Effect of non-uniform temperature distributions on nonlocal vibration and buckling of inhomogeneous size-dependent beams

Farzad Ebrahimi* and Erfan Salari

Department of Mechanical Engineering, Faculty of Engineering, Imam Khomeini International University, Qazvin, Iran

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Abstract. In the present investigation, thermal buckling and free vibration characteristics of functionally graded (FG) Timoshenko nanobeams subjected to nonlinear thermal loading are carried out by presenting a Navier type solution. The thermal load is assumed to be nonlinear distribution through the thickness of FG nanobeam. Thermo-mechanical properties of FG nanobeam are supposed to vary smoothly and continuously throughout the thickness based on power-law model and the material properties are assumed to be temperature-dependent. Eringen’s nonlocal elasticity theory is exploited to describe the size dependency of nanobeam. Using Hamilton’s principle, the nonlocal equations of motion together with corresponding boundary conditions based on Timoshenko beam theory are obtained for the thermal buckling and vibration analysis of graded nanobeams including size effect. Moreover, in following a parametric study is accompanied to examine the effects of the several parameters such as nonlocal parameter, thermal effect, power law index and aspect ratio on the critical buckling temperatures and natural frequencies of the size-dependent FG nanobeams in detail. According to the numerical results, it is revealed that the proposed modeling can provide accurate frequency results of the FG nanobeams as compared some cases in the literature. Also, it is found that the small scale effects and nonlinear thermal loading have a significant effect on thermal stability and vibration characteristics of FG nanobeams.

Keywords: thermal buckling; Timoshenko beam theory; vibration; functionally graded nanobeam; nonlocal elasticity theory

1. Introduction

Functionally graded material (FGM) is a composite material varying its microstructure from one material to another with a specific gradient, resulting in corresponding changes in the effective material properties (including elasticity modulus, thermal expansion coefficient and thermal conductivity) of the material (Ebrahimi and Rastgoo 2018). The FG materials can be designed to produce an optimum distribution of component materials for specific function and applications. Typically, FG materials are made of a mixture of two materials, mainly ceramic and metal phases, to achieve a composition with a certain functionality. In comparison with traditional composites, FGMs possess various advantages, for instance, higher fracture toughness, enhanced thermal...
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resistance, minimization or elimination of stress concentration, and increased bonding strength along the interface of two dissimilar materials. During the past decade, beams made of FGMs have found wide applications as structural elements in modern industries such as aeronautics/ astronautics manufacturing industry, mechanical engineering and engine combustion chamber, nuclear engineering and reactors. Motivated by these engineering applications, ceramic-metal FGMs have also attracted intensive research interests. Sofiyev and Kuruoglu (2015) studied buckling analysis of non-homogeneous orthotropic truncated conical shells subjected to combined loading of axial compression and external pressure. The superposition and Galerkin methods are used to achieve the expressions for critical loads of non-homogeneous orthotropic truncated conical shells with simply supported boundary conditions. Nanobeams are one of the basic components in micro/nano electromechanical systems (MEMS/NEMS), biomedical sensors and atomic force microscopy (AFM). Therefore understanding the mechanical and physical properties of nanostructures is necessary for its practical applications. For instance, Alizada et al. (2012) presented stress analysis of a substrate coated by nanomaterials with a vacancy under uniform extension load. They assumed that vacancies do not vary along the width of the substrate.

Nanoscale engineering materials have attracted great interest in modern science and technology after the invention of carbon nanotubes (CNTs) by Iijima (1991). They have significant mechanical, thermal and electrical performances that are superior to the conventional structural materials. In recent years, nanobeams and CNTs hold a wide variety of potential applications (Zhang et al. 2004) such as sensors, actuators, transistors, probes, and resonators in NEMSs. For example, in MEMS/NEMS; nanostructures have been used in many areas including communications, machinery, information technology and biotechnology technologies. The classical continuum theory is quite efficient in the mechanical analysis of the macroscopic structures, but its applicability to the identification of the size effect on the mechanical behaviors on micro- or nano-scale structures is questionable. This limitation of the classical continuum theory is partly due to the fact that the classical continuum theory does not admit the size dependence in the elastic solutions of inclusions and inhomogeneities. However the classical continuum models need to be extended to consider the nanoscale effects and this can be achieved through the nonlocal elasticity theory proposed by Eringen (1983) which consider the size-dependent effect. According to this theory, the stress state at a reference point is considered as a function of strain states of all points in the body. This nonlocal theory is proved to be in accordance with atomic model of lattice dynamics and with experimental observations on phonon dispersion. Moreover, in recent years the application of nonlocal elasticity theory, in micro and nanomaterials has received a considerable attention within the nanotechnology community. Wang and Liew (2007) carried out the static analysis of micro- and nano-structures based on nonlocal continuum mechanics using Euler-Bernoulli beam theory and Timoshenko beam theory. Aydogdu (2009) proposed a generalized nonlocal beam theory to study bending, buckling, and free vibration of nanobeams based on Eringen model using different beam theories.

To improve the performance of composite structures, the development of the FG materials are being accelerated to optimize some certain functional properties of structures by tailoring the material architecture at nano/micro scale. The rapid developments of MEMS and NEMS make the FG materials possible to be applied in nano/micro scaled systems such as thin films in the form of shape memory alloys, atomic force microscopes (AFMs), micro sensors, micro piezoactuator and nano-motors to achieve high sensitivity and desired performance. In such applications, size effects or small scale effects play major role which should be considered to study the mechanical behaviors of such small scale structures. FG nanobeams are one of the most important
nanostructures which are commonly used as components in MEMS, NEMS and AFMS with the order of microns or sub-microns, and their properties are closely related to their microstructures. Nevertheless, the possible applications rely on a good understanding of the vibration and thermal stability characteristics of FG nanobeams at small-scale. Therefore, establishing an accurate model of FG nanobeams is very important for successful NEMS design.

Recently, FGMs find increasing applications in micro- and nano-scale structures such as thin films in the form of shape memory alloys, atomic force microscopes, micro sensors, micro piezoelectric actuator and nano-motors (Fu et al. 2003). Moreover, with the development of the material technology, FGMs have also been employed in MEMS/NEMS. In all of these applications, the size effect plays major role which should be considered to study the mechanical behaviors of such small scale structures. Beams are the core structures widely used in MEMS, NEMS and AFMS with the order of microns or sub-microns, and their properties are closely related to their microstructures. On the other hands, FG nanobeams are important structural elements and hence, because of high sensitivity of MEMS/ NEMS to external stimulations, understanding mechanical properties and vibration behavior of them are of significant importance to the design and manufacture of FG MEMS/NEMS. Furthermore, different fabrication processes of nanoscale functionally graded material have been focused by the several researchers. The design and fabrication of stepwise functionally graded synthetic nanocomposites using combined powder stacking and compression molding techniques were studied by Bafekrpour et al. (2012). They concluded that the electrical and thermal properties of nanocomposites could be manipulated by changing the gradient patterns. Therefore, establishing an accurate model of FG nanobeams is a key issue for successful NEMS design. The free vibration analysis of FG microbeams was presented by Ansari et al. (2011) based on the strain gradient Timoshenko beam theory. They also concluded that the value of gradient index plays an important role in the vibrational response of the FG microbeams of lower slenderness ratios. Ebrahimi and Salari (2015) also studied the thermo-electrical buckling of the piezoelectric nanobeams subjected to in-plane thermal loads and applied electric voltage. Recently, Eltaher et al. (2012) presented a finite element formulation for free vibration analysis of FG nanobeams based on nonlocal Euler beam theory. Using nonlocal Timoshenko and Euler–Bernoulli beam theory, Simsek and Yurtcu (2013) investigated bending and buckling of FG nanobeam by analytical method. More recently, vibration behaviour of simply supported Timoshenko FG nanobeams were investigated by Rahmani and Pedram (2014). Material properties are assumed to be temperature independent in these works and the thermal environment effects were not considered. Furthermore, the common use of FGMs in high temperature environment leads to considerable changes in material properties. For example, Young’s modulus usually decreases when temperature increases in FGMs. To predict the behavior of FGMs under high temperature more accurately, it is necessary to consider the temperature dependency on material properties.

It can be evaluated from the literature survey that there is no study on the thermal buckling and vibration of FG Timoshenko nanobeams under nonlinear temperature rise via nonlocal elasticity theory. In fact, it is crucial to consider simultaneously nonlinear temperature rise and nonlocal effects for more accurate thermal buckling and vibration analysis and design of temperature dependent inhomogeneous nanobeams.

In view of the above, the aim of the present article is to develop a inhomogeneous nanobeam under nonlinear temperature distribution for thermal buckling and vibration analysis of Timoshenko nanobeams within the framework of nonlocal elasticity theory. An analytical method called Navier solution is employed for vibration and thermal buckling analysis of size-dependent
FG nanobeams. It is assumed that material properties of the beam, vary continuously through the beam thickness according to power-law form and are temperature dependent. Nonlocal Timoshenko beam model and Eringen’s nonlocal elasticity theory are employed. Governing equations and boundary conditions for the free vibration of a nonlocal FG beam have been derived via Hamilton’s principle. These equations are solved using Navier type method and numerical solutions are obtained. The detailed mathematical derivations are presented while the emphasis is placed on investigating the effect of several parameters such as thermal effects, constituent volume fractions, mode number, aspect ratio and small scale on critical buckling temperature and vibration characteristics of FG nanobeams. Comparisons with analytical solutions and the results from the existing literature are provided for two-constituents metal–ceramic nanobeams and the good agreement between the results of this article and those available in literature validated the presented approach. Numerical results are presented to serve as benchmarks for the application and the design of nanoelectronic and nano-drive devices, nano-oscillators, and nanosensors, in which nanobeams act as basic elements. They can also be useful as valuable sources for validating other approaches and approximate methods.

2. Theory and formulation

2.1 Nonlocal power-law FG nanobeam equations

Consider a FG nanobeam of length \( L \), width \( b \) and uniform thickness \( h \) in the unstressed reference configuration. The coordinate system for FG nanobeam is shown in Fig. 1. The nanobeam is made of elastic and isotropic functionally graded material with properties varying smoothly in the \( z \) thickness direction only. The effective material properties of the FG nanobeam such as Young’s modulus \( E_f \), shear modulus \( G_f \) and mass density \( \rho_f \) are assumed to vary continuously in the thickness direction (\( z \)-axis direction) according to a power function of the volume fractions of the constituents. According to the rule of mixture, the effective material properties, \( P_f \), can be expressed as Şimşek and Yurtcu (2013)

\[
P_f = P_c V_c + P_m V_m
\]

where \( P_m, P_c, V_m \) and \( V_c \) are the material properties and the volume fractions of the metal and the ceramic constituents related by

\[
V_c + V_m = 1
\]

The volume fraction of the ceramic constituent of the beam is assumed to be given by

\[
V_c = \left( \frac{z}{h} + \frac{1}{2} \right)^p
\]

here \( p \) is the non-negative variable parameter (power-law index) which determines the material distribution through the thickness of the beam and \( z \) is the distance from the mid-plane of the FG nanobeam.

The FG nanobeam becomes a fully ceramic beam when \( p \) is set to be zero. Therefore, from Eqs.
Table 1 Temperature dependent coefficients of Young’s modulus, thermal expansion coefficient, mass density and Poisson’s ratio for Si$_3$N$_4$ and SUS304 (Tang et al. 2018)

<table>
<thead>
<tr>
<th>Material</th>
<th>Properties</th>
<th>$P_0$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Si$_3$N$_4$</td>
<td>$E$ (Pa)</td>
<td>348.43e+9</td>
<td>0</td>
<td>-3.070e-4</td>
<td>2.160e-7</td>
</tr>
<tr>
<td></td>
<td>$\alpha$ (K$^{-1}$)</td>
<td>5.8723e-6</td>
<td>0</td>
<td>9.095e-4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\rho$ (Kg/m$^3$)</td>
<td>2370</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\kappa$ (W/mK)</td>
<td>13.723</td>
<td>0</td>
<td>-1.032e-3</td>
<td>5.466e-7</td>
</tr>
<tr>
<td></td>
<td>$v$</td>
<td>0.24</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>SUS304</td>
<td>$E$ (Pa)</td>
<td>201.04e+9</td>
<td>0</td>
<td>3.079e-4</td>
<td>-6.534e-7</td>
</tr>
<tr>
<td></td>
<td>$\alpha$ (K$^{-1}$)</td>
<td>12.330e-6</td>
<td>0</td>
<td>8.086e-4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\rho$ (Kg/m$^3$)</td>
<td>8166</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\kappa$ (W/mK)</td>
<td>15.379</td>
<td>0</td>
<td>-1.264e-3</td>
<td>2.092e-6</td>
</tr>
<tr>
<td></td>
<td>$v$</td>
<td>0.3262</td>
<td>0</td>
<td>-2.002e-4</td>
<td>3.797e-7</td>
</tr>
</tbody>
</table>

(1)-(2), the effective material properties of the FG nanobeam such as Young’s modulus ($E$), mass density ($\rho$), thermal expansion ($\alpha$), thermal conductivity ($\kappa$) and Poisson’s ratio ($v$) can be expressed as follows

\[
E(z) = (E_c - E_m) \left( \frac{z}{h} + \frac{1}{2} \right)^p + E_m, \quad \rho(z) = (\rho_c - \rho_m) \left( \frac{z}{h} + \frac{1}{2} \right)^p + \rho_m
\]

\[
\alpha(z) = (\alpha_c - \alpha_m) \left( \frac{z}{h} + \frac{1}{2} \right)^p + \alpha_m, \quad \kappa(z) = (\kappa_c - \kappa_m) \left( \frac{z}{h} + \frac{1}{2} \right)^p + \kappa_m
\]

\[
v(z) = (v_c - v_m) \left( \frac{z}{h} + \frac{1}{2} \right)^p + v_m
\]

To predict the behavior of FGMs under high temperature more accurately, it is necessary to consider the temperature dependency on material properties. Temperature dependency of the FGM constituents is frequently expressed based on Touloukian formula which captures the higher order dependencies. The temperature-dependent case, on the other hand, represents the conditions where properties are calculated at current temperature based on the Touloukian model described by Eq. (4). The nonlinear equation of thermo-elastic material properties in function of temperature $T(K)$
can be expressed as

\[ P = P_0 (P_{-1} T^{-1} + P_1 T + P_2 T^2 + P_3 T^3) \]  

where \( P_0, P_{-1}, P_1, P_2 \) and \( P_3 \) are the temperature dependent coefficients which can be seen in the table of materials properties (Table 1) for \( \text{Si}_3\text{N}_4 \) and \( \text{SUS}304 \). The bottom surface \((z = -h/2)\) of FG nanobeam is pure metal (SUS304), whereas the top surface \((z = h/2)\) is pure ceramics (\( \text{Si}_3\text{N}_4 \)).

### 2.2 Kinematic relations

The equations of motion is derived based on the Timoshenko beam theory according to which the displacement field at any point of the beam can be written as

\[
\begin{align*}
\begin{align*}
\mathbf{u}(x, z, t) &= \mathbf{u}(x, t) + z \varphi(x, t) \\
\mathbf{w}(x, z, t) &= w(x, t)
\end{align*}
\end{align*}
\]

where \( t \) is time, \( \varphi \) is the total bending rotation of the cross-section, \( u \) and \( w \) are displacement components of the mid-plane along \( x \) and \( z \) directions, respectively. Therefore, according to the Timoshenko beam theory, the nonzero strains are obtained as

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + z \frac{\partial \varphi}{\partial x} \\
\gamma_{xz} &= \frac{\partial w}{\partial x} + \varphi
\end{align*}
\]

where \( \varepsilon_{xx} \) and \( \gamma_{xy} \) are the normal strain and shear strain, respectively. Based on the Hamilton’s principle, which states that the motion of an elastic structure during the time interval \( t_1 < t < t_2 \) is such that the time integral of the total dynamics potential is extremum

\[
\int_{t_1}^{t_2} \left( U - T + V \right) dt = 0
\]

here \( U \) is strain energy, \( T \) is kinetic energy and \( V \) is work done by external forces. The virtual strain energy can be calculated as

\[
\delta U = \int \sigma_{ij} \delta \varepsilon_{ij} dV = \int (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{zz} \delta \varepsilon_{zz} + \sigma_{xz} \delta \gamma_{xz} + \sigma_{zy} \delta \gamma_{zy}) dV
\]

Substituting Eqs. (6) and (7) into Eq. (9) yields

\[
\delta U = \int_0^L \left[ N (\delta \frac{\partial u}{\partial x}) + M (\delta \frac{\partial \varphi}{\partial x}) + Q (\delta \frac{\partial w}{\partial x} + \varphi) \right] dx
\]

In which \( N \) is the axial force, \( M \) is the bending moment and \( Q \) is the shear force. These stress resultants used in Eq. (10) are defined as
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\[ N = \int_A \sigma_{sz} \, dA, \quad M = \int_A \sigma_{sz} z \, dA, \quad Q = \int_A K \sigma_z \, dA \]  \hspace{1cm} (11)

The kinetic energy for Timoshenko beam can be written as

\[ T = \frac{1}{2} \int_0^L \rho(z, T) \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \, dA \, dx \]  \hspace{1cm} (12)

Also the virtual kinetic energy can be expressed as

\[ \delta T = \int_0^L \left[ I_0 \left( \frac{\partial u}{\partial t} \frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} \right) + I_1 \left( \frac{\partial \delta \varphi}{\partial t} + \frac{\partial u}{\partial t} \frac{\partial \varphi}{\partial t} \right) + I_2 \frac{\partial \varphi}{\partial t} \frac{\partial \delta \varphi}{\partial t} \right] dx \]  \hspace{1cm} (13)

where \((I_0, I_1, I_2)\) are the mass moment of inertias, defined as follows

\[(I_0, I_1, I_2) = \int_A \rho(z, T)(1, z, z^2) \, dA \]  \hspace{1cm} (14)

For a typical FG nanobeam which has been in high temperature environment for a long period of time, it is assumed that the temperature can be distributed nonlinearly across its thickness. So that the case of nonlinear temperature rise is taken into consideration. Hence, the first variation of the work done corresponding to temperature change can be written in the form

\[ \delta V = \int_0^L N^T \left( \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right) dx \]  \hspace{1cm} (15)

where \(N^T\) is thermal resultant can be expressed as

\[ N^T = \int_{-h/2}^{h/2} E(z, T) \alpha(z, T)(T - T_0) \, dz \]  \hspace{1cm} (16)

where \(T_0\) is the reference temperature. By Substituting Eqs. (10), (13) and (15) into Eq. (8) and setting the coefficients of \(\delta u, \delta w\) and \(\delta \varphi\) to zero, the following Euler–Lagrange equation can be obtained

\[ \frac{\partial N}{\partial x} = I_0 \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^2 \varphi}{\partial t^2} \]  \hspace{1cm} (17a)

\[ \frac{\partial Q}{\partial x} - N^T \frac{\partial^2 w}{\partial x^2} = I_0 \frac{\partial^2 w}{\partial t^2} \]  \hspace{1cm} (17b)

\[ \frac{\partial M}{\partial x} - Q = I_1 \frac{\partial^2 u}{\partial t^2} + I_2 \frac{\partial^2 \varphi}{\partial t^2} \]  \hspace{1cm} (17c)

Under the following boundary conditions

\[ N = 0 \quad \text{or} \quad u = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \]  \hspace{1cm} (18a)
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\[ Q = 0 \quad \text{or} \quad w = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \] (18b)

\[ M = 0 \quad \text{or} \quad \varphi = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \] (18c)

**2.3 The nonlocal elasticity model for FG nanobeam**

Based on Eringen nonlocal elasticity model (Eringen 1983), the stress at a reference point \( x \) in a body is considered as a function of strains of all points in the near region. This assumption is agreement with experimental observations of atomic theory and lattice dynamics in phonon scattering in which for a homogeneous and isotropic elastic solid the nonlocal stress-tensor components \( \sigma_{ij} \) at any point \( x \) in the body can be expressed as

\[
\sigma_{ij}(x) = \int_{\Omega} \alpha(|x' - x|, \tau) t_{ij}(x') d\Omega(x')
\] (19)

where \( t_{ij}(x') \) are the components of the classical local stress tensor at point \( x \) which are related to the components of the linear strain tensor \( \varepsilon_{kl} \) by the conventional constitutive relations for a Hookean material, i.e.

\[
t_{ij} = C_{ijkl} \varepsilon_{kl}
\] (20)

The meaning of Eq. (19) is that the nonlocal stress at point \( x \) is the weighted average of the local stress of all points in the neighborhood of \( x \), the size of which is related to the nonlocal kernel \( \alpha(|x' - x|, \tau) \). Here \( |x' - x| \) is the Euclidean distance and \( \tau \) is a constant given by

\[
\tau = \frac{\varepsilon_0 a}{l}
\] (21)

which represents the ratio between a characteristic internal length, \( a \) (such as lattice parameter, C–C bond length and granular distance) and a characteristic external one, \( l \) (e.g., crack length, wavelength) through an adjusting constant, \( \varepsilon_0 \), dependent on each material. The magnitude of \( \varepsilon_0 \) is determined experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics. According to Eringen (1983), for a class of physically admissible kernel \( \alpha(|x' - x|, \tau) \) it is possible to represent the integral constitutive relations given by Eq. (19) in an equivalent differential form as

\[
(1-(\varepsilon_0 a)^2 \nabla^2) \sigma_{kl} = t_{kl}
\] (22)

where \( \nabla^2 \) is the Laplacian operator. Thus, the scale length \( \varepsilon_0 a \) takes into account the size effect on the response of nanostructures. For an elastic material in the one dimensional case, the nonlocal constitutive relations may be simplified as

\[
\sigma_{xx} - (\varepsilon_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E \varepsilon_{xx}
\] (23)

\[
\sigma_{zz} - (\varepsilon_0 a)^2 \frac{\partial^2 \sigma_{zz}}{\partial x^2} = G \gamma_{zz}
\] (24)
where \( \sigma \) and \( \varepsilon \) are the nonlocal stress and strain, respectively. \( E \) is the Young’s modulus, \( G = E / (2(1 + v)) \) is the shear modulus. For Timoshenko nonlocal FG beam, Eqs. (23) and (24) can be rewritten as

\[
\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E(z) \varepsilon_{xx} \tag{25}
\]

\[
\sigma_{zz} - \mu \frac{\partial^2 \sigma_{zz}}{\partial x^2} = G(z) \gamma_{zz} \tag{26}
\]

where \( \mu = (\varepsilon_0 a)^2 \). Integrating Eqs. (25) and (26) over the beam’s cross-section area, the force-strain and the moment-strain of the nonlocal Timoshenko FG beam theory can be obtained as follows

\[
N - \mu \frac{\partial^2 N}{\partial x^2} = A_{ss} \frac{\partial u}{\partial x} + B_{ss} \frac{\partial \varphi}{\partial x} \tag{27}
\]

\[
M - \mu \frac{\partial^2 M}{\partial x^2} = B_{ss} \frac{\partial u}{\partial x} + D_{ss} \frac{\partial \varphi}{\partial x} \tag{28}
\]

\[
Q - \mu \frac{\partial^2 Q}{\partial x^2} = C_{ss} (\frac{\partial w}{\partial x} + \varphi) \tag{29}
\]

In which the cross-sectional rigidities are defined as follows

\[
(A_{ss}, B_{ss}, D_{ss}) = \int_A E(z, T)(1, z, z^2) dA \tag{30}
\]

\[
C_{ss} = K_s \int_A G(z) dA \tag{31}
\]

where \( K_s = 5/6 \) is the shear correction factor. The explicit relation of the nonlocal normal force can be derived by substituting for the second derivative of \( N \) from Eq. (17a) into Eq. (27) as follows

\[
N = A_{ss} \frac{\partial u}{\partial x} + B_{ss} \frac{\partial \varphi}{\partial x} + \mu(I_0 \frac{\partial^3 u}{\partial x \partial t^2} + I_1 \frac{\partial^2 \varphi}{\partial x \partial t} + N \frac{\partial^2 w}{\partial x \partial t}) \tag{32}
\]

Also the explicit relation of the nonlocal bending moment can be derived by substituting for the second derivative of \( M \) from Eq. (17c) into Eq. (28) as follows

\[
M = B_{ss} \frac{\partial u}{\partial x} + D_{ss} \frac{\partial \varphi}{\partial x} + \mu(I_0 \frac{\partial^3 w}{\partial x \partial t^3} + I_1 \frac{\partial^3 u}{\partial x \partial t^2} + I_2 \frac{\partial^3 \varphi}{\partial x \partial t} + N \frac{\partial^2 w}{\partial x \partial t^2}) \tag{33}
\]

By substituting for the second derivative of \( Q \) from Eq. (17b) into Eq. (29), the following expression for the nonlocal shear force is derived

\[
Q = C_{ss} (\frac{\partial w}{\partial x} + \varphi) + \mu(I_0 \frac{\partial^2 w}{\partial x \partial t^2} + N \frac{\partial^2 w}{\partial x \partial t}) \tag{34}
\]
The nonlocal governing equations of Timoshenko FG nanobeam in terms of the displacement can be derived by substituting for $N$, $M$ and $Q$ from Eqs. (32)-(34), respectively, into Eq. (17) as follows

$$A_{xx} \frac{\partial^2 u}{\partial x^2} + B_{xx} \frac{\partial^2 \varphi}{\partial x^2} + \mu \left( I_0 \frac{\partial^2 u}{\partial t^2 \partial x^2} + I_1 \frac{\partial^2 \varphi}{\partial t^2 \partial x^2} \right) - I_0 \frac{\partial^4 u}{\partial t^4} - I_1 \frac{\partial^2 \varphi}{\partial t^2} = 0$$  \hspace{1cm} (35a)

$$C_{xx} \left( \frac{\partial^4 w}{\partial x^4} + \frac{\partial \varphi}{\partial x} \right) + \mu (N^r \frac{\partial^4 w}{\partial x^4} + I_0 \frac{\partial^2 \varphi}{\partial t^2 \partial x^2}) - N^r \frac{\partial^4 w}{\partial x^4} - I_0 \frac{\partial^2 \varphi}{\partial t^2} = 0$$  \hspace{1cm} (35b)

$$B_{xx} \frac{\partial^2 u}{\partial x^2} + D_{xx} \frac{\partial^2 \varphi}{\partial x^2} - C_{xx} \left( \frac{\partial w}{\partial x} + \varphi \right) + \mu (I_1 \frac{\partial^4 u}{\partial t^4 \partial x^2} + I_2 \frac{\partial^4 \varphi}{\partial t^2 \partial x^2}) - I_1 \frac{\partial^2 u}{\partial t^2} - I_2 \frac{\partial^2 \varphi}{\partial t^2} = 0$$  \hspace{1cm} (35c)

2.4 Non-uniform temperature distribution

In this study, nonlinear temperature rise across the thickness is assumed. The steady-state one-dimensional heat conduction equation with the known temperature boundary conditions on bottom and top surfaces of the FG nanobeam can be obtained by solving the following equation (Zhang 2013)

$$- \frac{d}{dz} \left( \kappa(z, T) \frac{dT}{dz} \right) = 0 \quad T \left( \frac{h}{2} \right) = T_e, \quad T \left( -\frac{h}{2} \right) = T_m$$  \hspace{1cm} (36)

Solving this equation via polynomial series and taking the sufficient terms to assure the convergence, yields the temperature distribution across the nanobeam thickness as

$$T = T_m + \Delta T \left[ \sum_{i=0}^{\infty} \left[ (-1)^i \left( \frac{\kappa_e - \kappa_m}{\kappa_m} \right) \right] \right] \sum_{i=0}^{\infty} \left[ (-1)^i \left( \frac{1}{2} + \frac{z}{h} \right)^{pi+1} \left( \frac{\kappa_e - \kappa_m}{\kappa_m} \right) \right]$$  \hspace{1cm} (37)

where $\Delta T = T_e - T_m$.

3. Solution procedures

Here, based on the Navier type method, analytical solutions of the governing equations for free vibration and thermal buckling of a simply supported FG nanobeam is presented. The displacement functions are expressed as product of undetermined coefficients and known trigonometric functions to satisfy the governing equations and the conditions at $x = 0, L$. The following displacement fields are assumed to be of the form

$$u(x,t) = \sum_{n=1}^{\infty} U_n \cos \left( \frac{n \pi}{L} x \right) e^{i\omega_n t}$$  \hspace{1cm} (38)

$$w(x,t) = \sum_{n=1}^{\infty} W_n \sin \left( \frac{n \pi}{L} x \right) e^{i\omega_n t}$$  \hspace{1cm} (39)
Effect of non-uniform temperature distributions on nonlocal vibration and buckling of...  

\[ \varphi(x,t) = \sum_{n=1}^{\infty} \varphi_n \cos \left( \frac{n\pi x}{L} \right) e^{i\omega_n t} \]  

where \((U_n, W_n, \varphi_n)\) are the unknown Fourier coefficients to be determined for each \(n\) value. Boundary conditions for simply supported beam are as Eq. (41)

\[ u(0) = 0, \quad \frac{\partial u}{\partial x}(L) = 0 \]

\[ w(0) = w(L) = 0, \quad \frac{\partial w}{\partial x}(0) = \frac{\partial w}{\partial x}(L) = 0 \]

Substituting Eqs. (38)-(40) into Eqs. (35a)-(35c) respectively, leads to Eqs. (42)-(44)

\[ (-A_{xx} \left( \frac{n\pi}{L} \right)^2 + I_0(1 + \mu \left( \frac{n\pi}{L} \right)^2 \omega_n^2)U_n + (-B_{xx} \left( \frac{n\pi}{L} \right)^2 + I_1(1 + \mu \left( \frac{n\pi}{L} \right)^2 \omega_n^2) \phi_n = 0 \]  

\[ (-C_{xx} \left( \frac{n\pi}{L} \right)^2 + I_2(1 + \mu \left( \frac{n\pi}{L} \right)^2 \omega_n^2 + N^T \left( \frac{n\pi}{L} \right)^2 (1 + \mu \left( \frac{n\pi}{L} \right)^2) W_n - C_{xz} \left( \frac{n\pi}{L} \right) \phi_n = 0 \]  

\[ (-B_{xx} \left( \frac{n\pi}{L} \right)^2 + I_1(1 + \mu \left( \frac{n\pi}{L} \right)^2 \omega_n^2)U_n \]

\[ + (-D_{xx} \left( \frac{n\pi}{L} \right)^2 - C_{xz} + I_2(1 + \mu \left( \frac{n\pi}{L} \right)^2 \omega_n^2) \phi_n - C_{xz} \left( \frac{n\pi}{L} \right) W_n = 0 \]

By setting the determinant of the coefficient matrix of the above equations, the analytical solutions can be obtained from the following equations

\[ \begin{bmatrix} (K) + \Delta T \left[ K_T \right] - \omega^2[M] \end{bmatrix} \begin{bmatrix} U_n \\ W_n \\ \phi_n \end{bmatrix} = 0 \]  

where \([K]\) and \([K_T]\) are stiffness matrix and the coefficient matrix of temperature change, respectively, and \([M]\) is the mass matrix. By setting this polynomial to zero, we can find natural frequencies \(\omega_n\) and critical buckling temperature \(\Delta T_{cr}\).

4. Numerical results and discussions

This section is dedicated to discuss about the effects of nonlocal parameter as well as the power law index, temperature change and thickness ratios on the thermal buckling and vibration behavior of FG nanobeam based on the nonlocal elasticity theory. For this purpose, the FG nanobeam considered here consist of Steel (SUS304) and Silicon nitride (Si$_3$N$_4$) with thermo-mechanical material properties listed in Table 1. The bottom surface of the beam is pure Steel, whereas the top surface of the beam is pure Silicon nitride. The beam geometry has the following dimensions: \(L\)
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(length) = 10 nm, b (width) = 1 nm and h (thickness) = varied. It is assumed that the temperature increase in metal surface to reference temperature $T_0$ of the FG nanobeam is $T_m - T_0 = 5K$ (Kiani and Eslami 2013). Relation described in Eq. (46) are performed in order to calculate the non-dimensional natural frequencies

$$\hat{\omega} = \omega L^2 \sqrt{\rho A/EI_c} \tag{46}$$

where $I = bh^3/12$ is the moment of inertia of the cross section of the beam. To evaluate accuracy of the natural frequencies predicted by the present method, the non-dimensional natural frequencies of simply supported FG nanobeam with various nonlocal parameters previously analyzed by Navier method are reexamined. Table 2 compares the results of the present study and the results presented by Rahmani and Pedram (2014) which has been obtained by analytical method for FG nanobeam with different nonlocal parameters (varying from 0 to 5). The reliability of the presented method and procedure for FG nanobeam may be concluded from Table 2; where the results are in an excellent agreement as values of non-dimensional fundamental frequency are consistent with presented analytical solution.

After making certain about the credibility of the numerical results of current study, the effects of different parameters such as aspect ratio, nonlocality parameter and gradient index on the thermal buckling of FG nanobeam are investigated. In Table 3 critical buckling temperature of the simply supported FG nanobeams are presented for various values of the gradient index ($p = 0, 0.2, 0.5, 1, 2, 5$), nonlocal parameters ($\mu = 0, 1, 2, 3, 4$) and three different values of aspect ratio ($L/h = 40, 50, 60$) based on Navier solution method. It is indicated that increasing of nonlocal parameter leads to lower critical buckling temperature difference. In other words, by increasing the influence of small scale effect, the stiffness of FG nanobeams decreases. It can be also observed that the critical buckling temperature difference of FG nanobeams decreases by increasing the value of $L/h$ ratio and this behavior is the same for all values of power index. Moreover, an increase in the nonlocal parameter leads to the decrease of the buckling temperature and it is more tangible for the nanobeams with lower values of $L/h$ ratio. Also, it can be seen that an increase in the power law index lead to lower critical temperature by decrease the stiffness of nanobeam.

In order to highlight the effect of the nonlocal parameter and aspect ratio on the thermal buckling behavior of the FG nanobeam, the critical buckling temperature against the $L/h$ ratio with the assumption of $p = 0.5, 2$, is illustrated in Fig. 2. This figure indicates that the decrease of

Table 2 Comparison of the nondimensional fundamental frequency for a S-S FG nanobeam with various gradient indexes when $L/h = 50$

<table>
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<tr>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8.3515</td>
<td>8.35146095</td>
<td>7.3577</td>
<td>7.35772548</td>
<td>5.9201</td>
<td>5.92013713</td>
<td>5.0287</td>
<td>5.02869994</td>
</tr>
<tr>
<td>5</td>
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<td>8.07079327</td>
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<td>7.11045428</td>
<td>5.7212</td>
<td>5.72117899</td>
<td>4.8597</td>
<td>4.85970034</td>
</tr>
</tbody>
</table>
the critical buckling temperature resulting from the increase of the nonlocal parameter occurs in all the $L/h$ ratios so that at the lower $L/h$ ratios, the rate of variations is higher. It is also obvious that the effect of the nonlocal parameter becomes more pronounced for lower $L/h$ ratios. This illustrates the size-dependence of the nanoscale beams which plays an important role in the nonlocal theory.
Table 4: Temperature and material graduation effect on first three dimensionless frequency of a S-S FG nanobeam with different nonlocality parameters ($L/h = 20$)

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\Delta T = 10[K]$</th>
<th>$\Delta T = 30[K]$</th>
<th>$\Delta T = 60[K]$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta_k$</td>
<td>$\Delta_k$</td>
<td>$\Delta_k$</td>
</tr>
<tr>
<td></td>
<td>Gradient index</td>
<td>Gradient index</td>
<td>Gradient index</td>
</tr>
<tr>
<td>2</td>
<td>85.5836 69.3213 51.5487 42.0461 85.4002 69.1540 51.3939 41.8928 85.1245 68.9024 51.1608 41.6618</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>62.1630 50.3359 37.4093 30.4970 61.9103 50.1053 37.1957 30.2853 61.5294 49.7575 36.8730 29.9649</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>44.4503 35.9715 26.7025 21.7451 44.0963 35.6481 26.4025 21.4471 43.5599 35.1574 25.9458 20.9923</td>
<td></td>
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</tr>
</tbody>
</table>

In order to investigate the vibration characteristics of the FG nanobeam under nonlinear temperature rise, the first three non-dimensional fundamental frequencies of simply-supported FG nanobeam is presented in Table 4, which figures out the effect of nonlocal parameter (varying from 0 to 4 (nm)$^2$), gradient index (varying from 0 to 5) and three different values of linear temperature changes ($\Delta T = 10$, 30, 60) for $L/h = 20$ on the natural frequency characteristics of FG nanobeam.

First of all, when the two parameters vanish ($\mu = 0$ and $p = 0$) the classical isotropic beam theory is rendered. Furthermore, the effects of temperature change, nonlocal parameter and gradient indexes on the dimensionless frequencies are presented in this table. Also, there is no rigorous study made on estimating the value of small scale to simulate mechanical behaviour of functionally graded micro/nanobeams. Hence all researchers who worked on size-dependent mechanical behaviour of FG nanobeams based on the nonlocal elasticity method investigated the effect of small scale parameter on mechanical behaviour of FG nanobeams by changing the value of the small scale parameter. The nonlocal parameter, $\mu = (\epsilon_0\alpha L)^2$, is experimentally obtained for various materials. Therefore, in the present study, a conservative estimate of the nonlocal parameter is in the range of 0-4 (nm)$^2$ (Rahmani and Pedram 2014).

As seen in Table 4, the natural frequencies decreases for increasing values of the power index from 0 up to 5. This is due to the fact that for large values of power law index, the material properties of the nanobeam become similar to the material properties of a metal with higher thermal expansion coefficients in comparison to the ceramic. Thus, one could easily control the frequencies and critical buckling temperature of the FG nanobeams by tuning the power index. Also, it is revealed that the major effect of power law index is for $p < 5$. Whilst the variations of
natural frequencies with respect to $p$ for $p > 5$ are not significant when compared to $p < 5$. However, the increasing of nonlocal parameter causes the decreasing in fundamental frequency, at a constant material graduation index. In addition, it is seen that the first three dimensionless natural frequencies decrease by increasing temperature change and it can be stated that temperature change has a significant effect on the dimensionless natural frequencies, especially for lower mode numbers.

Variations of the first three dimensionless natural frequencies of the simply supported FG nanobeams with respect to temperature changes for different values of gradient indexes and nonlocal parameters are depicted in Figs. 3-5, respectively. It is seen from the figures that the

![Fig. 3 Variations of the first dimensionless natural frequency of the S-S FG nanobeam with respect to temperature change for different values of nonlocal parameters and gradient indexes ($L/h = 50$)](image)

![Fig. 4 Variations of the second dimensionless natural frequency of the S-S FG nanobeam with respect to temperature change for different values of nonlocal parameters and gradient indexes ($L/h = 50$)](image)
fundamental frequency of FG nanobeam decreases with the increase of temperature until it approaches to the critical buckling temperature. This is due to the reduction in total stiffness of the beam, since geometrical stiffness decreases when temperature rises. Frequency reaches to zero at the critical temperature point. Also, the increase in temperature yields in higher frequency after the branching point. One important observation within the range of temperature before the critical temperature, it is seen that the FG nanobeams with lower value of gradient index (higher percentage of ceramic phase) usually provide larger values of the frequency results. However, this behavior is opposite in the range of temperature beyond the critical temperature. It is also observable that the branching point of the FG nanobeam is postponed by consideration of the lower gradient indexes due to the fact that the lower gradient indexes result in the increase of
Effect of non-uniform temperature distributions on nonlocal vibration and buckling of...  

stiffness of the structure. It can be also seen that regardless of the values of power index, the natural frequencies are considerably reduced by increasing the nonlocal parameter. This implies that the nonlocal effects make the stiffness of the FG nanobeam diminish and so, the small scale impact must be accounted.

Depicted in Fig. 6 is the influences of $L/h$ ratio and temperature changes on dimensionless natural frequency of FG nanobeams at $p = 0.5, \mu = 2$. As expected, the increase in aspect ratio results in reduction of natural frequency of temperature-dependent FG nanobeams. Also, an increase in the temperature change leads to the decrease of the dimensionless natural frequency.

5. Conclusions

In the present research, the study of thermal buckling and vibrational behavior of the temperature-dependent FG nanobeams subjected to nonlinear temperature distribution through the thickness direction was carried out based on the nonlocal elasticity theory accounting for the small scale effects. The power law distribution is assumed for the variation of the material properties in the nanobeam thickness. Thermo-mechanical properties of the FG nanobeams are assumed to be functions of both temperature and thickness. Eringen’s theory of nonlocal elasticity together with Timoshenko beam theory is used to model the nanobeam. The governing differential equations and related boundary conditions in thermal environment are derived by implementing Hamilton’s principle. Accuracy of the results is examined using available data in the literature. After performing comparison studies, parametric studies are done to investigate the influences of nonlocal parameter, gradient index, mode number, nonlinear temperature rise and aspect ratio on the critical buckling temperature and natural frequencies of FG nanobeams.

It is concluded that various factors such as nonlocal parameter, gradient index, temperature-dependent material properties, thermal environment and aspect ratio play important roles in buckling and vibration behavior of FG nanobeams. It is illustrated that presence of nonlocality leads to reduction in natural frequency and buckling temperature. This behavior was more tangible for the nanobeams with lower values of $L/h$ ratio. Moreover it is revealed that critical buckling temperature decreases with the increase in aspFFPect ratio and this behavior is the same for all values of power index. It is also observed that the fundamental frequency decreases with the increase in temperature and tends to the minimum point closing to zero at the critical temperature. This decrease in frequency with thermal load is attributed to the fact that the thermally induced compressive stress weakens the beam stiffness. However, after the critical temperature region, the fundamental frequency increases with the increment of temperature. Also, it is concluded that an increase in the power law index lead to lower critical buckling temperature and natural frequencies by decrease the stiffness of nanobeam.

References


Thermophysical Properties Research Center (1967), Thermophysical properties of high temperature solid materials; Volume 1, Elements.-Pt. 1. Ed. Yeram Sarkis Touloukian, Macmillan.


*JL*
Nomenclature

\[\begin{align*}
A & \quad \text{area of the cross section} \\
A_{xx}, B_{xx}, D_{xx} & \quad \text{cross-sectional rigidities} \\
E & \quad \text{Young's modulus} \\
h & \quad \text{thickness of the nanobeam} \\
l & \quad \text{moment of inertia of the cross section} \\
I_0, I_1, I_2 & \quad \text{mass moment of inertias} \\
L & \quad \text{length of the nanobeam} \\
M & \quad \text{bending moment} \\
N & \quad \text{axial force} \\
N^T & \quad \text{thermal resultant} \\
p & \quad \text{power-law index} \\
P_c & \quad \text{material properties of the ceramic constituent} \\
P_m & \quad \text{material properties of the metal constituent} \\
P_0, P_{-1}, P_1, P_2, P_3 & \quad \text{temperature dependent coefficients} \\
T_0 & \quad \text{initial temperature} \\
\Delta T & \quad \text{temperature change} \\
u & \quad \text{axial displacement} \\
U & \quad \text{strain energy} \\
V & \quad \text{work done by external forces} \\
w & \quad \text{transverse displacement} \\
t & \quad \text{time} \\
T & \quad \text{kinetic energy} \\
\rho & \quad \text{mass density} \\
\alpha & \quad \text{thermal expansion coefficient} \\
\sigma_{ij} & \quad \text{nonlocal stress-tensor components} \\
\omega & \quad \text{natural frequency} \\
\hat{\omega} & \quad \text{non-dimensional natural frequency} \\
\mu & \quad \text{nonlocal parameter}
\end{align*}\]
Highlights

- Thermal buckling and free vibration analysis of Timoshenko nanobeams subjected to nonlinear temperature rise are studied.
- The material properties of the nanobeam vary continuously through the beam thickness according to power-law form and are temperature dependent based on the Touloukian model.
- Nonlocal equations are derived by using the Hamilton’s principle and they are solved by applying an analytical solution.
- Effects of nonlocal parameter, nonlinear temperature distribution, geometrical characteristics and power law index on the critical buckling temperatures and natural frequencies of size-dependent FG nanobeams are discussed.

Graphical abstract

[Graphical abstract image]