Vibration of axially functionally graded nano rods and beams with a variable nonlocal parameter

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Abstract. Vibration of axially functionally graded nano-rods and beams is investigated. It is assumed that the material properties change along the rod and beam length. The Ritz method with algebraic polynomials is used in the formulation of the problems. Stress gradient elasticity theory is utilized in order to include the nonlocal effects. Frequencies are obtained for different boundary conditions, geometrical and material properties. Nonlocal parameter is assumed as changing linearly or quadratically along the length of the nanostructure. Frequencies are compared to constant nonlocal parameter cases and considerable differences are observed between constant and variable nonlocal parameter cases. Mode shapes in various cases are depicted in order to explain the effects of axial grading.

Keywords: vibration; axially functionally graded; nanorod; nanobeam; nonlocal elasticity

1. Introduction

Functionally graded (FG) material is a compositional gradient of two or more component materials. Materials can be designed for specific applications with the variation of their composition and structure gradually. FG materials can have the desirable properties of each component because of a homogenous mixture of constituents.

Functionally graded material (FGM) structures have taken great interest of engineers recent years. They were designed for thermal isolation for aerospace structural applications and fusion reactors, especially. Metal-ceramic functionally graded materials are used in extremely high temperature environments as a structural element (Bharti et al. 2013).

Functionally graded nano rod and beam structures are used in some nano electromechanical system similar to their macro counterparts because of smooth variation of the material properties in preferred directions of structures. Although thickness wise grading is generally considered in the previous studies, axially functionally graded rods and beams can also be used at the nano length scale.

Continuous material property variation provides continuous stress distribution in the FG structures, where a discontinuous stress distribution appears in laminated composites. FGMs with

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material properties varying only in the thickness direction can be manufactured by the high speed centrifugal casting (Fukui 1990, Yamanouchi et al. 1990, Berger et al. 1994, Moya 1995) and those with properties varying in the plane of a sheet by the ultraviolet irradiation to alter chemical composition (Lambros et al. 1999). Studies related with structural analysis of FGM with properties varying in the thickness direction can be found in (Vel and Batra 2004, Qian et al. 2004, Ferreira et al. 2006, Uymaz and Aydogdu 2007, Aydogdu 2008, Şimşek 2010).

Some studies can also be obtained related to axially graded macro structures. Elishakoff and co-workers (Elishakoff and Guédé 2004, Wu et al. 2005, Elishakoff and Johnson 2005) studied vibration and buckling of inhomogeneous beams for some preselected polynomial mode shapes. Recently, (Anandakumar and Kim 2010) used the Rayleigh-Ritz method and the finite element method in analysis of free vibration of a three-dimensional functionally graded cantilever beam. Some specific material gradations are considered in the axial direction: Exponential, linear and hyperbolic tangent material gradations. (Huang and Li 2010) studied axially graded beams with non-uniform cross section in free vibration case using Fredholm integral equations. Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams investigated by (Shahba et al. 2011) on the classical and non-classical boundary conditions using a finite element formulation, recently. (Qian and Batra 2005) investigated optimal natural frequencies in bidirectional functionally graded plates. They obtained that functionally grading in axial direction provides good vibration results. Free (Janghorban and Zare 2011, Nejad and Hadi 2016), forced (Barati 2017) and nonlinear (Hosseini-Hashemi and Nazemnezhad 2013) vibrations of FGM nanostructures also studied by scientists. Dynamic instability (Sedighi et al. 2015) and magnetic-electric fields effect (Ebrahimi and Jafari 2017) investigated in detail.

Axially grading has been also considered in some of the previous nanoscale studies (Akgöz and Civalek 2013, Shafiei et al. 2016, Li et al. 2017, Shafiei et al. 2017). Ebrahimi and co-workers studied vibration of the axially graded nanobeams in thermal environment (Ebrahimi et al. 2017, Ebrahimi and Barati 2017a, c, Ebrahimi et al. 2018) and embedded in elastic medium (Ebrahimi and Barati 2017b, d). Using a bottom-to-up approach by suitable arranging atoms axially grading may also be possible artificially. Continuously graded thick Si and Ge nanowires along the its length shows local variations in Raman phonon bands (Yang et al. 2008).

Free longitudinal vibration of axially functionally graded tapered nanorods has been studied by (Şimşek 2012) using the Galerkin method. (Huang et al. 2013) have investigated the vibration of axially functionally graded Timoshenko beam with non-uniform cross section.

In the nanoscale continuum modeling of the nanostructures the stress gradient elasticity has been used. Nonlocal elasticity has been firstly proposed by (Eringen 1976, 1983) and has been used in statics and dynamics analysis of nanorods and beams (Peddieson et al. 2003, Sudak 2003, Wang 2005). Molecular dynamic simulations and nonlocal continuum models are compared for wave propagation in carbon nanotubes (Hu et al. 2008). Good agreement is observed between two results.

In the previous studies related to nano functionally graded materials, the nonlocal parameter is generally assumed as a constant. But similar to other material properties like elasticity modulus, density and Poisson ratio, nonlocal parameter should also be variable depending on the grading directions.

In the present study, the vibration of axially functionally graded nano rods and beams has been investigated by using the Classical Beam Theory (CBT). The Ritz method is used in the solution of vibration problems. A linear variation of material properties (Young modulus and density) is assumed in the length direction of beams. For nonlocal parameter linear and quadratic variations
are considered. It is assumed that the elasticity modulus, density and nonlocal parameter of rods and beams are changing in the axial direction. The frequency parameters are obtained for various material and geometrical properties, nonlocal parameter and boundary conditions.

2. Analysis

2.1 The nonlocal elasticity model for nano beam

An axially functionally graded beam of length $L$ and radius $R$ is considered. A cartesian coordinate system is chosen in which $x$ axis in the length direction; $z$ and $y$ axes are cross sectional coordinates of the beam, respectively (Fig. 1). It is assumed that the material properties of the nano beam are changing along the length direction.

In the frame work of the Euler Bernoulli beam theory displacement field can be written as

\[ U(x,y) = u(x,t) - zw, \]
\[ V(x,y,t) = 0 \]
\[ W(x,t) = w(x,t) \]  

(1)

where $U$ and $W$ are the axial and transverse displacement of a typical point of the beam and $u$ and $w$ are the corresponding in-plane displacements, $t$ is the time. It should be noted that deformation in the $y$ direction is neglected. Assuming small deformations, linear strain components are

\[ \varepsilon_x = \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2} = \varepsilon_x^0 + z \kappa \]
\[ \varepsilon_x^0 = \frac{\partial u}{\partial x} \]
\[ \kappa = \frac{\partial^2 w}{\partial x^2} \]  

(2)

where $\varepsilon$ is the axial strain and $\kappa$ is the curvature. Stress and moment resultant of the beam, without any dimensional or material limitation, can be defined as

\[ N = \int_A \sigma_{xx} \, dA, \quad M = \int_A z \sigma_{xx} \, dA \]  

(3)
where \( N \) and \( M \) are the force and moment resultants, respectively and \( A \) is the cross sectional area of the beam. Virtual displacement principle can be written as

\[
\int_A \sigma_{ij} \delta \varepsilon_{ij} \, dV = \int_{S_2} \hat{t}_i \delta u_i \, dS_2 + \int_V f_i \delta u_i \, dV
\]  

(4)

where \( \sigma_{ij} \) is the components of the stress tensor, \( V \) is the volume, \( S_2 \) is the boundary, \( f_i \) is the body force, \( \delta \) is the variational symbol and \( \varepsilon_{ij} \) is the surface stress. Using Eq. (1-3) in Eq. (4) following relation can be obtained

\[
\int_0^L \int_0^L \left[ N \delta \varepsilon_{xx} + M \delta \kappa - m_0 \left( \frac{\partial u}{\partial t} \frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} \right) - m_2 \frac{\partial^2 w}{\partial x \partial t} \frac{\partial^2 \delta w}{\partial x \partial t} - f \delta u - q \delta w \right] \, dx \, dt = 0
\]  

(5)

where \( m_i = \int_A \rho z^i \, dA \) \((i = 0, 2)\) and \( f, q \) and \( N_e \) are the axial, flexural and in-plane external forces, respectively. Following equations of motion can be obtained from Eq. (5)

\[
\frac{\partial N}{\partial x} + f = m_0(x) \frac{\partial^2 u}{\partial t^2}
\]  

(6a)

\[
\frac{\partial^2 M}{\partial x^2} + q - \frac{\partial}{\partial x} \left( N \frac{\partial w}{\partial x} \right) = m_0(x) \frac{\partial^2 w}{\partial t^2} - m_2(x) \frac{\partial^4 w}{\partial x^2 \partial t^2}
\]  

(6b)

Here Eqs. (6a) and (6b) are the axial and transverse equations of motion of a beam structure. For uncoupled problems, Eq. (6a) is also called the classical rod equation of motion. Following boundary conditions are also obtained at \( x = 0 \) and \( x = L \).

\[
u \quad or \quad N
\]  

(7a)

\[
w \quad or \quad \frac{\partial M}{\partial x} - N_e \frac{\partial w}{\partial x} + m_2 \frac{\partial^3 w}{\partial x \partial t^2} = V_s
\]  

(7b)

\[
- \frac{\partial w}{\partial x} \quad or \quad M
\]  

(7c)

where \( V_s \) is the shear force. The nonlocal constitutive relations of stress gradient elasticity has been proposed as (Lu et al. 2007, Aydogdu 2009a)

\[
(1 - \mu \beta^2) \tau_{kl} = \lambda e_{rr} \delta_{kl} + 2G e_{kl}
\]  

(8)

where \( \tau_{kl} \) is the nonlocal stress tensor, \( e_{kl} \) is the strain tensor, \( \lambda \) and \( G \) are the Lame constants, \( \mu = (\epsilon_0 a)^2 \) is called the nonlocal parameter, \( a \) is an internal characteristic length (nm) and \( \epsilon_0 \) is a constant. Physical interpretation and choice of this parameter has been discussed in some of the previous studies (Aydogdu 2009a, b).
In this study, $0 \leq (e_0 a)^2 \leq 2 \text{ nm}^2$ is chosen in order to investigate nonlocality effects as suggested in (Wang 2005). It should be noted that for axially graded (or tapered) nano structure nonlocal parameter $\mu$ cannot be a constant. This is due to variation of the micro structure of the solid along the axial direction. It means that the atomic distance changes along the length of the nano structure or there are different atoms along the length of the structure. Therefore, the nonlocal parameter is assumed as a function of axial coordinate in this study.

2.2 Equation of motion of a nanorod

For the axial vibration of an axially functionally graded nanorod Eq. (8) can be written in the following one dimensional form

$$\left(1 - \mu(x) \frac{\partial^2}{\partial x^2}\right) \tau_{xx} = E(x) \varepsilon \tag{9}$$

where $E(x)$ is the modulus of elasticity. Integrating Eq. (9) with respect to the area of nanorod, and multiplying by $z$ and integrating again with respect to area gives the following relations

$$N - \mu(x) \frac{\partial^2 N}{\partial x^2} = N^L \tag{10a}$$
$$M - \mu(x) \frac{\partial^2 M}{\partial x^2} = M^L \tag{10b}$$

where the local force and moment resultants are defined in the conventional manner

$$N^L = EA \varepsilon_{xx}^0 \tag{11a}$$
$$M^L = EI \kappa \tag{11b}$$

From Eqs. (4) and (6) and Eq. (8) axial force can be written as

$$N = EA \frac{\partial u}{\partial x} + \mu(x) \frac{\partial}{\partial x} \left(m_0(x) \frac{\partial^2 u}{\partial t^2} - f\right) \tag{12}$$

Using Eq. (8) in Eq. (6) gives the equation of motion for the axial free-vibrating nanorod in the nonlocal elasticity in terms of displacement

$$\frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x} \left[\mu(x) \frac{\partial}{\partial x} \left(m_0(x) \frac{\partial^2 u}{\partial t^2} - f\right)\right] + f = m_0(x) \frac{\partial^2 u}{\partial t^2} \tag{13}$$

Eq. (13) is the consistent fundamental equation of the axially graded nonlocal rod model. This equation leads to second order variable coefficient ordinary differential equation. Unfortunately it is not possible to obtain a general solution of this kind of equations. When $\mu = 0 \text{ nm}^2$, it is reduced to the equation of the classical rod model. When material properties are uniform and homogeneous along the length direction of nano rod previously obtained equations of motion of nonlocal
nanorod is obtained.

### 2.3 Equation of motion of a nanobeam

Using Eq. (6b) and (10b) the moment relation of nonlocal elasticity can be written as

\[
M = -EI \frac{\partial^2 w}{\partial x^2} + \mu(x) \left[ \frac{\partial}{\partial x} \left( Ne \frac{\partial w}{\partial x} \right) - q + m_0 \frac{\partial^2 w}{\partial x^2} - m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} \right]
\]  

Inserting Eq. (14) into Eq. (10b) leads to following equation of motion for a nanobeam

\[
\frac{\partial^2}{\partial x^2} \left( -EI \frac{\partial^2 w}{\partial x^2} + \mu(x) \left[ \frac{\partial}{\partial x} \left( Ne \frac{\partial w}{\partial x} \right) - q + m_0 \frac{\partial^2 w}{\partial x^2} - m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} \right] \right) + q - \frac{\partial}{\partial x} \left( N \frac{\partial w}{\partial x} \right) = m_0 \frac{\partial^2 w}{\partial x^2} - m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2}
\]  

(15)

Again, this equation is a variable coefficient differential equation. The equation of motion of the conventional Euler Bernoulli beam theory is obtained from Eq. (15) by setting \(\mu = 0\).

### 2.4 Functionally graded materials

The functionally graded materials are produced by mixing two or more materials. In the present study variation of the material properties (elasticity modulus, density and nonlocal parameter) are assumed in the following forms (It should be noted that these material properties can be chosen as any function in the present formulation)

\[
E(x) = \alpha_1 x + \alpha_2
\]  

(16a)

\[
\rho(x) = \beta_1 x + \beta_2
\]  

(16b)

\[
\mu(x) = \gamma_0 x^2 + \gamma_1 x + \gamma_2
\]  

(16c)

where related parameters are

\[
\alpha_1 = E_1 - E_0, \quad \alpha_2 = E_0
\]  

(17a)

\[
\beta_1 = \rho_1 - \rho_0, \quad \beta_2 = \rho_0
\]  

(17b)

\[
\gamma_0 = \gamma_1 = \mu_1 / 2, \quad \alpha_2 = \mu_0
\]  

(17c)

here \(E_0, \rho_0, \mu_0\) and \(E_1, \rho_1, \mu_0\) are the material properties at the left and right end of the beam respectively. Material property variations are shown in Fig. 2.

### 2.5 Ritz formulation

Analytical solution of equations of motion of axially functionally graded nanorods and beams are difficult or not possible, therefore an approximate variational method, Ritz method will be used...
Vibration of axially functionally graded nano rods and beams with a variable nonlocal parameter

2.5 Material properties of FGM nanostructures

in the formulation of the present problem.

2.5.1 Nano rod

The strain energy of an axially functionally graded rod can be (Adali 2008, 2009, 2015, Robinson and Adali 2016)

\[ U_s = \frac{1}{2} \int_0^L E(x) A u_x^2 \, dx \]  

where \( u \) and \( A \) are the displacement in the \( x \) direction and cross-sectional area respectively and \( L \) is the length of the rod. The kinetic energy of the rod is (Adali 2008, 2009, 2015, Robinson and Adali 2016)

\[ T = \frac{1}{2} \int_0^L \left( m(x) \ddot{w}^2 + \mu(x) m(x) \ddot{w}_x^2 \right) \, dx \]

where a dot denotes derivative with respect to time. In this relation the second term of the kinetic energy is due to the nonlocal elasticity.

2.5.2 Nano beam

A functionally graded beam is an inhomogeneous structure consisting of a mixture of isotropic materials. Therefore, for thin beams say \( L/h > 20 \) classical beam theory can be used. Since axially grading leads to a variable coefficient differential equation for vibration of beams, the Ritz method (Aydogdu and Filiz 2011) is used in the present study. Strain energy and kinetic energy of the beam for the Ritz formulation can be defined as below. The strain energy of the beam
where \( w \) and \( I \) are the displacement in the \( z \) direction and moment of inertia of cross-section respectively and \( L \) is the length of the beam. The kinetic energy of the beam

\[
T = \frac{1}{2} \int_0^L \left( m(x) \dot{w}^2 + \mu(x) m(x) \ddot{w}^2 \right) dx
\]

where \( \dot{\text{ }} \) denotes the time derivative. In the Ritz method, displacement component is defined in the following form

\[
w(x, t) = \sin \omega t \sum_{i=0}^I A_i \psi_i(x)
\]

where \( A_i \)'s are the unknown coefficients, \( \omega \) is the angular frequency and \( \psi_i(x) \) is a function which satisfies at least geometric boundary conditions of the beam. Convergence of this function is guaranteed if this function is mathematically complete set. To determine the vibration frequencies of the axially graded beams following functional is defined

\[
F = T_{\text{max}} - U_{\text{max}}
\]

This functional should be minimized with respect to unknown coefficients given in Eq. (22)

\[
\frac{\partial F}{\partial A_i} = 0
\]

This yields a total of \( IX \) simultaneous, linear, homogeneous equations in an equal number of unknowns \( A_i \). Those equations can be described as an eigen-value problem

\[
(K - \Omega^2 M)\Delta = 0
\]

where \( K \) and \( M \) are the stiffness and mass matrix respectively; \( \Omega \) is the dimensionless frequency parameter defined as \( \Omega^4 = (m_0 \omega^2 L^4)/(E_0 I) \) and \( \Omega^2 = (m_0 \omega^2 L^2)/(E_0 A) \) for the nano beam and the nano rod respectively and \( \Delta \) is the column vector of unknown coefficients \( A_i \). Here \( \omega \) (rad/s) is the angular frequency of vibrations. The mode shapes corresponding to any \( \Omega \) is found by substituting that value into Eq. (25) and solving for the eigenvector components \( A_i/A_1 \). Inserting these components into Eq. (22) gives mode shape of nanorod or nano beam. In the present study, mode shape amplitudes are normalized with respect to highest amplitude value of the corresponding mode, therefore the highest amplitude in each mode equals to unity.

After the defining non-dimensional coordinate (\( \xi = x/L \)), the following simple algebraic polynomials are used

\[
w(\xi, t) = \sum_{i=I_0}^M A_i X_i (\xi) \sin \omega t
\]
Vibration of axially functionally graded nano rods and beams with a variable nonlocal parameter

where the polynomial is defined as

\[ X_i = \xi^i (\xi - 1)^B \]  (27)

and \( A_i \)'s are unknown undetermined coefficients. The values of \( B = 0, 1 \) and \( 2 \) correspond to the free, hinged and clamped edge, respectively. It should be noted that rod kinetic and strain energy equations (Eqs. (18)-(19)) should be used in Eq.(23) when solving rod vibration problem.

3. Results and discussions

Investigation on vibration of axially functionally graded nano rods and nano beams with various boundary conditions are carried out. Following boundary conditions are assumed in the present study.

**Nanorod:** Clamped-Clamped (C-C), Clamped-Free (C-F)

**Nanobeam:** Clamped-Free (C-F), Clamped-Simply Supported (C-SS)

In the Ritz method, displacement field should be chosen as satisfy at least the geometric boundary conditions. The kinematic boundary conditions of the nanorod and nanobeam are given in Table 1. It should be noted that free edge boundary conditions are approximately satisfied in this study.

In order to see the effect of nonlocal parameter five different nonlocal parameter values are chosen. For axially functionally graded rods and beams \( \mu = 1 \text{ nm}^2 \) and \( \mu = 2 \text{ nm}^2 \) are assumed at the left and right end of nano structure, respectively. The variations of \( \mu \) parameter are assumed in the following forms.

\[ \mu = \mu_0 (\xi + 1) \] (nonlocal with linear \( \mu \) variation)
\[ \mu = \mu_0 (0.5 \xi^2 + 0.5 \xi + 1) \] (nonlocal with quadratic \( \mu \) variation)

Sample convergence studies are carried out for the non-dimensional frequency parameter \( \Omega \) and

<table>
<thead>
<tr>
<th>Table 1 Kinematic boundary conditions of rods and beams</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Nanorod</strong></td>
</tr>
<tr>
<td>( \xi ) = 0</td>
</tr>
<tr>
<td>C-C</td>
</tr>
<tr>
<td>( u = 0 )</td>
</tr>
<tr>
<td>( \frac{\partial u}{\partial \xi} = 0 )</td>
</tr>
</tbody>
</table>
Table 2 Convergence of non-dimensional frequencies for nano rods and nano beams  
\( (L = 5 \text{ nm}, \mu = (0.5\xi^2 + 0.5\xi+1)) \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Nanorod C-C</th>
<th>Nanorod C-F</th>
<th>Nanobeam C-F</th>
<th>Nanobeam C-S</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Omega_1 )</td>
<td>2.479</td>
<td>3.385</td>
<td>3.965</td>
<td>1.304</td>
</tr>
<tr>
<td>( \Omega_2 )</td>
<td>2.479</td>
<td>3.385</td>
<td>3.668</td>
<td>1.304</td>
</tr>
</tbody>
</table>

Table 3 Convergence of non-dimensional frequencies for nano rods and nano beams  
\( (L = 30 \text{ nm}, \mu = (0.5\xi^2 + 0.5\xi+1)) \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Nanorod C-C</th>
<th>Nanorod C-F</th>
<th>Nanobeam C-F</th>
<th>Nanobeam C-S</th>
</tr>
</thead>
</table>

It is seen that the convergence is better for the lower modes of vibration and long nano rods and beams. The highest difference between 6 and 7 terms is 0.33% for the third mode of C-C nanorod at \( L = 5 \) nm. The non-dimensional frequency parameters obtained with 6 and 7 term are very close to each other. Consequently, all of the remaining results presented have been obtained using 7 terms in Eq. (26).

In Table 4, the present non-dimensional frequency parameters are compared with classical analytical results for axially graded rods and beams for different boundary conditions. The maximum percentage difference is less than 3%. It can be concluded that, there is a good agreement between the results.

After verifying the convergence and accuracy of the present Ritz analysis, the non-dimensional frequency parameters are obtained for axially graded nano rods and nano beams with different material properties, nonlocal parameter and boundary conditions. In Tables 5-8 percentage errors are tabulated in Tables 2 and 3.

Table 4 Comparison of Non-Dimensional Frequencies for linear material property variations  
\( (E = (-\xi + 2), \rho = (-\xi + 2), \mu = 0) \)

<table>
<thead>
<tr>
<th></th>
<th>Nanorod C-C</th>
<th>Nanorod C-F</th>
<th>Nanobeam C-F</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Omega_1 )</td>
<td>Present (Aydogdu and Filiz 2011)</td>
<td>3.123</td>
<td>3.1235</td>
</tr>
<tr>
<td>( \Omega_2 )</td>
<td>Present (Aydogdu and Filiz 2011)</td>
<td>1.794</td>
<td>1.7932</td>
</tr>
<tr>
<td>( \Omega_3 )</td>
<td>Present (Aydogdu and Filiz 2011)</td>
<td>2.077</td>
<td>2.0245</td>
</tr>
<tr>
<td>( \Omega_4 )</td>
<td>Present (Aydogdu and Filiz 2011)</td>
<td>4.849</td>
<td>4.715</td>
</tr>
<tr>
<td>( \Omega_5 )</td>
<td>Present (Aydogdu and Filiz 2011)</td>
<td>7.950</td>
<td>7.738</td>
</tr>
</tbody>
</table>
of non-dimensional frequencies using constant nonlocal parameters are compared with that of variable nonlocal linear and parabolic parameters. Percentage errors are computed according to the following equation

\[
Error \% = \frac{\Omega_{\text{constant } \mu} - \Omega_{\text{variable } \mu}}{\Omega_{\text{variable } \mu}} \times 100
\]  

From these tables, it is seen that non-dimensional frequencies obtained by using \( \mu = 1 \text{ nm}^2 \) are always lower than non-dimensional frequencies obtained by using variable \( \mu \), whereas the reverse is true for \( \mu = 2 \text{ nm}^2 \). \( \mu = 1.5 \text{ nm}^2 \) results are lower than variable case results except some frequencies of C-F nano rods and C-S nano beams. The percentage errors are higher in the rod case when compared with the beam vibration case. Also the errors are increases with increasing the mode number. Highest errors are obtained with more constrained boundary conditions (CC rod). Errors are changing between 0.686% and 24.947% for the rod problem and between 0.339% and 9.225% for the beam problem when \( \mu = 1 \text{ nm}^2 \).

When \( \mu = 2 \text{ nm}^2 \), the errors are between -2.348% and -9.055% for the rod problem and between -0.994% and -6.608% for the beam problem. It is natural to assume the nonlocal parameter \( \mu \) as an average of the end values. So in the present problem average value of the \( \mu \) is 1.5 nm\(^2\). The percentage errors are higher than 5% for some cases and these results suggest to use a variable nonlocal parameter for the axially graded nano rods and nano beams.

In Figs. (3)-(6) the variation of dimensionless frequency parameter with \( \mu_0 \) is given. It is seen that the dimensionless frequency parameters are decreasing with the increasing nonlocal parameter. Higher order frequencies are more sensitive to the nonlocal elasticity. Order of frequencies is obtained as local, non-local (\( \mu = 1 \text{ nm}^2 \)), non-local (quadratic), non-local (linear) and non-local (\( \mu = 2 \text{ nm}^2 \)). The higher order frequencies of the variable nonlocal parameters are much closer to

<table>
<thead>
<tr>
<th>( \mu = 1 \text{ nm}^2 )</th>
<th>( \mu = 1.5 \text{ nm}^2 )</th>
<th>( \mu = 2 \text{ nm}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear error (%)</td>
<td>Quadratic error (%)</td>
<td>Linear error (%)</td>
</tr>
<tr>
<td>( \Omega_1 )</td>
<td>7.576</td>
<td>6.839</td>
</tr>
<tr>
<td>( \Omega_2 )</td>
<td>17.123</td>
<td>15.513</td>
</tr>
<tr>
<td>( \Omega_3 )</td>
<td>24.947</td>
<td>23.874</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \mu = 1 \text{ nm}^2 )</th>
<th>( \mu = 1.5 \text{ nm}^2 )</th>
<th>( \mu = 2 \text{ nm}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear error (%)</td>
<td>Quadratic error (%)</td>
<td>Linear error (%)</td>
</tr>
<tr>
<td>( \Omega_1 )</td>
<td>0.962</td>
<td>0.686</td>
</tr>
<tr>
<td>( \Omega_2 )</td>
<td>11.757</td>
<td>10.014</td>
</tr>
<tr>
<td>( \Omega_3 )</td>
<td>20.694</td>
<td>19.277</td>
</tr>
</tbody>
</table>
Table 7 Percentage frequency errors of constant nonlocal parameters compared to variable nonlocal parameters for C-F nanobeam ($L = 5 \text{ nm}$)

<table>
<thead>
<tr>
<th></th>
<th>$\mu = 1 \text{ nm}^2$</th>
<th>$\mu = 1.5 \text{ nm}^2$</th>
<th>$\mu = 2 \text{ nm}^2$</th>
<th>Linear error (%)</th>
<th>Quadratic error (%)</th>
<th>Linear error (%)</th>
<th>Quadratic error (%)</th>
<th>Linear error (%)</th>
<th>Quadratic error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_1$</td>
<td>2.360</td>
<td>2.077</td>
<td>0.618</td>
<td>0.339</td>
<td>-0.994</td>
<td>-1.268</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Omega_2$</td>
<td>8.688</td>
<td>8.091</td>
<td>2.465</td>
<td>1.902</td>
<td>-2.137</td>
<td>-2.675</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Omega_3$</td>
<td>8.998</td>
<td>7.961</td>
<td>1.295</td>
<td>0.331</td>
<td>-4.181</td>
<td>-5.094</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8 Percentage frequency errors of constant nonlocal parameters compared to variable nonlocal parameters for C-S nanobeam ($L = 5 \text{ nm}$)

<table>
<thead>
<tr>
<th></th>
<th>$\mu = 1 \text{ nm}^2$</th>
<th>$\mu = 1.5 \text{ nm}^2$</th>
<th>$\mu = 2 \text{ nm}^2$</th>
<th>Linear error (%)</th>
<th>Quadratic error (%)</th>
<th>Linear error (%)</th>
<th>Quadratic error (%)</th>
<th>Linear error (%)</th>
<th>Quadratic error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_1$</td>
<td>5.018</td>
<td>4.570</td>
<td>1.208</td>
<td>0.777</td>
<td>-2.004</td>
<td>-2.421</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Omega_2$</td>
<td>7.956</td>
<td>6.864</td>
<td>0.861</td>
<td>-0.158</td>
<td>-4.363</td>
<td>-5.330</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Omega_3$</td>
<td>9.225</td>
<td>7.882</td>
<td>0.577</td>
<td>-0.659</td>
<td>-5.445</td>
<td>-6.608</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\mu = 2 \text{ nm}^2$ curve. The variable nonlocal parameter is more effective for the C-F boundary conditions when compared to the C-C boundary conditions. The quadratic nonlocal parameter variation gives higher results then the linear variation especially for the lower modes of vibration.

Fig. 3 Variation of first three non-dimensional frequency parameter with nonlocal parameter for variable nonlocal parameters for C-C nanorod ($L = 5 \text{ nm}$)
The variation of the non-dimensional frequency parameters with length of nano structure are shown in Figs. 7-10. It is seen that with decreasing nanotube length difference between the classical and the nonlocal model increases. The difference between different $\mu_0$ models is more pronounced for the lower nanotube lengths. The variable $\mu$ predicts frequencies between two
constant values predicted. Frequencies are closer to \( \mu_0 = 2 \text{ nm}^2 \) value especially for the beam problems. Also the difference between the classical elasticity and the nonlocal elasticity is more pronounced for the higher modes of vibration.

Normalized mode shapes equations (highest amplitude equals to unity) of nano-rods and beams obtained from Eq. (25) are given below for various boundary conditions and isotropic and FGM nano rods and nano beams.

**C-C nanorod**
\[ w_{ISO,1} = -3.141\xi - 0.0002\xi^2 + 5.170\xi^3 - 0.022\xi^4 - 2.462\xi^5 - 0.200\xi^6 + 0.875\xi^7 - 0.218\xi^8 \]

\[ w_{ISO,2} = -6.314\xi + 0.756\xi^2 - 32.723\xi^3 + 46.041\xi^4 - 210.184\xi^5 + 191.767\xi^6 - 54.790\xi^7 + 0\xi^8 \]

\[ w_{ISO,3} = -9.586\xi + 7.101\xi^2 - 37.765\xi^3 + 676.543\xi^4 - 3011.278\xi^5 + 4631.704\xi^6 - 3109.667\xi^7 + 777.416\xi^8 \]

\[ w_{FGM,1} = -3.462\xi + 1.449\xi^2 + 3.514\xi^3 - 0.722\xi^4 - 0.812\xi^5 + 0.011\xi^6 - 0.006\xi^7 + 0.028\xi^8 \]

\[ w_{FGM,2} = -4.884\xi + 1.893\xi^2 + 9.963\xi^3 + 25.986\xi^4 - 78.819\xi^5 + 96.576\xi^6 - 73.471\xi^7 + 22.755\xi^8 \]

\[ w_{FGM,3} = -6.926\xi + 81.283\xi^2 - 845.196\xi^3 + 4254.318\xi^4 - 10976.366\xi^5 - 15275.630\xi^6 - 10904.819\xi^7 + 3122.076\xi^8 \]

(30)

where subscript “ISO” denotes the isotropic case whereas “FGM” denotes the functionally graded case and “i” denotes the mode number:

C-F nanorod

\[ w_{ISO,1} = 1.570\xi - 0.000038\xi^2 - 0.645\xi^3 - 0.0013\xi^4 + 0.082\xi^5 - 0.0030\xi^6 - 0.0032\xi^7 \]

\[ w_{ISO,2} = 4.721\xi - 0.270\xi^2 - 14.917\xi^3 - 10.882\xi^4 + 43.998\xi^5 - 29.649\xi^6 + 6\xi^7 \]

\[ w_{ISO,3} = 7.623\xi + 5.537\xi^2 - 119.540\xi^3 + 92.236\xi^4 + 270.585\xi^5 - 406.551\xi^6 + 151.108\xi^7 \]

\[ w_{FGM,1} = 1.973\xi - 0.950\xi^2 + 0.052\xi^3 - 0.185\xi^4 + 0.158\xi^5 - 0.059\xi^6 + 0.011\xi^7 \]

\[ w_{FGM,2} = 4.451\xi - 1.453\xi^2 - 10.579\xi^3 + 3.834\xi^4 - 0.691\xi^5 + 5.747\xi^6 - 2.310\xi^7 \]

\[ w_{FGM,3} = 2.323\xi + 1.858\xi^2 - 29.331\xi^3 + 74.077\xi^4 - 128.509\xi^5 + 121.740\xi^6 - 41.159\xi^7 \]

(31)

Fig. 8 Variation of first three non-dimensional frequency parameter with nano tube length \( L \) for various nonlocal parameters for C-F nanorod
Metin Aydogdu, Mustafa Arda and Seckin Filiz

Fig. 9 Variation of first three non-dimensional frequency parameter with nano tube length $L$ for various nonlocal parameters for C-F nanobeam

Fig. 10 Variation of first three non-dimensional frequency parameter with nano tube length $L$ for various nonlocal parameters for C-S nanobeam

C-F nanobeam

\[
\begin{align*}
\psi_{IS,1} &= 1.614\xi^2 - 0.573\xi^3 - 0.095\xi^4 + 0.018\xi^5 + 0.045\xi^6 - 0.009\xi^7 + 0.0002\xi^8 \\
\psi_{IS,2} &= 10.589\xi^2 - 10.630\xi^3 - 8.910\xi^4 + 2.530\xi^5 + 14.493\xi^6 - 11.110\xi^7 + 2.334\xi^8 \\
\psi_{IS,3} &= 22.556\xi^2 - 11.618\xi^3 - 162.909\xi^4 + 240.779\xi^5 - 0.189\xi^6 - 150.696\xi^7 + 62.514\xi^8 \\
\psi_{FG,1} &= 1.946\xi^2 - 1.372\xi^3 + 0.633\xi^4 - 0.350\xi^5 + 0.199\xi^6 - 0.068\xi^7 + 0.010\xi^8 \\
\psi_{FG,2} &= 12.740\xi^2 - 20.186\xi^3 + 4.535\xi^4 - 1.279\xi^5 + 8.305\xi^6 - 5.840\xi^7 + 1.123\xi^8
\end{align*}
\] (32)
Vibration of axially functionally graded nano rods and beams with a variable nonlocal parameter

\[ w_{FGM,3} = 25.664\xi^2 - 42.702\xi^3 - 55.419\xi^4 + 81.520\xi^5 + 100.678\xi^6 - 167.920\xi^7 + 58.441\xi^8 \]  

\[ w_{ISO,1} = -8.868\xi^2 + 8.035\xi^3 + 7.109\xi^4 - 3.225\xi^5 - 7.093\xi^6 + 4.235\xi^7 + 0.156\xi^8 - 0.349\xi^9 \]

C-S nanobeam

Fig. 11 First three mode shapes of C-C isotropic and axially functionally graded nanorods

Fig. 12 First three mode shapes of C-F isotropic and axially functionally graded nanorods
Fig. 13 First three mode shapes of C-F isotropic and axially functionally graded nanobeams

Fig. 14 First three mode shapes of C-S isotropic and axially functionally graded nanobeams

\[
\begin{align*}
w_{ISO,2} &= -23.639\xi^2 + 29.713\xi^3 + 45.470\xi^4 + 141.851\xi^5 - 684.748\xi^6 \\
&+ 843.902\xi^7 - 435.520\xi^8 + 82.969\xi^9 \\

w_{ISO,3} &= -40.325\xi^2 - 13.487\xi^3 + 814.222\xi^4 - 1550.392\xi^5 - 423.082\xi^6 \\
&+ 3420.276\xi^7 - 3085.544\xi^8 + 878.332\xi^9 \\

w_{FGM,1} &= -10.969\xi^2 + 16.359\xi^3 - 3.664\xi^4 + 0.637\xi^5 - 5.002\xi^6 + 2.897\xi^7 \\
&- 0.124\xi^8 - 0.134\xi^9
\end{align*}
\]
Vibration of axially functionally graded nano rods and beams with a variable nonlocal parameter

\[ w_{FGM,2} = -26.338\xi^2 + 51.458\xi^3 + 3.508\xi^4 + 83.137\xi^5 - 391.421\xi^6 \\
+ 459.133\xi^7 - 213.771\xi^8 + 34.292\xi^9 \\
w_{FGM,3} = -44.591\xi^2 + 87.371\xi^3 + 152.017\xi^4 + 305.968\xi^5 - 3008.848\xi^6 \\
+ 5251.263\xi^7 - 3677.971\xi^8 + 934.790\xi^9 \] (33)

Some typical mode shapes are shown in Figs. 11-14 for different boundary conditions and isotropic and quadratically graded nanorods and nanobeams. It is seen that the first mode shapes are approximately identical. The difference between the mode shapes of isotropic and axially FG rods and beams are more apparent for the higher modes and at the right end of the nanorod or nano beam. All the materials properties, young modulus, density and nonlocal parameter increase to the right end of the structure and this makes the structure more flexible at the right end. Considerable changes are observed for the nodes of the mode shapes. This gives an opportunity to the designer to tailor the mode shapes of the nano structures.

4. Conclusions

In the present study, the vibration of axially functionally graded nanorods and nanobeams was presented. The material properties are graded in the axial direction of the rod and beam linearly. The Euler-Bernoulli beam theory and Ritz method are employed for the analysis. The dimensionless frequency parameters are obtained for various material properties. The following main conclusions can be drawn from the present analysis.

- The percentage errors of using a constant nonlocal parameter leads to errors higher than 5% for some cases and these results suggest using a variable nonlocal parameter for the axially graded nano rods and nano beams. Errors are more apparent for the higher modes of vibration.
- Mode shapes of the axially functionally graded nanorods and nano beams are different from the isotropic nanorods and nanobeams and this may give various design opportunities.
- The Ritz method can be easily used in the formulation of axially functionally graded nanorods and beams with the variable nonlocal parameter.

One could easily vary the frequency parameters of the FG beam by varying the volume fraction of the constituents. Present study can be extended to the statics and dynamics analysis of nano plates or nano shells.

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