds. O. Faugeras and G. Giralt, MIT Press, Cambridge, MA.

Dhingra, A.K., Cheng, J.C. and Kohli, D. (1994), “Synthesis of six-link, slider-crank and four-link mechanisms for function, path and motion generation using homotopy with m-Homegenization”, *J. Mech. Des.*, ASME, **116**, 1122-1130.

Lee, E. and Mavroidis, C. (2003), “Four precision points geometric design of spatial 3R manipulators”, *Proceedings of IFTOMM’s 11th World Congress in Mechanism and Machine Science*, 2003 Tianjin-China.

Raghavan, M. and Roth, B. (1995), “Solving polynomial systems for the kinematic analysis and synthesis of mechanisms and robot manipulators”, *J. Mech. Des.*, ASME, **117**(B), 71-78.

Varedi, S.M., Daniali, H.M. and Ganji, D.D. (2009), “Kinematics of an offset 3-UPU translational parallel manipulator by the homotopy continuation method”, *Nonlin. Anal. Real World Appl.*, **10**(3), 1767-1774.

Wu, T.M. (2006), “Solving the nonlinear equations by the Newton-homotopy continuation method with adjustable auxiliary homotopy function”, *Appl. Math. Comput.*, **173**(1), 383-388.

Lee, E. and Mavroidis, C. (2001), “Solving the geometric design problem of spatial 3R robot manipulators using polynomial homotopy continuation”, *J. Mech. Des.*, ASME, **124**(4), 652-661.

Yang, A.T. and Freudenstein, F. (1964), “Application of dual-number quaternion algebra to the analysis of spatial mechanisms”, *J. Appl. Mech.*, ASME, 300-308.

Bottema, O. and Roth, B. (1979), *Theoretical Kinematics*, Dover Publications, New York.

McCarthy, J.M. (1990), *Introduction to Theoretical Kinematics*, The MIT Press, Cambridge, MA.

Funda, J. and Paul, R.P. (1990), “A computational analysis of screw transformations in Robotics”, *IEEE Trans. Robot. Automat.*, **6**(3), 348-356.

Funda, J., Taylor, R.H. and Paul, R.P. (1990), “On homogenous transformations, quaternions, and computational efficiency”, *IEEE Trans. Robot. Automat*., **6**, 382-388.

Gouasmi, M., Ouali, M., Brahim, F. (2012), “Robot Kinematics Using Dual Quaternions”, *Int J. Robot. Automat. (IJRA)*, **1**(1), 13-30.

Sommese, A.J., Wampler, C.W., Hauenstein, J.D. and Bates, D.J. (2006), *Bertini: Software for Numerical Algebraic Geometry*, 1-14.

Wu, T.M. (2006), "The inverse kinematics problem of spatial 4P3R robot manipulator by the homotopy continuation method with an adjustable auxiliary homotopy function", *Nonlin. Anal.*, **64**(10), 2373-2380.

Morgan, A.P., Wampler, C.W. and Sommese, A.J. (1990), "Numerical continuation methods for solving polynomial systems arising in kinematics", *J. Mech. Des.*, ASME, **112**(1), 59-68.

Sommese, A.J. and Wampler, C.W. (2005), *Numerical solution of systems of polynomials arising in engineering and science*, World Scientific Press, Singapore.

Mohammadi Daniali, H.R., Zsombor-Murray, P.J. and Angeles, J. (1995), "Direct kinematics of double-triangular parallel manipulators", *Math. Pannonica*, **7**(1), 79-96.

Gan, D., Liao, Q., Wei, S., Dai, J.S. and Qiao, S. (2008), "Dual quaternion-based inverse kinematics of the general spatial 7R mechanism", *Proceedings of the IMechE, Part C: J. Mech. Eng.*, **222**(8), 1593-1598.

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Appendix

Once the numerical values of the variables are calculated using the polynomial continuation method, V_1, V_2, V_3 and V_4 in Eq. (18) can be found using back-substitution procedure. Moreover, one can calculate θ_1 and θ_2 as follows

$$\theta_1 = Arctan\frac{V_3 + V_4}{V_1 - V_2} + Arctan\frac{V_3 - V_4}{V_1 + V_2} \tag{29}$$

$$\theta_2 = Arctan\frac{V_3 + V_4}{V_1 - V_2} - Arctan\frac{V_3 - V_4}{V_1 + V_2} \tag{30}$$