Dynamic analysis of nanosize FG rectangular plates based on simple nonlocal quasi 3D HSDT

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Abstract. In the present work the dynamic analysis of the functionally graded rectangular nanoplates is studied. The theory of nonlocal elasticity based on the quasi 3D high shear deformation theory (quasi 3D HSDT) has been employed to determine the natural frequencies of the nanosize FG plate. In HSDT a cubic function is employed in terms of thickness coordinate to introduce the influence of transverse shear deformation and stretching thickness. The theory of nonlocal elasticity is utilized to examine the impact of the small scale on the natural frequency of the FG rectangular nanoplate. The equations of motion are deduced by implementing Hamilton’s principle. To demonstrate the accuracy of the proposed method, the calculated results in specific cases are compared and examined with available results in the literature and a good agreement is observed. Finally, the influence of the various parameters such as the nonlocal coefficient, the material indexes, the aspect ratio, and the thickness to length ratio on the dynamic properties of the FG nanoplates is illustrated and discussed in detail.

Keywords: nonlocal elasticity theory; FG nanoplate; free vibration; refined theory; elastic foundation

1. Introduction


The plate-as nanostructures like nanoplates or nano-scale sheets are very important kinds of the nanostructures with 2D shapes (Shahadat et al. 2018). They contain important mechanic properties (Iijima 1991, Miller and Shenoy 2000, Shen and Zhang 2010, Pradhan and Phadikar 2009, Eltaheer et al. 2012, 2016, Ebrahimi and Salari 2015, Khoshshid et al. 2015, Chemi et al. 2015, Akbaş 2016, Ghorbanpour Arani et al. 2012, Janghorban 2016, Wu et al. 2018) and with these unique characteristics they become ideal candidates for multifarious field of nanotechnology industry incorporating energy storage (Ma et al. 2008), nano electrome-chanical systems, strain, mass and pressure sensors (Sakhaee-Pour et al. 2008a, b), solar cells (Aagesen and Sorensen 2008), photo-catalytic degradation of organic dye (Ye et al. 2006), composite materials (Rafiee et al. 2010) and ect. The size-dependent continuum modeling of the nanostructures has taken a wide attention by the scientific community because the controlled experimentations in nanosize are difficult and molecular dynamic simulations are highly expensive computationally. We can found in the literature various size dependent continuum models such as modified couple stress theory (Koiter 1969, Mindlin and Tiersten 1962, Toupin 1962), strain gradient elasticity theory (Nix and Gao 1998, Lam et al. 2003, Aifantis 1999, Li et al. 2016) and nonlocal elasticity theory (Eringen 1972). Among these models, the
theory of nonlocal elasticity has been widely employed (Peddisen et al. 2003, Reddy 2007, Reddy and Pang 2008, Heireche et al. 2008, Murmu and Pradhan 2009a, b, Wang 2009). To overcome the shortcomings of the conventional elasticity theory, Eringen and Edelen (1972) proposed the nonlocal elasticity model in 1972. They modified the conventional continuum mechanics to consider the small scale influences. It should be noted that in the nonlocal elasticity theory, the tensor of stress at an arbitrary point in the continuum of nano-material is related not only on the tensor of strain at that point but also on the tensor of strain at all other points in the continuum. Both the atomistic simulation data and the experimental studies on phonon dispersion indicated the accuracy of this remark (Eringen 1983, Chen et al. 2004).


In addition, structural complements such as plates, beams and membranes in micro or nano-length size are often employed as elements in micro/nano electromechanical systems (MEMS/NEMS). Thus understanding the mechanics and physics characteristics of nanostructures is necessary for its practical uses. In past decades, the dynamic of FGMs has been employed extensively. Malekzadeh and Heydarpour (2012) studied the dynamic behavior of rotating FG cylindrical shells under thermal environment by using the first-order shear deformation theory (FSDT) of shells. Unghbakorn and wattanasakulpong (2013) examined the thermo-elastic dynamic response of FG plates carrying distributed patch mass based on HSDT. Kumar and Lal (2013) examined the first three natural frequencies of the free axisymmetric vibration of the 2D FG annular plates resting on Winkler foundation by employing differential quadrature technique and Chbyshhev collocation method. Based on the 3D theory of elasticity and considering that the mechanical characteristics of the materials changed continuously in the direction of thickness, the 3D free and forced vibration investigation of FG circular plate with various boundary conditions was established by Nie and Zhong (2007). 3D elasticity theory was utilized, and novel sets of admissible functions for the kinematics were developed to improve the effectiveness of the Ritz technique in modeling the behavior of the cracked plates. Matsunaga (2008) analyzed the buckling stresses and the natural frequencies of FG plates by considering the influences of transverse shear and normal deformations. Ke et al. (2013) proposed a non-conventional micro-plate model for the axisymmetric nonlinear dynamic analysis of annular FG micro-plates by using the modified couple stress theory, FSDT and von-Karman geometric nonlinearity theory. Ke et al. (2012) also investigated the bending, stability and dynamic of annular FG micro-plates based on the modified couple stress theory and FSDT. Asghari and Taati (2013) employed a size-dependent approach for mechanical investigations of FG micro-plates based on the modified theory of couple stress. Kocaturk and Akbas (2012) examined the thermal influence on post-buckling response of FGM beams based on Timoshenko beam theory and by employing finite element formulation. The vibration characteristics of beam with power law properties graduation in the transversal or the axial directions was reported by Alshorbagy et al. (2011). Recently, Eltaher et al. (2012, 2013a) used a finite element approach for dynamic investigation of FG nanoscale beams based on nonlocal Euler-Bernoulli beam theory. They also discussed the size-dependent bending-buckling response of FG nanobeams by using the nonlocal continuum theory (Eltaher et al. 2013b). Dynamic behavior of simply supported Timoshenko FG nanoscale beams were studied by Rahman and Pedram (2014). Zemri et al. (2015) investigated the mechanical response of FG nanoscale beam using a refined nonlocal shear deformation theory beam theory. Belkorissat et al. (2015) examined the dynamic properties of FG nano-plate using a new nonlocal refined four variable theory. Ahouel et al. (2016) studied the size-dependent mechanical behavior of FG trigonometric shear deformable nanobeams including neutral surface position concept. Bououara et al. (2016) presented a nonlocal zeroth-order shear deformation theory for free vibration of FG nanoscale plates resting on elastic foundation. Khetir et al. (2017) developed a novel nonlocal trigonometric shear deformation theory for thermal buckling analysis of embedded nanosize FG plates. Bouafia et al. (2017) proposed a nonlocal quasi-3D theory for bending and free flexural vibration behaviors of FG nanobeams. Besseghei et al. (2017) analyzed the dynamic response of embedded nanosize FG plates using a new nonlocal trigonometric shear deformation theory. Mouffoki et al. (2017) examined the dynamic response of nonlocal advanced nanobeams in hygro-thermal environment using a new two-unknown trigonometric shear deformation beam theory. Karami et al. (2019c) investigated the wave propagation of FG anisotropic nanoplates resting on Winkler-Pasternak foundation. Recently, several authors proposed advanced plate/beam theories to study the mechanical behavior of nano- or macro-structures (Belabed et al. 2014, Hamidi et al. 2015, Kar and Panda 2016a, b, Bousahla et al. 2014, Beldjelil et al. 2016, Sahoo et al. 2016, Draiche et al. 2016, Bouazza et al. 2016, Mehar and Panda 2016, Becheri et al. 2016, Katariya et al. 2017a, b, c, El-Haina et al. 2017, Fahsi et al. 2017, Mehar et al. 2017, Ebrahimii et al. 2017, Chikh et al. 2017, Sahoo et al. 2017, Abdelaziz et al. 2017, Singh and Panda 2017, Hirwani et al. 2017, Katariya and Panda.
In the current work, the dynamic of FG nanoscale plates is studied based on the cubic quasi 3D high shear deformation theory in the conjunction with the nonlocal elasticity model. By considering the integral term in the kinematic led to a reduction in the number of variables and equations of motion. The Navier solution is employed to investigate the dynamic behavior of the FG nanoplates. It is considered that the material characteristics are varying within the thickness according to the power law variation. Numerical results are provided to be utilized as benchmarks for the application and the design of nanoelectronic and nano-drive devices, nano-oscillators, and nanosensors, in which nanoplates act as basic elements. They can also be useful as valuable sources for validating other approximate methods and formulations.

2. Theory and formulation

2.1 Nonlocal power-law FG nanoplate equations

Consider a rectangular nanoscale plate of length $a$, width $b$, and total thickness $h$ and composed of FGMs within the thickness as demonstrated in Fig. 1.

$$E(z) = (E_c - E_m)V_f(z) + E_m$$
$$\rho(z) = (\rho_c - \rho_m)V_f(z) + \rho_m$$

(1)

(2)

where the subscripts $c$ and $m$ denote the ceramic and metallic constituents, respectively, and $V_f$ is the volume fraction that is given by the following expression

$$V_f(z) = \left( \frac{z}{h} + \frac{1}{2} \right)^n$$

(3)

where $n$ is the gradient index and takes only positive values. Poisson’s ratio $\nu$ is the same for all the ceramic/metal materials that are employed here, so it is considered to be constant and is assumed to be equal to 0.3 throughout the investigation (Reddy 2011). The typical values for metals and employed in the FG nanoscale plate are reported in Table 1.

2.2 The nonlocal elasticity theory

In nonlocal theory, the field of stress at each point body is a function of the field of strain. So stress plays a considerable role in the model which is presented by the following expression (Khorshidi et al. 2015)

$$t_{ij} = \int_V \alpha(|X' - X|)\sigma_{ij}(X')dV'$$

(4)

where $X$ is a point on the body that the tensor of stress on its efficacy, $X'$ can be any point else in the body, $V$ is the volume of a region of the body that integral is considered on it, $\sigma_{ij}$ is the tensor of classical stress, $\alpha(|X' - X|)$ is the nonlocal kernel function related to the internal characteristic length. With respect to characteristics of nonlocal kernel function $\alpha(|X' - X|)$ that are presented by Eringen (1983), taking in a Greens function of a linear differential operator, $\Im$, can be defined as following

$$\Im\alpha(|X' - X|) = \delta\alpha(|X' - X|)$$

(5)

Substituting Eq. (5) into Eq. (4), the primary expression (1) form of the following differential equation is determined as

$$\Im t_{ij} = \sigma_{ij}$$

(6)

For the nonlocal linear elastic solids, the equations of motion have the following form (Narendar 2011)

$$t_{ij,j} + f_i = \rho(z)\ddot{u}_i$$

(7)

where $\rho$ is the mass density, $f_i$ body loads and $u_i$ is the vector of displacement. Substituting Eq. (7) into Eq. (6) yields to the following relation

$$\sigma_{ij} + \Im (f_i - \rho(z)\ddot{u}_i) = 0$$

(8)

The nonlocal theory with the linear differential operator for the 3D case is presented by the following expression (Sakhaee-Pour et al. 2008a)

$$\Im = 1 - \mu^2\nabla^2$$

(9)
where $V^2$ is the Laplace operator, which in Cartesian coordinates is defined by $V^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2$ and $\mu = \varepsilon_0 a$. $a$ is the internal property length and $\varepsilon_0$ is the material constant which is predicted by the experiment. The value of the nonlocal parameter is related to the boundary condition, the chirality, the mode shapes, the number of walls, and the nature of motions (Hosseini-Hashemi et al. 2013a). There is no accurate way to compute this parameter, but it is considered that the factor be obtained by conducting a comparison of dispersion curves from nonlocal elasticity and lattice dynamics of nano-material crystal structure (Hosseini-Hashemi et al. 2013a).

2.3 The assumptions made in the present theory

(1) The components of displacement $u$ and $v$ are the axial displacements of the middle plane in $x$ and $y$ directions respectively, and $w$ is the vertical displacement of the middle plane in $z$ direction. The magnitude of the vertical displacement $w$ is not of the same order as the thickness $h$ of the plate and is small with respect to the plate thickness.

(2) The axial displacements, $u$ and $v$ incorporate three parts:

- A displacement part equivalent to the displacement used in the classical plate theory (CPT).
- A displacement component owing to the shear deformation which is included via undetermined integral.
- The shear strains in $z$ direction are zero in the bottom and top faces of the plates.

(1) The vertical displacement $w$ in $z$ direction is considered to be a function of $y$ and $x$ coordinates.

(2) The nanoplate is subjected to the vertical load only.

The displacement field of the cubic shear deformation model is expressed as below (Abualnour et al. 2018)

\[ u(x, y, z) = u_0(x, y) - z \frac{\partial w_0}{\partial x} + k_1 f(z) \int \theta(x, y) \, dx \]  
\( \text{(10a)} \)

\[ v(x, y, z) = v_0(x, y) - z \frac{\partial w_0}{\partial y} + k_2 f(z) \int \theta(x, y) \, dy \]  
\( \text{(10b)} \)

\[ w(x, y, z) = w_0(x, y) + g(z) \phi_z(x, y) \]  
\( \text{(10c)} \)

The coefficients $k_1$ and $k_2$ depends on the geometry. In this work, the shape function is considered based on the cubic function given by

\[ f(z) = \frac{5}{4} \left( z - \frac{4z^3}{3h^2} \right) \]  
\( \text{(11)} \)

and $u_0 (x, y)$, $v_0 (x, y)$, $w_0 (x, y)$, $\theta (x, y)$ and $\varphi_z (x, y)$ are the five variables displacement functions of middle surface of the plate.

With the linear supposition of von-Karman strain, the displacement strain field will be as follows

\[ \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} k_x^0 \\ k_y^0 \\ k_y^x \end{bmatrix} + f(z) \begin{bmatrix} k_x^1 \\ k_y^1 \\ k_y^x \end{bmatrix} \]  
\( \text{(12)} \)

\[ \begin{bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = g(z) \begin{bmatrix} k_z^{0y} \\ k_z^{0x} \end{bmatrix}, \quad \varepsilon_z = g'(z) \varepsilon_z^0 \]

\( \text{(13a)} \)

\[ \begin{bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{bmatrix} = \begin{bmatrix} k_1 \theta \\ k_2 \theta \\ k_1 \frac{\partial \theta}{\partial y} + k_2 \frac{\partial \phi_z}{\partial x} \end{bmatrix} \]  
\( \text{(13b)} \)

The integrals presented in the above equations shall be resolved by a Navier type solution and can be expressed as follows

\[ \begin{aligned} \frac{\partial}{\partial y} \int \theta \, dx &= A' \frac{\partial^2 \theta}{\partial x \partial y}, \\
\frac{\partial}{\partial x} \int \theta \, dy &= B' \frac{\partial^2 \theta}{\partial x \partial y}, \end{aligned} \]  
\( \text{(14)} \)

where the coefficients $A'$ and $B'$ are expressing according to the type of solution employed, in this case by using Navier. Therefore, $A'$ and $B'$ are written as follows

\[ A' = -\frac{1}{\alpha^2}, \quad B' = -\frac{1}{\beta^2}, \quad k_1 = \alpha^2, \quad k_2 = \beta^2 \]  
\( \text{(15)} \)

where $\alpha$ and $\beta$ are defined in expression (29).


\[ 0 = \int_0^T \delta (U - K) \, dt \]  
\( \text{(16)} \)
where \( \delta \) is the variation operator, \( U \) is the strain energy, and \( K \) is the kinetic energy.

The variation of strain energy of the plate is given by

\[
\delta U = \int_{A} \left( \sigma_{x} \delta \varepsilon_{x} + \sigma_{y} \delta \varepsilon_{y} + \sigma_{xy} \delta \varepsilon_{xy} \right) dA dz
\]

\[
+ \tau_{xy} \delta \gamma_{xy} + \tau_{x} \delta \gamma_{xz} + \tau_{z} \delta \gamma_{zx} \right) dA dz
\]

\[
= \int_{A} \left[ N_{x} \delta \varepsilon_{x}^{0} + N_{y} \delta \varepsilon_{y}^{0} + N_{xy} \delta \varepsilon_{xy}^{0} \right. \\
+ M_{x} \delta \varepsilon_{x}^{b} + M_{y} \delta \varepsilon_{y}^{b} + M_{xy} \delta \varepsilon_{xy}^{b} + M_{x} \delta b_{x}^{b} + M_{y} \delta b_{y}^{b} + M_{xy} \delta b_{xy}^{b} + M_{zx} \delta \gamma_{zx}^{0} + M_{zy} \delta \gamma_{zy}^{0} \\
+ \left. M_{x} \delta \gamma_{zx}^{b} + M_{y} \delta \gamma_{zy}^{b} + M_{xy} \delta \gamma_{xy}^{b} + S_{x} \delta \gamma_{x}^{0} + S_{y} \delta \gamma_{y}^{0} \right] dA
\]

where \( A \) is the top surface and the stress resultants \( N, M \), and \( S \) are expressed by

\[
\begin{align*}
(N, M, S) &= \int_{-h/2}^{h/2} (1, z, f) \sigma_{d} dz \quad (i = x, y, xy) \\
N_{x} &= \int_{-h/2}^{h/2} g(z) \sigma_{x} dz
\end{align*}
\]

and

\[
\left( S_{x}^{z}, S_{y}^{z} \right) = \int_{-h/2}^{h/2} g(z) \left( \tau_{x}, \tau_{z} \right)
\]

The variation of kinetic energy is expressed as

\[
\delta K = \int_{A} \left[ \frac{\partial \delta u}{\partial x} v + \frac{\partial \delta v}{\partial y} w + \frac{\partial \delta w}{\partial z} w \right] \rho(z) dV
\]

\[
= \int_{A} \left[ I_{0} \left( \frac{\partial \delta \bar{u}}{\partial x} + \frac{\partial \delta \bar{v}}{\partial y} + \frac{\partial \delta \bar{w}}{\partial z} \right) + \right. \frac{\partial \delta \bar{u}}{\partial x} \delta u_{0} + \frac{\partial \delta \bar{v}}{\partial y} \delta v_{0} + \frac{\partial \delta \bar{w}}{\partial z} \delta w_{0} \\
+ J_{1} \left( k_{1} A \right) \left( \frac{\partial \delta \bar{u}}{\partial x} + \frac{\partial \delta \bar{v}}{\partial y} + \frac{\partial \delta \bar{w}}{\partial z} \right) + \right. \frac{\partial \delta \bar{u}}{\partial x} \delta \bar{v} + \frac{\partial \delta \bar{v}}{\partial y} \delta \bar{w} + \frac{\partial \delta \bar{w}}{\partial z} \delta \bar{u} \\
+ J_{2} \left( k_{2} B \right) \left( \frac{\partial \delta \bar{v}}{\partial y} + \frac{\partial \delta \bar{w}}{\partial z} + \frac{\partial \delta \bar{u}}{\partial x} \right) + \right. \frac{\partial \delta \bar{v}}{\partial y} \delta \bar{w} + \frac{\partial \delta \bar{w}}{\partial z} \delta \bar{v} + \frac{\partial \delta \bar{u}}{\partial x} \delta \bar{v} \\
+ J_{3} \left( k_{3} C \right) \left( \frac{\partial \delta \bar{w}}{\partial z} + \frac{\partial \delta \bar{v}}{\partial y} + \frac{\partial \delta \bar{u}}{\partial x} \right) + \right. \frac{\partial \delta \bar{w}}{\partial z} \delta \bar{u} + \frac{\partial \delta \bar{u}}{\partial x} \delta \bar{u} + \frac{\partial \delta \bar{u}}{\partial x} \delta \bar{v} + \frac{\partial \delta \bar{v}}{\partial y} \delta \bar{v} + \frac{\partial \delta \bar{w}}{\partial z} \delta \bar{w}
\]

where dot-superscript convention indicates the differentiation with respect to the time variable \( t \); \( \rho(\bar{z}) \) is the mass density; and \((I_{0}, J_{0}, I_{1}, I_{2}, J_{1}, J_{2}, K_{1}, K_{2})\) are mass inertias expressed as

\[
(I_{0}, J_{0}, I_{1}, I_{2}) = \int_{-h/2}^{h/2} \left( 1, g(z), z, z^{2} \right) \rho(\bar{z}) d\bar{z}
\]

\[
(J_{1}, J_{2}, K_{1}, K_{2}) = \int_{-h/2}^{h/2} \left( f(z), z f(z), f^{2}(z), g^{2}(z) \right) \rho(\bar{z}) d\bar{z}
\]

Substituting the expressions for \( \delta U \) and \( \delta K \) from Eqs (18) and (19) into Eq. (20) and integrating by parts and collecting the coefficients of \( \delta \bar{u}_{0}, \delta \bar{v}_{0}, \delta \bar{w}_{0}, \delta \theta, \) and \( \delta \varphi_{c} \), the following equations of motion of the plate are obtained as

\[
\begin{align*}
\frac{\partial \delta \bar{u}}{\partial x} &: \frac{\partial \delta \bar{v}}{\partial y} + \frac{\partial \delta \bar{w}}{\partial z} \frac{\partial \bar{w}}{\partial \bar{t}} = I_{0} \delta \bar{u}_{0} - I_{1} \delta \bar{u}_{1} + k_{1} A \delta \bar{\theta} \delta \bar{u}_{0} \\
\frac{\partial \delta \bar{v}}{\partial x} + \frac{\partial \delta \bar{v}}{\partial y} \frac{\partial \bar{v}}{\partial \bar{t}} + \frac{\partial \delta \bar{w}}{\partial z} \frac{\partial \bar{w}}{\partial \bar{t}} = I_{0} \delta \bar{v}_{0} - I_{1} \delta \bar{v}_{1} + k_{1} B \delta \bar{\theta} \delta \bar{v}_{0} + k_{2} B \delta \bar{\theta} \delta \bar{v}_{1} \\
\frac{\partial \delta \bar{w}}{\partial x} : &+ \frac{\partial \delta \bar{w}}{\partial y} \frac{\partial \bar{w}}{\partial \bar{t}} + \frac{\partial \delta \bar{w}}{\partial z} \frac{\partial \bar{w}}{\partial \bar{t}} = I_{0} \delta \bar{w}_{0} + I_{1} \delta \bar{w}_{1} + k_{2} B \delta \bar{\theta} \delta \bar{w}_{0} + J_{0} \delta \bar{\varphi}_{c}
\end{align*}
\]

\[
\delta \bar{\theta} : = -k_{1} M_{x}^{1} - k_{2} M_{y}^{1} - (k_{1} A + k_{2} B) \delta \bar{\theta} \delta \bar{u}_{0} + k_{2} B \delta \bar{\theta} \delta \bar{v}_{0} + k_{2} B \delta \bar{\theta} \delta \bar{w}_{0} + J_{0} \delta \bar{\varphi}_{c}
\]

2.4 The nonlocal elasticity model for FG nano-plate

The constitutive relations of nonlocal theory for a FG nano-plate using Eq. (6) can be written as

\[
| \sigma_{x} | \sigma_{x} | \sigma_{y} | \sigma_{y} | \sigma_{z} | \sigma_{z} | \tau_{xy} | \tau_{xy} | \tau_{xyz} | \tau_{xyz} | \tau_{zx} | \tau_{zx} | \tau_{zy} | \tau_{zy} | \tau_{yz} | \tau_{yz}| = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{12} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{66} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{55}
\end{bmatrix}
\]

Density; and \((I_{0}, J_{0}, I_{1}, I_{2}, J_{1}, J_{2}, K_{1}, K_{2})\) are mass inertias expressed as

\[
(I_{0}, J_{0}, I_{1}, I_{2}) = \int_{-h/2}^{h/2} \left( 1, g(z), z, z^{2} \right) \rho(z) d\bar{z}
\]

\[
(J_{1}, J_{2}, K_{1}, K_{2}) = \int_{-h/2}^{h/2} \left( f(z), z f(z), f^{2}(z), g^{2}(z) \right) \rho(z) d\bar{z}
\]
where

\[
\begin{align*}
C_{11} &= C_{22} = C_{33} = \frac{E(z)(1 - \nu)}{(1 - 2\nu)(1 + \nu)}, \\
C_{12} &= C_{13} = C_{23} = \frac{E(z)\nu}{(1 - 2\nu)(1 + \nu)}, \\
C_{44} &= C_{55} = C_{66} = \frac{E(z)}{2(1 + \nu)}.
\end{align*}
\]

Integrating Eq. (20) over the plate’s cross-section area yields the force–strain and the moment–strain of the nonlocal refined FG nano-plates as follows

\[
\begin{align*}
\{N_i\} &= \left[ \begin{array}{c}
N_{1} \\
N_{2} \\
N_{3} \\
N_{4} \\
N_{5} \\
N_{6} \\
N_{7} \\
N_{8} \\
N_{9}
\end{array} \right], \\
\{M_i\} &= \left[ \begin{array}{c}
M_{1} \\
M_{2} \\
M_{3} \\
M_{4} \\
M_{5} \\
M_{6} \\
M_{7} \\
M_{8} \\
M_{9}
\end{array} \right], \\
\{v_i\} &= \left[ \begin{array}{c}
v_{11} \\
v_{12} \\
v_{13} \\
v_{14} \\
v_{15} \\
v_{16} \\
v_{17} \\
v_{18} \\
v_{19}
\end{array} \right],
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix} N_{11} \\ N_{12} \\ N_{13} \\ N_{14} \\ N_{15} \\ N_{16} \\ N_{17} \\ N_{18} \\ N_{19} \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{77} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{88} & 0 \\ X_{14} & X_{15} & X_{16} & X_{17} & X_{18} & X_{19} & 0 & 0 & 0 \end{bmatrix}, \\
\begin{bmatrix} M_{11} \\ M_{12} \\ M_{13} \\ M_{14} \\ M_{15} \\ M_{16} \\ M_{17} \\ M_{18} \\ M_{19} \end{bmatrix} &= \begin{bmatrix} A_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{54} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{64} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{74} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{84} & 0 \\ X_{14} & X_{15} & X_{16} & X_{17} & X_{18} & X_{19} & 0 & 0 & 0 \end{bmatrix}, \\
\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \\ v_{16} \\ v_{17} \\ v_{18} \\ v_{19} \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{77} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{88} & 0 \\ X_{14} & X_{15} & X_{16} & X_{17} & X_{18} & X_{19} & 0 & 0 & 0 \end{bmatrix},
\end{align*}
\]

The cross-sectional rigidities are defined as follows

\[
\begin{align*}
\begin{bmatrix} A_{ij} \\ A_{ij}^b \\ B_{ij} \\ B_{ij}^b \\ D_{ij} \\ D_{ij}^b \\ H_{ij} \end{bmatrix} &= \int_{-h/2}^{h/2} C_{ij} \left[ \begin{array}{c}
[1, g^2(z), z, z^2, f(z), f(z), f(z^2)]
\end{array} \right] dz,
\end{align*}
\]

\[
\begin{align*}
\left[ \begin{array}{c}
X_{ij} \\ Y_{ij} \\ Y_{ij}^b \\ Z_{ij}
\end{array} \right] &= \int_{-h/2}^{h/2} \left[ \begin{array}{c}
[1, 0, g^2(z), g^2(z), g(z, z), g(z)]
\end{array} \right] C_{ij} dz
\end{align*}
\]

The nonlocal equations of motion of FG nano-plates in terms of the displacement can be obtained by substituting Eqs. (24a) and (24b), into Eq. (21) as follows

\[
\begin{align*}
A_{22} d_{22} v_0 + A_{66} d_{11} v_0 + (A_{12} + A_{66}) d_{12} u_0 + X_{23} d_2 \theta_z \\
- B_{22} d_{22} w_0 - (B_{12} + 2B_{66}) d_{12} w_0 \\
+ B_{66}' \left( k_1 A + k_2 B \right) d_{12} \theta + B_{22}' d_{22} k_2 + B_{22}' d_{22} k_1) d_2 \theta
\end{align*}
\]

\[
\begin{align*}
= \{1 - \mu \nu^2\} \left[ \begin{array}{c}
I_0 v_0 - I_1 d_2 w_0 + J_1 B' k_2 d_2 \theta
\end{array} \right]
\end{align*}
\]

where

\[
\begin{align*}
A_{22} &= d_{22} v_0 + A_{66} d_{11} v_0 + (A_{12} + A_{66}) d_{12} u_0 + X_{23} d_2 \theta_z \\
- B_{22} &= d_{22} w_0 - (B_{12} + 2B_{66}) d_{12} w_0
\end{align*}
\]

\[
\begin{align*}
+ B_{66}' \left( k_1 A + k_2 B \right) d_{12} \theta + B_{22}' d_{22} k_2 + B_{22}' d_{22} k_1) d_2 \theta
\end{align*}
\]

\[
\begin{align*}
= \{1 - \mu \nu^2\} \left[ \begin{array}{c}
I_0 v_0 - I_1 d_2 w_0 + J_1 B' k_2 d_2 \theta
\end{array} \right]
\end{align*}
\]
where \((U_{mn}, V_{mn}, W_{mn}, X_{mn}, Y_{mn})\) are the unknown Fourier coefficients.

with \(\alpha = m\pi / a\) and \(\beta = n\pi / b\)

Inserting Eq. (28) into Eqs. (26), leads to

\[
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} \\
S_{12} & S_{22} & S_{23} & S_{24} & S_{25} \\
S_{13} & S_{23} & S_{33} & S_{34} & S_{35} \\
S_{14} & S_{24} & S_{34} & S_{44} & S_{45} \\
S_{15} & S_{25} & S_{35} & S_{45} & S_{55}
\end{bmatrix}
\begin{bmatrix}
M_{11} & 0 & M_{13} & M_{14} & 0 \\
0 & M_{22} & M_{23} & M_{24} & 0 \\
M_{13} & M_{23} & M_{33} & M_{34} & M_{35} \\
M_{14} & M_{24} & M_{34} & M_{44} & 0 \\
0 & 0 & M_{55} & 0 & M_{55}
\end{bmatrix}
\begin{bmatrix}
U_{mn} \\
V_{mn} \\
W_{mn} \\
X_{mn} \\
Y_{mn}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

(30)

S_{11} = -(A_{11}\alpha^2 + A_{66}\beta^2),
\quad S_{12} = -\alpha\beta (A_{12} + A_{66}),
\quad S_{13} = \alpha(B_{11}\alpha^2 + B_{12}\beta^2 + 2B_{66}\beta^2),
\quad S_{14} = -(A_{11}\alpha^2 + A_{66}\beta^2),
\quad S_{15} = -(A_{11}\alpha^2 + A_{66}\beta^2),
\quad S_{22} = -\beta(B_{22}\beta^2 + B_{12}\alpha^2 + 2B_{66}\alpha^2),
\quad S_{24} = \beta(k_2B_{22} + k_1B_{12} - (k_1\alpha + k_2B)B_{66}\alpha^2)
\quad S_{25} = X_{23}\beta
\quad S_{33} = -(D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4),
\quad S_{34} = -k_2(D_{11}\alpha^2 + D_{66}\alpha^2\beta^2)
\quad + 2(k_1\alpha + k_2B)D_{66}\alpha^2\beta^2
\quad - k_2(D_{22}\beta^3 + D_{66}\alpha^2\alpha^2)
\quad S_{35} = -(Y_{12}\alpha^2 - (Y_{23})\beta^2)
\quad S_{44} = -k_2(H_{11}k_1 + H_{12}k_2) - (k_1\alpha + k_2B)^2H_{66}\alpha^2\beta^2
\quad - k_2(H_{11}k_1 + H_{22}k_2) - (k_1\alpha)^2A_{55}\alpha^2
\quad - (k_2B)^2A_{55}\beta^2
\quad S_{45} = -(k_1\alpha)A_{55}\alpha^2 - (k_2B)A_{44}\beta^2 + k_1Y_{12} + k_2Y_{23}
\quad S_{55} = -(A_{11}\alpha^2 - (A_{44}\alpha^2\beta^2 - Z_{33})
\quad M_{11} = -I_0, \quad M_{13} = \alpha I_1, \quad M_{14} = -k_1I_1A\alpha,
\quad M_{15} = 0, \quad M_{22} = -I_0, \quad M_{23} = \beta I_1,
\quad M_{24} = -k_2B\beta I_1, \quad M_{25} = 0
\quad M_{33} = -I_0 - I_2(\alpha^2 + \beta^2),
\quad M_{34} = J_2(k_1\alpha^2 + k_2B\beta^2), \quad M_{35} = -J_0,
\quad M_{44} = -K_2((k_1\alpha)^2\alpha^2 + (k_2B)^2\beta^2), \quad M_{45} = 0
\quad M_{55} = -K_3, \quad \lambda = (1 + \mu(\alpha^2 + \beta^2))
\]

(31)

4. Numerical results and discussions

In this work, two separate parts are considered; in the first part, have been examined and validated isotropic rectangular nano-plate, and in the second part, it does for FG one.

4.1 Isotropic rectangular nano-plate

Only homogeneous plate \((n = 0)\) is employed in this part for the verification.

Tables 2-4 provide the first three non-dimensional frequency and Frequency Ratios (FR) for simply supported boundary condition with different values of aspect ratio \((n = b/a)\), specified values of non-dimensional scale parameter \((\zeta = \mu a)\) and the thickness to length ratio \(h/a = 0.1\) on rectangular nano-plates. The natural frequency parameters written in non-dimensional form \(\beta = \omega\alpha^2\sqrt{\rho h / D}\), \(D = Eh^3 / 12(1 - \nu^2)\) are the flexural rigidity. The nano-plate is made of the following material properties: \(E = 210\) GPa, \(v = 0.3\) and \(\rho = 7800\) (kg/m\(^3\)). The computed frequencies based on the proposed nonlocal cubic shear deformation theory are compared with those given by Hosseini-Hashemi et al. (2013b) based on Mindlin Plate Theory (MPT) and those reported by Khorsheed et al. (2015) based on exponential shear deformation theory. Also, the Frequency Ratio (FR) expression between the nonlocal and local non-dimensional frequencies is given as what follows

\[
FR = \frac{\beta_{nl}}{\beta_{lc}}
\]

where \(\beta_{nl}\) is the non-dimensional nonlocal frequency parameter, and \(\beta_{lc}\) is the non-dimensional local frequency parameter.

It can be seen from Tables 2-4, that the obtained values for non-dimensional nonlocal frequency parameter \(\beta_{nl}\) are in good agreement with those provided by Khorsheed et al. (2015) and Hosseini-Hashemi et al. (2013b). The introduction of stretching thickness effect makes the nanoplate more stiffness.

4.2 FGM plate

Table 5 presents a comparison of the frequency parameters \(\beta = \omega h \sqrt{\rho c / E_c}\) for AL/AL\(_2\)O\(_3\) square moderately thick plates with those provided by Hosseini-
Table 2 The variations of the non-dimensional frequency ($\beta = \omega a^2 \sqrt{\rho h/D}$) and the frequency ratio (FR) for the nonlocal plate ($m = 1, n = 1$)

<table>
<thead>
<tr>
<th>Method</th>
<th>$\zeta = 0$</th>
<th>$\zeta = 0.2$</th>
<th>$\zeta = 0.4$</th>
<th>$\zeta = 0.6$</th>
<th>$\zeta = 0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta_{nl}$</td>
<td>FR</td>
<td>FR</td>
<td>FR</td>
<td>FR</td>
</tr>
<tr>
<td>$\eta = 0.6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present ($\varepsilon \neq 0$)</td>
<td>35.0858</td>
<td>1.0000</td>
<td>0.6335</td>
<td>0.3789</td>
<td>0.2633</td>
</tr>
<tr>
<td>Present ($\varepsilon = 0$)</td>
<td>35.0045</td>
<td>1.0000</td>
<td>0.6335</td>
<td>0.3789</td>
<td>0.2633</td>
</tr>
<tr>
<td>Khorshidi et al. (2015)</td>
<td>35.015</td>
<td>1.0000</td>
<td>0.6335</td>
<td>0.3789</td>
<td>0.2633</td>
</tr>
<tr>
<td>Hosseini-Hashemi et al. (2013b)</td>
<td>35.0643</td>
<td>1.0000</td>
<td>0.6335</td>
<td>0.3789</td>
<td>0.2633</td>
</tr>
<tr>
<td>$\eta = 0.8$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present ($\varepsilon \neq 0$)</td>
<td>24.2431</td>
<td>1.0000</td>
<td>0.7051</td>
<td>0.4451</td>
<td>0.3146</td>
</tr>
<tr>
<td>Present ($\varepsilon = 0$)</td>
<td>24.2034</td>
<td>1.0000</td>
<td>0.7051</td>
<td>0.4451</td>
<td>0.3146</td>
</tr>
<tr>
<td>Khorshidi et al. (2015)</td>
<td>24.2084</td>
<td>1.0000</td>
<td>0.7051</td>
<td>0.4451</td>
<td>0.3146</td>
</tr>
<tr>
<td>Hosseini-Hashemi et al. (2013b)</td>
<td>24.2330</td>
<td>1.0000</td>
<td>0.7050</td>
<td>0.4451</td>
<td>0.3146</td>
</tr>
<tr>
<td>$\eta = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present ($\varepsilon \neq 0$)</td>
<td>19.0902</td>
<td>1.0000</td>
<td>0.7475</td>
<td>0.4904</td>
<td>0.3512</td>
</tr>
<tr>
<td>Present ($\varepsilon = 0$)</td>
<td>19.0653</td>
<td>1.0000</td>
<td>0.7475</td>
<td>0.4904</td>
<td>0.3512</td>
</tr>
<tr>
<td>Khorshidi et al. (2015)</td>
<td>19.0684</td>
<td>1.0000</td>
<td>0.7475</td>
<td>0.4904</td>
<td>0.3512</td>
</tr>
<tr>
<td>Hosseini-Hashemi et al. (2013b)</td>
<td>19.0840</td>
<td>1.0000</td>
<td>0.7475</td>
<td>0.4904</td>
<td>0.3512</td>
</tr>
</tbody>
</table>

Table 3 The variations of the non-dimensional frequency ($\beta = \omega a^2 \sqrt{\rho h/D}$) and the frequency ratio (FR) for the nonlocal plate ($m = 2, n = 1$)

<table>
<thead>
<tr>
<th>Method</th>
<th>$\zeta = 0$</th>
<th>$\zeta = 0.2$</th>
<th>$\zeta = 0.4$</th>
<th>$\zeta = 0.6$</th>
<th>$\zeta = 0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta_{nl}$</td>
<td>FR</td>
<td>FR</td>
<td>FR</td>
<td>FR</td>
</tr>
<tr>
<td>$\eta = 0.6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present ($\varepsilon \neq 0$)</td>
<td>60.3530</td>
<td>1.0000</td>
<td>0.5216</td>
<td>0.2923</td>
<td>0.1997</td>
</tr>
<tr>
<td>Present ($\varepsilon = 0$)</td>
<td>60.1243</td>
<td>1.0000</td>
<td>0.5216</td>
<td>0.2923</td>
<td>0.1997</td>
</tr>
<tr>
<td>Khorshidi et al. (2015)</td>
<td>60.1556</td>
<td>1.0000</td>
<td>0.5216</td>
<td>0.2923</td>
<td>0.1997</td>
</tr>
<tr>
<td>Hosseini-Hashemi et al. (2013b)</td>
<td>60.2869</td>
<td>1.0000</td>
<td>0.5216</td>
<td>0.2923</td>
<td>0.1997</td>
</tr>
<tr>
<td>$\eta = 0.8$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present ($\varepsilon \neq 0$)</td>
<td>50.3554</td>
<td>1.0000</td>
<td>0.5594</td>
<td>0.3197</td>
<td>0.2194</td>
</tr>
<tr>
<td>Present ($\varepsilon = 0$)</td>
<td>50.1930</td>
<td>1.0000</td>
<td>0.5594</td>
<td>0.3197</td>
<td>0.2194</td>
</tr>
<tr>
<td>Khorshidi et al. (2015)</td>
<td>50.2147</td>
<td>1.0000</td>
<td>0.5594</td>
<td>0.3197</td>
<td>0.2194</td>
</tr>
<tr>
<td>Hosseini-Hashemi et al. (2013b)</td>
<td>50.3100</td>
<td>1.0000</td>
<td>0.5594</td>
<td>0.3197</td>
<td>0.2194</td>
</tr>
<tr>
<td>$\eta = 1$</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Present ($\varepsilon \neq 0$)</td>
<td>45.6216</td>
<td>1.0000</td>
<td>0.5799</td>
<td>0.3353</td>
<td>0.2308</td>
</tr>
<tr>
<td>Present ($\varepsilon = 0$)</td>
<td>45.4869</td>
<td>1.0000</td>
<td>0.5799</td>
<td>0.3353</td>
<td>0.2308</td>
</tr>
<tr>
<td>Khorshidi et al. (2015)</td>
<td>45.5048</td>
<td>1.0000</td>
<td>0.5799</td>
<td>0.3353</td>
<td>0.2308</td>
</tr>
<tr>
<td>Hosseini-Hashemi et al. (2013b)</td>
<td>45.5845</td>
<td>1.0000</td>
<td>0.5799</td>
<td>0.3353</td>
<td>0.2308</td>
</tr>
</tbody>
</table>

Table 4 The variations of the non-dimensional frequency ($\beta = \omega a^2 \sqrt{\rho h/D}$) and the frequency ratio (FR) for the nonlocal plate ($m = 2, n = 2$)

<table>
<thead>
<tr>
<th>Method</th>
<th>$\zeta = 0$</th>
<th>$\zeta = 0.2$</th>
<th>$\zeta = 0.4$</th>
<th>$\zeta = 0.6$</th>
<th>$\zeta = 0.8$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$\beta_{nl}$</td>
<td>FR</td>
<td>FR</td>
<td>FR</td>
<td>FR</td>
</tr>
<tr>
<td>$\eta = 0.6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present ($\varepsilon \neq 0$)</td>
<td>122.0595</td>
<td>1.0000</td>
<td>0.3789</td>
<td>0.2005</td>
<td>0.1352</td>
</tr>
<tr>
<td>Present ($\varepsilon = 0$)</td>
<td>121.2246</td>
<td>1.0000</td>
<td>0.3789</td>
<td>0.2005</td>
<td>0.1352</td>
</tr>
<tr>
<td>Khorshidi et al. (2015)</td>
<td>121.356</td>
<td>1.0000</td>
<td>0.3789</td>
<td>0.2005</td>
<td>0.1352</td>
</tr>
<tr>
<td>Hosseini-Hashemi et al. (2013b)</td>
<td>121.7770</td>
<td>1.0000</td>
<td>0.3789</td>
<td>0.2006</td>
<td>0.1352</td>
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<tr>
<td>$\eta = 0.8$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present ($\varepsilon \neq 0$)</td>
<td>87.3788</td>
<td>1.0000</td>
<td>0.4451</td>
<td>0.2412</td>
<td>0.1635</td>
</tr>
<tr>
<td>Present ($\varepsilon = 0$)</td>
<td>86.9235</td>
<td>1.0000</td>
<td>0.4451</td>
<td>0.2412</td>
<td>0.1635</td>
</tr>
<tr>
<td>Khorshidi et al. (2015)</td>
<td>86.9989</td>
<td>1.0000</td>
<td>0.4451</td>
<td>0.2412</td>
<td>0.1635</td>
</tr>
<tr>
<td>Hosseini-Hashemi et al. (2013b)</td>
<td>87.2357</td>
<td>1.0000</td>
<td>0.4451</td>
<td>0.2412</td>
<td>0.1635</td>
</tr>
</tbody>
</table>
The comparison of the natural frequency parameter continues...

<table>
<thead>
<tr>
<th>Method</th>
<th>$\zeta = 0$</th>
<th>$\zeta = 0.2$</th>
<th>$\zeta = 0.4$</th>
<th>$\zeta = 0.6$</th>
<th>$\zeta = 0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present ($\varepsilon \neq 0$)</td>
<td>70.1122</td>
<td>1.0000</td>
<td>0.4904</td>
<td>0.2708</td>
<td>0.1843</td>
</tr>
<tr>
<td>Present ($\varepsilon = 0$)</td>
<td>69.8093</td>
<td>1.0000</td>
<td>0.4904</td>
<td>0.2708</td>
<td>0.1843</td>
</tr>
<tr>
<td>Khorshidi et al. (2015)</td>
<td>69.8517</td>
<td>1.0000</td>
<td>0.4904</td>
<td>0.2708</td>
<td>0.1843</td>
</tr>
<tr>
<td>Hosseini-Hashemi et al. (2013b)</td>
<td>70.0219</td>
<td>1.0000</td>
<td>0.4904</td>
<td>0.2708</td>
<td>0.1844</td>
</tr>
</tbody>
</table>

Table 5: The comparison of the natural frequency parameter ($\tilde{\beta} = \omega h \sqrt{\rho h / E_i}$) for Al$_2$O$_3$ square plates ($\eta = 1$)

<table>
<thead>
<tr>
<th>$h/a$</th>
<th>$(m,n)$</th>
<th>Method</th>
<th>$n$</th>
<th>$0$</th>
<th>$0.5$</th>
<th>$1$</th>
<th>$4$</th>
<th>$10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>(1, 1)</td>
<td>Present ($\varepsilon \neq 0$)</td>
<td>0.0148</td>
<td>0.0126</td>
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<td>0.2976</td>
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<td>0.4381</td>
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<td>0.5807</td>
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<td>Matsunaga (2008)</td>
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<td>0.5891</td>
<td>0.5444</td>
<td>0.4362</td>
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In addition, the corresponding mode shapes (2015) and Matsunaga (2008) when element FSDT method (Hosseini-Hashemi series procedure (Vel and Batra 2004), finite element HSDT (Matsunaga 2008), 3D theory by using the power series procedure (Vel and Batra 2004), finite element HSDT method (Pradyumna and Bandyopadhay 2008), finite element FSDT method (Hosseini-Hashemi et al. 2008), an analytical FSDT solution (Hosseini-Hashemi et al. 2010) and HSDT solution Khoshidi et al. (2015) is demonstrated.

From Tables 5 and 6, it can be confirmed that there is a very good agreement among the results confirming the high accuracy of the proposed analytical formulation. The effect of the geometric ratio \( \eta = b/a \) on the frequency parameters \( \beta = \omega a^2 \sqrt{\rho_c h/E_c} \) of a rectangular Al/ZrO\(_2\) plate \( \delta = b/a = 0.2, n = 1 \) is shown in Table 7.

From Table 7, it can be deduced that with a reduction in the aspect ratio, the frequency parameter increases due to the increase of the stiffness of the plate. It can be also observed that the stretching effect increases the frequency parameter.

In Table 8, the influences of different parameters on the non-dimensional frequencies of the rectangular FG nanoplate are presented. From these results, it is found that by increasing the scale parameter, the rate of variation of non-dimensional frequencies diminishes, because by increasing the scale parameter, the strain energy diminishes, and it causes a reduction of the rigidity of the plates.

In Table 6, a comparison of the results \( \bar{\beta} = \omega h \sqrt{\rho_c h/E_c} \) for Al/ZrO\(_2\) square plates with those for the higher modes.

It is observed that the frequency ratios for the lower modes are more than those for the higher modes.

From Tables 5 and 6, it can be deduced that with a reduction in the aspect ratio, the frequency parameter increases due to the increase of the stiffness of the plate. It can be also observed that the stretching effect increases the frequency parameter.

Fig. 2 The influences of the aspect ratio and the scale parameter on the non-dimensional frequency.
Table 8 The effect of the non-dimensional nonlocal parameter $\zeta$ and the gradient index $n$ on the non-dimensional frequencies $\beta = ak\sqrt{\rho h/E_c}$ of the rectangular FG nanoplate (AL/AL$_2$O$_3$)

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<th>$\zeta$</th>
<th>$a/b$</th>
<th>$h/a$</th>
<th>Method</th>
<th>Gradient index</th>
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gradient index leads to a decrease in the stiffness of the FG nano-plate.

5. Conclusions

The size-dependent dynamic properties of FG nano-plate are analytically studied by using a simple cubic refined plate model based on the nonlocal differential constitutive relations of Eringen. The kinematic of the present theory is modified by considering undetermined integral terms in in-plane displacements which results in a reduced number of variables compared with other HSDT of the same order. Comparing the obtained results with those found in the literature for FG nano-plates demonstrates a high stability and accuracy of the present solution. What presented herein

![Fig. 3 The effects of the aspect ratio and the nonlocal parameter on the non-dimensional frequency](image1.png)

![Fig. 4 Influence of the gradient index (n) and the scale parameter (μ) on dimensionless frequency for a simply supported square FG plate with a / h = 10: (a) first frequency; (b) second frequency](image2.png)
Fig. 4 Continued

demonstrates the influences of the variations of the scale parameter, the ratio of the thickness to the length, the gradient indexes and the aspect ratio on the frequency values of a FG nano-plate. It is demonstrated that the frequency ratio diminishes with increasing the mode number and the value of the scale parameter, and also increasing the gradient index causes the non-dimensional frequencies to decrease.

References


