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# Study of viscoelastic model for harmonic waves in non-homogeneous viscoelastic filaments

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**Abstract.** A five parameter viscoelastic model is developed to study harmonic waves propagating in the non-homogeneous viscoelastic filaments of varying density. The constitutive relation for five parameter model is first developed and then it is applied for harmonic waves in the specimen. In this study, it is assumed that density, rigidity and viscosity of the specimen i.e., rod are space dependent. The specimen is non-homogeneous, initially unstressed and at rest. The method of non-linear partial differential equation has been used for finding the dispersion equation of harmonic waves in the rods. A simple method is presented for reflections at the free end of the finite non-homogeneous viscoelastic rods. The harmonic wave propagation in viscoelastic rod is also presented numerically with MATLAB.

**Keywords:** harmonic waves; viscoelastic media; friedlander series; inhomogeneous; varying density

## 1. Introduction

The theory of viscoelasticity is useful in the field of solid mechanics, engineering, seismology, exploration and geophysics. In harmonic waves, the particles vibrated perpendicular to the direction of propagation of the wave. Many literatures are available on wave propagation in homogeneous media. But in recent years sufficient interest has arisen towards non-homogeneous bodies.

Many researchers Alfrey (1944), Barberan (1966), Achenbach (1967), Bhattacharya (1978) and Acharya (2008) formulated and developed the theory of elasticity. Bert (1969), Biot (1940), Batra (1998) successfully applied this theory to wave-propagation in homogeneous elastic media. On the basis of the theory of elasticity, the propagation of harmonic waves in isotropic or anisotropic materials has been evaluated numerically by White (1981), Mirsky (1965) and Tsai (1991). Murayama (1981) and Schiffman *et al*. (1964) have proposed higher order five parameter and seven parameter viscoelastic models for soil behavior. Lei *et al*. (2008) discussed the response of a rocksalt crystal to electromagnetic wave modeled by a multiscale field theory. Kumar (2010) analyzed wave motion in micro-polar transversely isotropic thermoelastic half space without energy dissipation. Gurdarshan (1980), Kakar *et al*. (2012) analyzed five parameter models under dynamic loading and Kaur *et al*. (2012, 2013) purposed four parameter non-homogeneous

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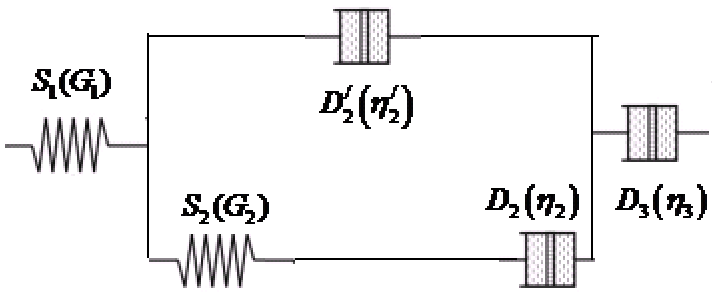


Fig. 1 Five parameter viscoelastic model

viscoelastic models for longitudinal wave propagation and three parameter non-linear viscoelastic model. Ponnusamy (2012) has given a view on wave propagation in a generalized thermo elastic plate embedded in elastic medium. Recently, Kakar and Kaur (2013) have studied the response of non-homogeneous five parameter viscoelastic model for cylindrical shear waves.

We have considered the harmonic wave propagation in non-homogeneous media, when density '$\rho$' rigidity '$G$' and viscosity '$\eta$' of the material are space dependent such that the harmonic wave velocity is also space dependent. Here it is assumed that density, rigidity and viscosity of the rods obey the law $\rho = \rho_0(1+\cos\alpha_1 x), G = G_0(1+\cos\alpha_2 x), \eta = \eta_0(1+\cos\alpha_3 x)$ respectively. The wave equation is first approximated by using WKB theory, and then the problem is solved with Eikonal equation. The displacements are taken so small such that the linear constitutive laws hold under isothermal conditions. The paper ends with numerical analysis by taking material parameters.

## 2. Five parameter viscoelastic model

We have considered a five parameter model consists of two springs $S_1(G_1), S_2(G_2)$ where $G_1$ and $G_2$ are the modulli of elasticity associated to them and three dash-pots $D_2(\eta_2), D_{2'}(\eta_{2'}), D_3(\eta_3)$ where $\eta_2$, $\eta_{2'}$ and $\eta_3$ are the Newtonian viscosity coefficients associates to these elements. The module of elasticity and viscosity coefficients are assumed to be space dependent i.e., functions of '$x$' in inhomogeneous case taken into consideration. Unidirectional problem is formed by taking the material in the form of filament of non-homogeneous viscoelastic material by taking one end at $x$ = 0. The co-ordinate $x$ is measured positive in the direction of the axis of the filament. Time is specified by $t$, and $\sigma, \gamma$ and $u$ respectively specify the only non-zero components of stress, shearing strain and particle displacement. The model has be divided into three Sections, I, II and III. In Fig. 1, the Section I, Section II and Section III has one spring $S_1(G_1)$, two dash-pots $D_2(\eta_2), D_{2'}(\eta_{2'})$ one spring $S_2(G_2)$ and one dash-pot $D_3(\eta_3)$ respectively.

Under the supper- supposition principle strains are added in the case of series connections and stresses are added when they are in parallel. Now if $a_1$, $a_2$, $a_3$ be the three shearing strains elongations in respective sections connected in series, then total elongation is $a = a_1 + a_2 + a_3$. The total stress in the network remains the same. In each section but in the case of section II which is sub-divided into two sections is added i.e., $\sigma = \sigma_1 + \sigma_2$, where $\sigma_1$ and $\sigma_2$ are the stresses in the sub-sections. Relation for stress and strain for $D_{2'}(\eta_{2'})$ for Section II (represented by single dash-pot) is

$$\sigma_1 = \eta_{2'} a_2 \tag{1}$$

Since the sub-section II is represented by a Maxwell- element, then the relation is expressed as

$$\left(\frac{D}{G_2} + \frac{1}{\eta_2}\right)\sigma_2 = D(a_2) \tag{2}$$

Since, $\sigma = \sigma_1 + \sigma_2$ for Section II, therefore

$$\left(D + \frac{G_2}{\eta_2}\right)_\sigma = \eta_2\left[D^2 + G_2\left(\frac{1}{\eta_2'} + \frac{1}{\eta_2}\right)D\right]a_2 \tag{3}$$

For Section I, for the spring $S_1(G_1)$, the stress-strain relation is given by

$$\sigma = G_1 a_1 \tag{4}$$

For Section III; for the dash-pot $D_3(\eta_3)$, the stress –strain relation is given by

$$\sigma = \eta_3 \dot{a}_3 \tag{5}$$

The Stress-strain relation for the model representing the viscoelastic body for total stress ($\sigma$) and strain $(a)$ can be obtained from Eqs. (3)-(5) as

$$\left[D^2 + \left\{\frac{G_1}{\eta_{2'}} + \frac{G_1}{\eta_3} + \frac{G_2}{\eta_{2'}} + \frac{G_2}{\eta_2}\right\}D + \left\{\frac{G_1}{\eta_{2'}}\frac{G_2}{\eta_2} + \frac{G_1}{\eta_3}\left(\frac{G_2}{\eta_2} + \frac{G_2}{\eta_2}\right)\right\}\right]\sigma = G_1\left\{D^2 + \left(\frac{G_2}{\eta_{2'}} + \frac{G_2}{\eta_2}\right)D\right\}a \tag{6}$$

Now we take

$$\tau_{ij}^{-1} = \theta_{ij} = \frac{G_i}{\eta_j} = \frac{S_i(G_i)}{Dj(\eta_j)} \tag{7}$$

Where, $S_i(G_i)$ elastic modulus of spring and $D_j(\eta_j)$= viscosity of dash-pot,

$$\tau_{ij} = \frac{\eta_j}{G_i}, (i = 1, 2;\ j = 2, 2, 3) \tag{8}$$

Using, Eqs. (6) and (7), we get

$$\left[D^2 + \{(\theta_{12'} + \theta_{13}) + (\theta_{22} + \theta_{22'})\}D + \{\theta_{12'}\theta_{22} + \theta_{13}(\theta_{22} + \theta_{22'})\}\right]\sigma = G_1\{D^2 + (\theta_{22} + \theta_{22'})D\}a \tag{9}$$

Put R$_1$ = $\theta_{12'} + \theta_{13}$, R$_2$ = $\theta_{22} + \theta_{22'}$, R$_3$ = $\theta_{12'}.\theta_{22}$, R$_4$ = $\theta_{13}$ R$_2$ in Eq. (8), we get

$$\left[D^2 + (R_1 + R_2)D + (R_3 + R_4)\right]\sigma = G_1\left\{D^2 + R_2 D\right\}a = (G_1 D^2 + G_1 R_2 D)a \tag{10}$$

The Eq. (10) can be written in terms of differential operator form as

$$\sum_{n=1}^{2} \alpha_n D^n a(x,t) = \sum_{m=0}^{2} \beta_m D^m \sigma(x,t) \tag{11}$$

Where, the order *m* and *n* of sums on right hand side and left hand side in the Eq. (11) depends upon the structure of the mechanical model representing the viscoelastic body. $\alpha_n$ and $\beta_m$ are the combinations of the material constants and $\alpha_2 = G_1, \alpha_1 = G_1 R_{2,}\ \beta_2 = 1, \beta_1 = R_1 + R_{2,}$ $\beta_0 = R_3 + R_4, D \equiv \frac{d}{dt}$.

Eq. (11) is the required differential operator form of constitutive relation for the model for viscoelastic material

## 3. Formulation of the problem

We have considered the propagation of waves along the five parameter viscoelastic model and accordingly it as the uniaxial complex modulus $G^*(i\omega)$ that involved the mechanical property. Let the filament is very long and its one end is subjected to a steady state harmonic oscillation condition, then harmonic waves are propagated along the filament with a reduction in the direction of propagation.

The equation of motion is

$$\frac{\partial^2 \overline{u}}{\partial x^2} = \frac{-\rho\omega^2 \overline{u}}{G^*(i\omega)} \tag{12}$$

where, $\overline{u}(x,\omega)$ is Fourier transformed displacement.

The uniaxial complex modulus $G^*(i\omega)$ is

$$\left[G^*(i\omega)\right]^{\frac{1}{2}} = G_3(\omega) + iG_4(\omega) \tag{13}$$

Let

$$\frac{1}{c} = \frac{\rho^{\frac{1}{2}} G_3}{G_3^2 + G_4^2} \tag{14}$$

where, $c$ is phase velocity of the harmonic waves.

Let reduction is

$$\zeta = \frac{\rho^{\frac{1}{2}} \omega G_4}{G_3^2 + G_4^2} \tag{15}$$

where, $\zeta$ is reduction

The solution of Eq. (12) is of the form

$$u(x,t) = Pe^{-\frac{\rho^{\frac{1}{2}}\omega G_4}{\left(G_3^2+G_4^2\right)}x} e^{i\omega\left(t-\frac{\rho^{\frac{1}{2}}\omega G_3}{\left(G_3^2+G_4^2\right)}x\right)} \tag{16}$$

It can also be written as

$$u(x,t) = Pe^{-\zeta x} e^{i\omega\left(t-\frac{x}{c}\right)} \tag{17}$$

Thus the solution (17) represent s the propagation of a harmonic wave moving in the positive $x$ direction with phase velocity $c$ and reduction $\zeta$ and the complex constant $P$ is to be determined from boundary conditions. This method has widely used to determine high frequency properties, not only with regard of $G^*(i\omega)$ but for stress states as well.

Considering the propagation of transient disturbance along the filament and using the Fourier integral of synthesize the velocity from the solution given in Eq. (17). The resulting velocity is given by

$$\dot{u}(x,t) = \int_{-\infty}^{\infty} F(\omega) e^{-\zeta x} e^{i\omega\left(t-\frac{x}{c}\right)} d\omega \tag{18}$$

here, $F(\omega)$ is a complex function of frequency to be determined from specified end conditions

Therefore we can rewrite the expression for phase velocity c and reduction $\zeta$ in Eqs. (14) and (15) as

$$c = \left(\left|G^*\right|^{\frac{1}{2}} \rho^{\frac{1}{2}}\right) \sec\left(\frac{\delta}{2}\right) \tag{19}$$

$$\zeta = c^{-1}\left\{\omega \tan\left(\frac{\delta}{2}\right)\right\} \tag{20}$$

To give proper dispersion and attenuation of a pulse as it passes from one point of measurement to the other particularly, let

$$\dot{u}(x,t)\big|_{x=x_1} = v_1(t) \tag{21}$$

and

$$\dot{u}(x,t)\big|_{x=x_2} = v_2(t) \tag{22}$$

From Eq. (21), Eq. (22) and the Fourier transform inversion formula, we have

$$F(\omega) = e^{[\alpha + (i\omega/c)]x_1} \overline{v}_1(\omega) \tag{23}$$

Where,

$$\overline{v}_1(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v_1(t) e^{-i\omega t} dt \tag{24}$$

Using Eq. (23) in Eq. (18), and evaluating the result at $x = x_2$, using Eq. (22), we get

$$\int_{-\infty}^{\infty} \overline{v}_1(\omega) e^{-\alpha(x_2 - x_1)} e^{i\omega[t - (x_2 - x_1)/c]} d\omega = v_2(t) \tag{25}$$

It is to be assumed that the duration of the disturbing pulse is very short that can be represented by a Fourier series rather than the Fourier integral and it would be expected an appropriate procedure for the cases in which very short duration pulses are started by a dispersion effects have broadened the pulse too much. Therefore, a basic time interval is taken which includes the main part of the plus, and in this interval pulse is represented by a Fourier series.

Let $\omega = np$ , where $p$ is the preselected basic frequency.

Eq. (18) can be written as

$$\dot{u}(x,t) = \int_{-\infty}^{\infty} F(\omega) e^{-\zeta x} \left\{ \cos\omega\left(t - \frac{x}{c}\right) + i \sin\omega\left(t - \frac{x}{c}\right) \right\} d\omega$$

Using the formula for sine and cosine of the difference of two angles, the Fourier series solution corresponding to the Fourier integral Solution given by Eq. (18) is of form

$$\dot{u}(x,t) = \sum_{n=0}^{\infty} e^{-\zeta_n x [A_n \cos(npx/c_n) - B_n \sin(npx/c_n)] \cos npt} + \sum_{n=0}^{\infty} e^{-\zeta_n x [A_n \cos(npx/c_n) - B_n \sin(npx/c_n)] \cos npt} \tag{26}$$

Where $A_n$ and $B_n$ are unknown constants.

For the sake of convenience, let

$$t - \frac{x}{c} = \left(t - \frac{x}{v}\right) + \left(-\frac{x}{c} + \frac{x}{v}\right). \tag{27}$$

In Eq. (26), replacing $t$ by $t - \dfrac{x}{v}$ and $\dfrac{x}{c}$ by $\dfrac{x}{c} - \dfrac{x}{v}$ , we get

$$\dot{u}(x,t) = \sum_{n=0}^{\infty} C_n \cos npt' + \sum_{n=1}^{\infty} D_n \sin npt' \tag{28}$$

Where

$$t' = t - \left(\frac{x}{v}\right) \tag{29}$$

$$C_n = e^{-\zeta_n x}\left[A_n \cos\left\{npx(c_n^{-1} - v_n^{-1}\right\} - B_n \sin\left\{npx(c_n^{-1} - v_n^{-1}\right\}\right] \tag{30}$$

and

$$D_n = e^{-\zeta_n x}\left[A_n \sin\left\{npx(c_n^{-1} - v_n^{-1}\right\} + B_n \cos\left\{npx(c_n^{-1} - v_n^{-1}\right\}\right] \tag{31}$$

The velocity $v$ should be taken to be near an average value of propagation of the pulse so that the plus will maintained within the basic interval of the Fourier series representation. In the Fourier integral representation, the velocity is measured at two locations, $x_1$ and $x_2$, as given by Eqs. (21) and (22). To evaluate Eq. (28) at $x = x_1$, using Eq. (21), and in the usual way the coefficients are given by

$$C_{n1} = \frac{p}{\pi}\int_{-\pi/p}^{\pi/p} v_1\left(t'\right)\cos npt' dt', \qquad n = 1, 2, \ldots\ldots \tag{32}$$

$$D_{n1} = \frac{p}{\pi}\int_{-\pi/p}^{\pi/p} v_1\left(t'\right)\sin npt' dt', \qquad n = 1, 2, \ldots\ldots \tag{33}$$

The coefficients $A_n$ and $B_n$ are obtained from Eqs. (30) and (31) as

$$A_n = e^{\zeta_n x_1}\left[C_{n1} \cos\left\{npx_1(c_n^{-1} - v_n^{-1}\right\} + D_{n1} \sin\left\{npx_1(c_n^{-1} - v_n^{-1}\right\}\right] \tag{34}$$

and

$$B_n = e^{\zeta_n x_1}\left[-C_{n1} \sin\left\{npx_1(c_n^{-1} - v_n^{-1}\right\} + D_{n1} \cos\left\{npx_1(c_n^{-1} - v_n^{-1}\right\}\right] \tag{35}$$

Thus Eqs. (28)-(31) specify the response with $A_n$ and $B_n$ given by Eqs. (32)-(35).

At $x = x_2$, using Eq. (22) to synthesize the plus into Fourier series

$$\dot{u}(x,t)\big|_{x=x_2} = \sum_{n=0}^{\infty} P_n \cos npt' + \sum_{n=1}^{\infty} Q_n \sin npt' \tag{36}$$

where,

$$P_n = \frac{p}{\pi}\int_{-\pi/p}^{\pi/p} v_2\left(t'\right)\cos npt' dt', \qquad n = 1,2,\ldots\ldots \tag{37}$$

$$Q_n = \frac{p}{\pi}\int_{-\pi/p}^{\pi/p} v_2\left(t'\right)\sin npt' dt', \qquad n = 1,2,\ldots\ldots \tag{38}$$

Now, evaluate the solution of Eq. (28) at $x = x_2$ and equate the result to Eq. (36). Using Eqs (30)-(35), (37) and (38), the term by term equivalence of these two forms gives the two general relations

$$\begin{bmatrix} \cos\left\{npx_2(c_n^{-1} - v_n^{-1}\right\}\sin\left\{npx_1(c_n^{-1} - v_n^{-1}\right\} \\ -\sin\left\{npx_2(c_n^{-1} - v_n^{-1}\right\}\cos\left\{npx_1(c_n^{-1} - v_n^{-1}\right\} \end{bmatrix} \int_{-\pi/p}^{\pi/p} v_1\left(t'\right)\sin npt'dt'$$
$$+\begin{bmatrix} \cos\left\{npx_2(c_n^{-1} - v_n^{-1}\right\}\cos\left\{npx_1(c_n^{-1} - v_n^{-1}\right\} \\ -\sin\left\{npx_2(c_n^{-1} - v_n^{-1}\right\}\sin\left\{npx_1(c_n^{-1} - v_n^{-1}\right\} \end{bmatrix} \int_{-\pi/p}^{\pi/p} v_1\left(t'\right)\sin npt'dt' = e^{-\zeta_n(x_1-x_2)}\int_{-\pi/p}^{\pi/p} v_2\left(t'\right)\cos npt'dt',$$

Where $n$ =1, 2,……. (39)

and

$$\begin{bmatrix} \sin\left\{npx_2(c_n^{-1} - v_n^{-1}\right\}\sin\left\{npx_1(c_n^{-1} - v_n^{-1}\right\} \\ +\cos\left\{npx_2(c_n^{-1} - v_n^{-1}\right\}\cos\left\{npx_1(c_n^{-1} - v_n^{-1}\right\} \end{bmatrix} \int_{-\pi/p}^{\pi/p} v_1\left(t'\right)\sin npt'dt'$$
$$+\begin{bmatrix} \sin\left\{npx_2(c_n^{-1} - v_n^{-1}\right\}\cos\left\{npx_1(c_n^{-1} - v_n^{-1}\right\} \\ -\cos\left\{npx_2(c_n^{-1} - v_n^{-1}\right\}\sin\left\{npx_1(c_n^{-1} - v_n^{-1}\right\} \end{bmatrix} \int_{-\pi/p}^{\pi/p} v_1\left(t'\right)\cos npt'dt' = e^{-\zeta_n(x_1-x_2)}\int_{-\pi/p}^{\pi/p} v_2\left(t'\right)\sin npt'dt',$$

Where $n$ =1, 2,……. (40)

Considering with $v_1\left(t'\right)$ and $v_2\left(t'\right)$, the particle velocities at two different locations on the filament are known from experimental measurements, Eqs. (39) and (40) include two equations in two unknowns, $c_n$ and $\zeta_n$, for each value of $n$. Then using Eqs. (14) and (15), we get

$$G_{3n} = \rho^{\frac{1}{2}} c_n \left[\left(\zeta_n^2 c_n^2 / \omega_n^2\right) + 1\right]^{-1} \tag{41}$$

and

$$G_{4n} = \rho^{\frac{1}{2}} \zeta_n c_n^2 \omega_n^{-1} \left[\left(\zeta_n^2 c_n^2 / \omega_n^2\right) + 1\right]^{-1} \tag{42}$$

Where $\omega_n = np$ , $G_{3n}$ and $G_{4n}$ are values of $G_3$ and $G_4$ right to the $n^{\text{th}}$ frequency component. The real and imaginary parts of the complex modulus, $G^*(i\omega)$ may be then found directly from Eq. (28) as

$$G_n' = \left(G_{3n}\right)^2 - \left(G_{4n}\right)^2 \tag{43}$$

and

$$G_n'' = 2G_{3n}G_{4n} \tag{44}$$

The described procedure for determining $G^*(i\omega)$ in the high frequency range has not been explicitly used. The simplest assumptions are to be made taking particular forms for phase velocity and reduction. The simplest assumption is

$$c = \text{ Constant} \tag{45}$$

and

$$\tan\delta = \frac{G^*}{G'} = \text{Constant} \tag{46}$$

The reduction can be found from Eq. (20). For this, the frequency range of relevance is narrow enough, then $G^*(i\omega)$ is effectively constant in this region and therefore $c$ and $\tan\delta$ can then be found from a slight modification of the described procedure.

### *3.1 Wave propagation for the non-homogeneous cases*

The stress strain relation for five parameter viscoelastic model is given by the Eq. (6).
The equation of motion and strain-displacement relations are given by

$$\frac{\partial\sigma}{\partial x} = \rho\frac{\partial^2 U}{\partial t^2} \tag{47}$$

$$a = \frac{\partial U}{\partial x} \tag{48}$$

where, $\rho = \rho(x)$ is the variable density of the material
Differentiating Eq. (47) w.r.t. $x$, we get

$$-\frac{1}{\rho^2}\frac{\partial\rho}{\partial x}\frac{\partial\sigma}{\partial x} + \frac{1}{\rho}\frac{\partial^2\sigma}{\partial x^2} = u_{,xtt} \tag{49}$$

Differentiating Eq. (48) w.r.t. $t$, we get

$$\frac{\partial a}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) \tag{50}$$

Again differentiating w.r.t. $t$,

$$\frac{\partial^2 a}{\partial t^2} = \frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\right\} = u_{,ttx} = u_{,xtt} \tag{51}$$

Using Eq. (64) and Eq. (66), Eq. (8) gives

$$\sigma_{,ttt} + \left\{\frac{G_1}{\eta_2} + \frac{G_1}{\eta_3} + \frac{G_2}{\eta_2} + \frac{G_2}{\eta_2}\right\}\sigma_{,tt} + \left\{\frac{G_1}{\eta_2}\frac{G_2}{\eta_2} + \frac{G_1}{\eta_3}\left(\frac{G_2}{\eta_2} + \frac{G_2}{\eta_2}\right)\right\}\sigma_{,t} = \frac{G_1}{\rho}\left\{\frac{\partial}{\partial t} + \frac{G_2}{\eta_2}\left(1 + \frac{\eta_2}{\eta_2}\right)\right\}\left\{\frac{\partial}{\partial x} - \frac{\rho'}{\rho}\right\}\frac{\partial\sigma}{\partial x} \tag{52}$$

## 4. Solution of the problem

Let the solution $\sigma(x,t)$ of Eq. (52) may be represented by the series (Friedlander (1947))

$$\sigma(x,t) = \sum_{n=0}^{\infty} A_n(x) F_n\{t - h(x)\}, \qquad A_0 \neq 0 \tag{53}$$

where,

$$F'_n = F_{n-1} \quad \text{(where, } n = 1, 2, 3\ldots\ldots\ldots\ldots\text{) with } F_{n,t} = F_{n-1} \text{ and } F_{n,x} = -h_{,x} F_{n-1} \tag{54}$$

and for $n < 0$ assume that $A_n = 0$ and the derivatives of σ may be obtained by term-wise differentiation of Eq. (53), the prime in Eq. (54) denotes differentiation with respect to the argument concerned, and by using Eq. (54) and Eq. (11) we relate all $F_n^{'}$s to $F_0$ by successive integrations.

The Solution of equation Eq. (52) in the form of Eq. (53) can be obtained by taking a phase function $h(x)$ , $h(x)$ satisfies the Eikonal equation of geometrical optics [20]

$$\left(\frac{dh(x)}{dx}\right)^2 = \frac{\rho}{G_1} = \frac{1}{c^2} \tag{55}$$

where $c = c(x)$ is the variable wave speed for viscoelastic harmonic waves in a medium whose modulus of elasticity $G_1$. Using, Eq. (10), Eq. (11) and the successive derivatives of $\sigma(x,t)$ *w.r.t.* *'t'* and *'x'* in equation Eq. (52), we get the amplitude function satisfy the equation

$$2h'(x)A_n'(x) + \left\{\rho\left(\frac{1}{\eta_{2'}} + \frac{1}{\eta_3}\right) - (\log\rho)_{,x} h'(x) + h''(x)\right\} A_n(x) = Q_n, (n = 0, 1, 2...) \tag{56}$$

where,

$$Q_n = A''_{n-1} - \left\{(log\rho)_{,x} + 2\frac{G_2}{\eta_{2'}}\left(1 + \frac{\eta_2^{'}}{\eta_2}\right)h'\right\}A'_{n-1} +$$

$$\left\{\rho\left(\frac{G_2}{\eta_2\eta_2^{'}} + \frac{G_2}{\eta_2^{'}\eta_3}\left(1 + \frac{\eta_2^{'}}{\eta_2}\right)\right) + \frac{G_2}{\eta_2^{'}}\left(1 + \frac{\eta_2^{'}}{\eta_2}\right)h^{''} - \frac{G_2}{\eta_2^{'}}\left(1 + \frac{\eta_2^{'}}{\eta_2}\right)(log\rho)_{,x} h'\right\}A_{n-1} + \frac{G_2}{\eta_2^{'}}\left(1 + \frac{\eta_2^{'}}{\eta_2}\right)A_{n-2}^{''}$$

$$-\frac{G_2}{\eta_2^{'}}\left(1 + \frac{\eta_2^{'}}{\eta_2}\right)(log\rho)_{,x} A_{n-2}^{'} + \frac{G_2}{\eta_2}A''_{n-2} - \frac{G_2}{\eta_2}(log\rho)_{,x} A'_{n-2}$$

Since the wave is travelling along x-axis, therefore, integrating Eq. (55), we get

$$h(x) = h(0) \pm \int_0^x \frac{ds}{c(s)}$$

Since Eq. (56) is a first order linear differential equation in $A_n(x)$. Therefore the general solution of Eq. (56) can be obtained as

$$A_n(x) = A_n(0)\left\{\frac{l(x)}{l(0)}\right\}^{\frac{1}{2}} exp\left\{\mp\int_0^x m(s)\,ds\right\} \pm \frac{1}{2}\int_0^x c(s)\left\{\frac{l(x)}{l(s)}\right\}^{\frac{1}{2}} exp\left\{\pm\int_x^z m(z)\,dz\right\} \text{잛}_n^{\pm}(s)\,ds$$

$(n = 0,1,2\ldots)$ (57)

where, $l = (x) = \rho c$ and $m(x) = \dfrac{\rho c}{2}\left(\dfrac{1}{\eta_1} + \dfrac{1}{\eta_2}\right)$.

The plus $'+'$ sign is associated with wave traveling in the positive direction of *x* and the minus $'-'$ sign is associated with the waves travelling in the negative direction of *x*.

Let an impulse of magnitude $\sigma_{0?}$ suddenly applied at the end $x = 0$ of the rod and thereafter steadily maintained, that is

$$\sigma(0,t) = \sigma_0 H(t) \tag{58}$$

From Eq. (53) and Eq. (58), we have

$$\sum_{n=0}^{\infty} A_n(0) F_n\{t - h(0)\} = \sigma_0 H(t) \tag{59}$$

Thus we choose (Moodie (1973)

$$A_n(0) = \begin{vmatrix} \sigma_0 \cdots\cdots\cdots if\ n = 0 \\ 0 \cdots\cdots\cdots if\ n < 0\ or\ n > 0 \end{vmatrix} \tag{60}$$

$$h(0) = 0‘ \text{ and } F_0 = H(t)$$

For the waves travelling in the positive direction of *x*, solution of Eq. (52) is generated by boundary stress Eq. (58) as

$$\sigma(x,t) = \sum_{n=0}^{\infty} A_n(x) \frac{\{t - h(x)\}^n}{n!} H\{t - h(x)\} = \sum_{n=0}^{\infty} A_n(x) \frac{\{t - h(x)\}^n}{n!} H\left\{t - \int_0^x \frac{ds}{c(s)}?\right\} \tag{61}$$

Where,

$$h(x) = \int_0^x \frac{ds}{c(s)}$$

where, $A_n(x)$ are given recursively by Eq. (57) (with upper signs) in combination with Eq. (60).

The first–term approximation leads to Eq. (57) is

$$\sigma(x,t) = \sigma_0 \left\{ \frac{l(x)}{l(0)} \right\}^{\frac{1}{2}} \exp\left\{ -\int_0^x m(s)ds \right\} H\left\{ t - \int_0^x \frac{ds}{c(s)} \right\} \tag{62}$$

The Eq. (62) represents a transient stress wave which starts from the end '$x$=0' with amplitude '$\sigma_0$' and moves in the positive direction of '$x$' with velocity $c(x)$. Hence, it is modulated by the factor

$$\left\{ \frac{l(x)}{l(0)} \right\}^{\frac{1}{2}} exp\left\{ -\int_0^x m(s)\, ds \right\} \tag{63}$$

Further terms in the approximate solution may be obtained recursively from Eq. (57). The solution of Eq. (59) applies until the wave moving in the positive direction of '$x$' strikes either an interface (in the case of a composite rod) or at end (in the case of a finite rod).

## 5. Application of the problem

For the sake of concreteness and for studying the qualitative effect of non-homogeneity on the harmonic wave propagation in non-homogeneous five parameter viscoelastic rods, it is assumed that density '$\rho$', rigidity '$G$' and viscosity '$\eta$' of the specimen i.e., rod are space dependent and obey the laws

$$\rho = \rho_0(1+\cos\alpha_1 x), G = (G_0(1+\cos\alpha_2 x), \eta = \eta_0(1+\cos\alpha_3 x) \tag{64}$$

### *5.1 Wave propagation for the semi-homogeneous case*

In this case, we take $\alpha_1 = \alpha_2 = \alpha_3$, then from Eq. (63), we get

$$\rho = \rho_0(1+\cos\alpha x), G = G_0(1+\cos\alpha x), \eta = \eta_0(1+\cos\alpha x) \tag{65}$$

Therefore, from Eikonal equation of geometric optics

$$\left( \frac{dh(x)}{dx} \right)^2 = \frac{\rho}{G_1} = \frac{\rho_0(1+\cos\alpha x)}{G_{01}(1+\cos\alpha x)} = \frac{\rho_0}{G_{01}} = \frac{1}{c_0^{\,2}}$$

$$\Rightarrow c_0 = \sqrt{\frac{G_{01}}{\rho_0}} \tag{66}$$

Since, the exponential variation of modulus of rigidity $G$ and density $\rho$ is similar, therefore sound speed is constant i.e., non-homogeneous has no effect on speed and phase of the wave is given $h(x) = \frac{x}{c_0}$. So it becomes the case of semi non-homogeneous medium (a medium when characteristics are space dependent while the speed is independent of space variable).

The amplitude function $A_n(x)$ satisfies the equation

$$2h'(x)A_n'(x)+\left\{\rho_0\left(\frac{1}{\eta_{02'}}+\frac{1}{\eta_{0.}}\right)-\alpha h'(x)+h(x)\right\}A_n(x)=Q_{n,}\quad (n=0,1,2.........) \tag{67}$$

$$Q_n = \mathcal{A}_{n-1} - \left\{\alpha\frac{\sin\alpha x}{(1+\cos\alpha x)}+2\frac{G_{02}}{\eta_{02'}}\left(1+\frac{\eta_{02'}}{\eta_{02}}\right)h'\right\}A'_{n-1}-$$
$$\left\{\rho_0\left(\frac{G_{02}}{\eta_{02}\eta_{02'}}+\frac{G_2}{\eta_{02'}\eta_{03}}\left(1+\frac{\eta_{02'}}{\eta_2}\right)\right)+\frac{G_{02}}{\eta_{02'}}\left(1+\frac{\eta_{02'}}{\eta_{02}}\right)h''-\frac{G_{02}}{\eta_{02'}}\left(1+\frac{\eta_{02'}}{\eta_{02}}\right)\alpha\frac{\sin\alpha x}{(1+\cos\alpha x)}h'\right\}A_{n-1}+$$
$$\frac{G_{02}}{\eta'_{02}}\left(1+\frac{\eta_{02'}}{\eta_{02}}\right)A''_{n-2}-\frac{G_{02}}{\eta_{02'}}\left(1+\frac{\eta_{02'}}{\eta_{02}}\right)\alpha\frac{\sin\alpha x}{(1+\cos\alpha x)}A'_{n-2} \tag{68}$$

As the amplitude function is given by Eq. (57), for this case

$$l(x)=\sqrt{\rho_0G_{10}}\left(1+\cos\alpha x\right),\quad m(x)=\frac{\sqrt{\rho_0G_{01}}}{2}\left(\frac{1}{\eta_{02'}}+\frac{1}{\eta_{03}}\right)=m_0,\quad \int_0^x m(x)dx=m_0x \tag{69}$$

Hence,

$$A_n(x)=A_n(0)\sqrt{\frac{(1+\cos\alpha x)}{2}}\exp\left\{\mp\int_0^x m_0ds\right\}\pm\frac{1}{2}\int_0^x c_0\left(\frac{1+\cos\alpha x}{1+\cos\alpha s}\right)\exp\left\{\pm\int_x^s m_0dz\right\}Q_n'^{\pm}(s)ds \tag{70}$$

The value of first term approximation, the stress function is given by

$$\sigma(x,t)=\sigma_0\sqrt{\frac{(1+\cos\alpha x)}{2}}exp\left\{-\int_0^x m_0ds\right\}\mathcal{H}\left\{t-h(x)\right\} \tag{71}$$

The harmonic stress wave which starts from the end $x=0$ with amplitude $\sigma_0$ and moves with constant velocity $c_0=\sqrt{\frac{G_{01}}{\rho_0}}$ in the positive direction of $x$ is modulated by the factor

$$\sqrt{\frac{(1+\cos\alpha x)}{2}}exp\left\{-\int_0^x m_0ds\right\}. \tag{72}$$

Further terms in the approximation solution may be obtained from Eq. (70).

### *5.2 Wave propagation for the non-homogeneous case*

If $\alpha_1>\alpha_2>\alpha_3$ i.e., density > rigidity > viscosity, then from Eq. (64), we get

$$\rho = \rho_0(1+\cos\alpha_1 x), G = G_0(1+\cos\alpha_2 x), \eta = \eta_0(1+\cos\alpha_3 x) \tag{73}$$

From Eikonal equation of geometric optics

$$\left(\frac{dh(x)}{dx}\right)^2 = \frac{\rho}{G_1} = \frac{\rho_0\left(1+\cos\alpha_1 x\right)}{G_{10}\left(1+\cos\alpha_2 x\right)} = \frac{1}{c^2}$$

$$\Rightarrow c = \sqrt{\frac{G_{01}(1+\cos\alpha_2 x}{\rho_0(1+\cos\alpha_1 x)}} \tag{74}$$

The amplitude function $A_n\left(x\right)$ satisfies the equation

$$h'(x)A_n'(x) + \left\{\rho_0 \frac{(1+\cos a_1 x}{1+\cos a_3 x}\left(\frac{1}{\eta_{02}} + \frac{1}{\eta_{03}}\right) - \alpha_1 \frac{\sin\alpha_1 x}{(1+\cos\alpha_1 x)} h'(x) + h\,x\right\} A_n(x) = Q_n^m$$

$$(n = 0, 1, 2\ldots\ldots\ldots) \tag{75}$$

where,

$$A_{n-1} - \left\{\alpha_1 \frac{\sin\alpha_1 x}{(1+\cos\alpha_1 x} + 2\frac{G_{02}(1+\cos\alpha_2 x}{\eta_{02'}(1+\cos\alpha_3 x)}\left(1+\frac{\eta_{02'}}{\eta_{02}}\right)h'\right\}A_{n-1}' -$$

$$Q_n'' = \left\{\begin{array}{l} \rho_0(1+\cos\alpha_1 x)\left(\dfrac{G_{02}(1+\cos\alpha_2 x)}{\eta_{02}\eta_{02'}(1+\cos\alpha_3 x)^2} + \dfrac{G_{02}(1+\cos\alpha_2 x)}{\eta_{02'}\eta_{03}(1+\cos\alpha_3 x)^2}\left(1+\dfrac{\eta_{02'}}{\eta_{02}}\right)\right) + \\ \dfrac{G_{02}(1+\cos\alpha_2 x)}{\eta_{02'}(1+\cos\alpha_3 x)}\left(1+\dfrac{\eta_{02'}}{\eta_{02}}\right)h'' - \dfrac{G_{02}(1+\cos\alpha_2 x)}{\eta_{02'}(1+\cos\alpha_3 x)}\left(1+\dfrac{\eta_{02'}}{\eta_{02}}\right)\alpha\dfrac{\sin\alpha x}{(1+\cos\alpha x)}h' \end{array}\right\} A_{n-1} + \tag{76}$$

$$\frac{G_{02}(1+\cos\alpha_2 x)}{\eta_{02'}(1+\cos\alpha_3 x)}\left(1+\frac{\eta_{2'}}{\eta_2}\right)A_{n-2}'' - \frac{G_{02}(1+\cos\alpha_2 x)}{\eta_{02'}(1+\cos\alpha_3 x)}\alpha\frac{\sin\alpha x}{(1+\cos\alpha x)}A_{n-2}' \qquad (n = 0, 1, 2.........)$$

And Amplitude function $A_n(x)$ is given by Eq. (57).
For this case,

$$l(x) = \sqrt{\rho_0 G_0\left(1+\cos\alpha_1 x\right)\left(1+\cos\alpha_2 x\right)} = l_1(x)$$

$$m(x) = \frac{\sqrt{\rho_0 G_0\left(1+\cos\alpha_1 x\right)\left(1+\cos\alpha_2 x\right)}}{2\left(1+\cos\alpha_3 x\right)}\left\{\frac{1}{\eta_{02'}} + \frac{1}{\eta_{03}}\right\} = m_1(x),$$

$$\int_0^x m(x)dx = \frac{\sqrt{G_{01}\rho_0}}{2}\left\{\frac{1}{\eta_{10}} + \frac{1}{\eta_{20}}\right\}\int_0^x \frac{\sqrt{\left(1+\cos\alpha_1 x\right)\left(1+\cos\alpha_2 x\right)}}{\left(1+\cos\alpha_3 x\right)}dx$$

Therefore,

$$A_n(x)=A_n(0)\left[\frac{\sqrt{(1+\cos\alpha_1 x)(1+\cos\alpha_2 x)}}{2}\right]^{\frac{1}{2}}\exp\left\{\mp\int_0^x m_1(s)ds\right\}\pm\frac{1}{2}\int_0^x c(s)\left\{\frac{l_1(x)}{l_1(s)}\right\}^{\frac{1}{2}}\exp\left\{\pm\int_x^z m_1(z)dz\right\}Q_n'^{\pm}(s)ds \qquad (77)$$

The value of first term approximation, the stress function is given by

$$\sigma(x,t)=\sigma_0\left[\frac{\sqrt{(1+\cos\alpha_1 x)(1+\cos\alpha_2 x)}}{2}\right]^{\frac{1}{2}} exp\left\{-\frac{\sqrt{G_{01}\rho_0}}{2}\left\{\frac{1}{\eta_{10}}+\frac{1}{\eta_{20}}\right\}\int_0^x\frac{\sqrt{(1+\cos\alpha_1 x)(1+\cos\alpha_2 x)}}{(1+\cos\alpha_3 x)}dx\right\}H\{t-h(x)\} \qquad (78)$$

$$h(x)=\frac{1}{(\alpha_1-\alpha_2)}\sqrt{\frac{\rho_0}{G_{10}}}e^{(\alpha_1-\alpha_2)x}$$

Eq. (78) gives a harmonic stress wave which starts from the end $x=0$ with amplitude $\sigma_0$ and moves with positive direction of $x$ with velocity

$$c=\sqrt{\frac{G_{01}(1+\cos\alpha_2 x)}{\rho_0(1+\cos\alpha_1 x)}}$$

and modulated by the factor as

$$\left[\frac{\sqrt{(1+\cos\alpha_1 x)(1+\cos\alpha_2 x)}}{2}\right]^{\frac{1}{2}}\exp\left\{-\frac{\sqrt{G_{01}\rho_0}}{2}\left\{\frac{1}{\eta_{10}}+\frac{1}{\eta_{20}}\right\}\int_0^x\frac{\sqrt{(1+\cos\alpha_1 x)(1+\cos\alpha_2 x)}}{(1+\cos\alpha_3 x)}dx\right\} \qquad (79)$$

## 6. Numerical analysis

To see qualitative effect of non-homogeneity on the harmonic wave propagation in non-homogeneous five parameter viscoelastic rod, a graph is plotted between $\sigma/\sigma_0$ and $x$ for Eq. (71) (semi-homogeneous case), by taking $\alpha$ = (4, 3, 0, -3, -4). The material properties are as shown in Table 2. Figs. 2-5 represent this case. For non- homogeneous case, graph is plotted between $\sigma/\sigma_0$ v/s $x$ by taking Eq. (78). Figs. 6-9 are plotted for non-homogeneous case. It is quite obvious, from these curves the dispersion is not uniform for harmonic waves propagating in non-homogeneous viscoelastic rods.

Table 1 Material properties

| | $\rho_0$ | $G_{01}$ | $G_{02}$ | $G_{02}^{'}$ | $\eta_{03}$ | $\eta'_{02}$ |
|---|---|---|---|---|---|---|
| Material | 20 | 1.8 | 1.6 | 1.4 | 1.5 | 1.3 |

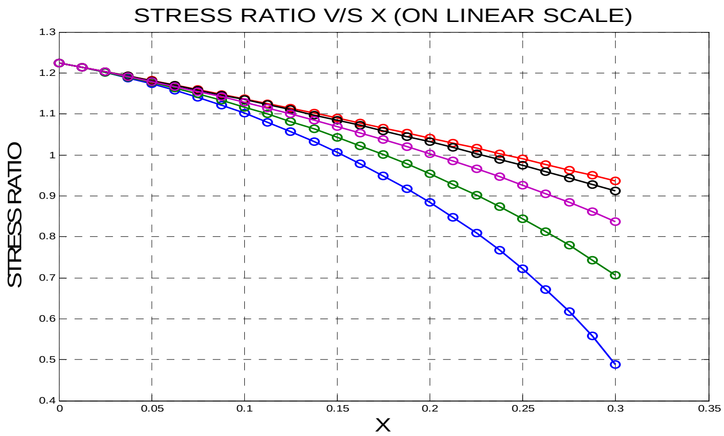


Fig. 2 Variation of stress ratio *v*/*s* distance for semi-homogeneous case by taking $\alpha$ = 4, 3, 0,-1 and -2

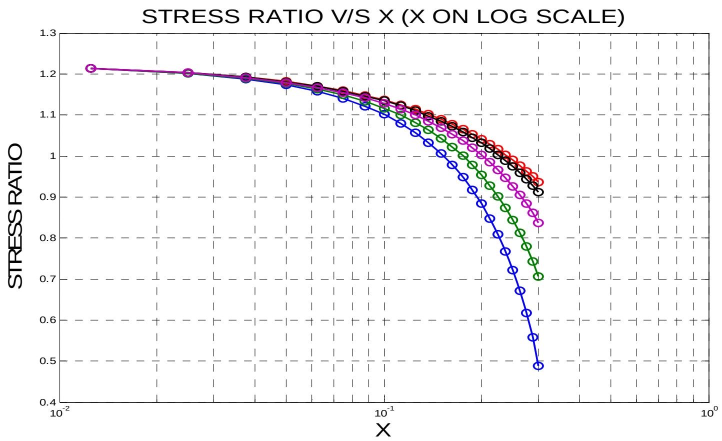


Fig. 3 Variation of stress ratio *v*/*s* distance for semi-homogeneous case by taking $\alpha$ = 4, 3, 0,-1 and -2

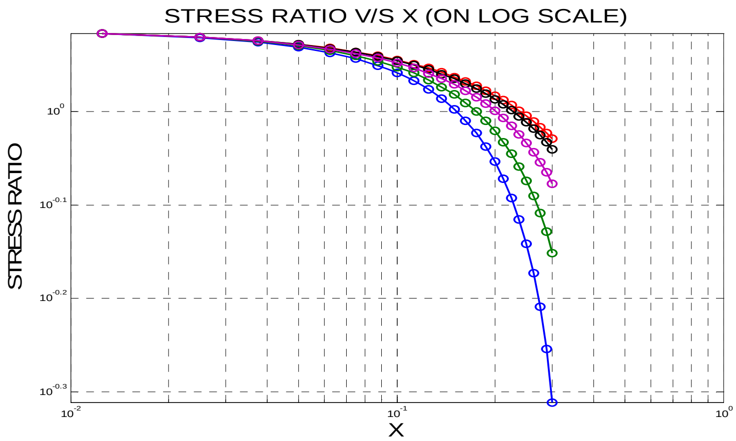


Fig. 4 Variation of stress ratio *v*/*s* distance for semi-homogeneous case by taking $\alpha$ = 4, 3, 0,-1 and -2

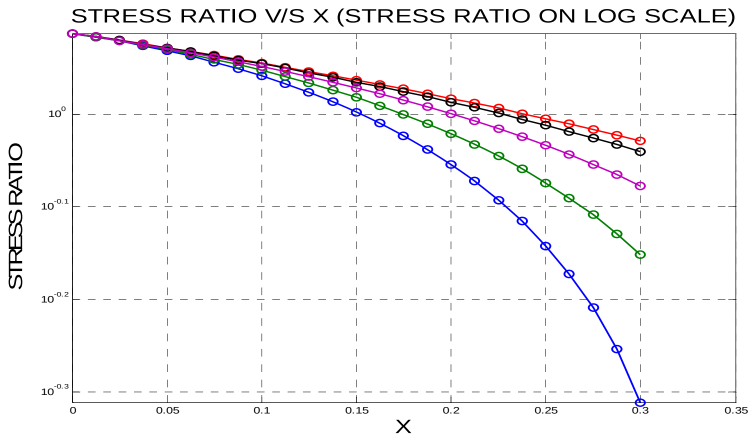


Fig. 5 Variation of stress ratio *v/s* distance for semi-homogeneous case by taking $\alpha = 4, 3, 0, -1$ and $-2$

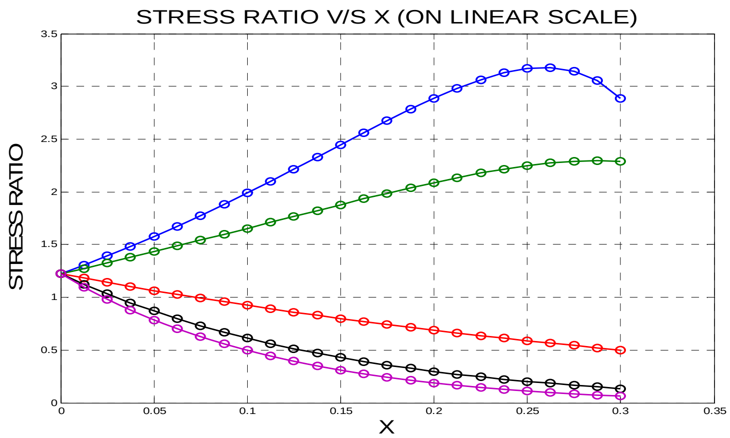


Fig. 6 Variation of stress ratio *v/s* distance for non-homogeneous case by taking $\alpha = 4, 3, 0, -1$ and $-2$

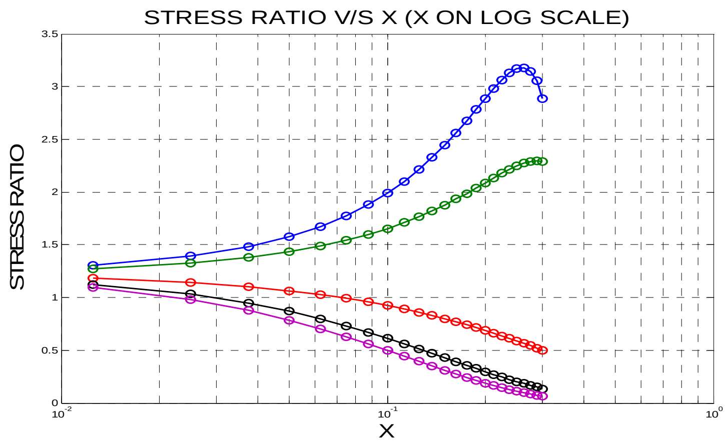


Fig. 7 Variation of stress ratio *v/s* distance for non-homogeneous case by taking $\alpha = 4, 3, 0, -1$ and $-2$

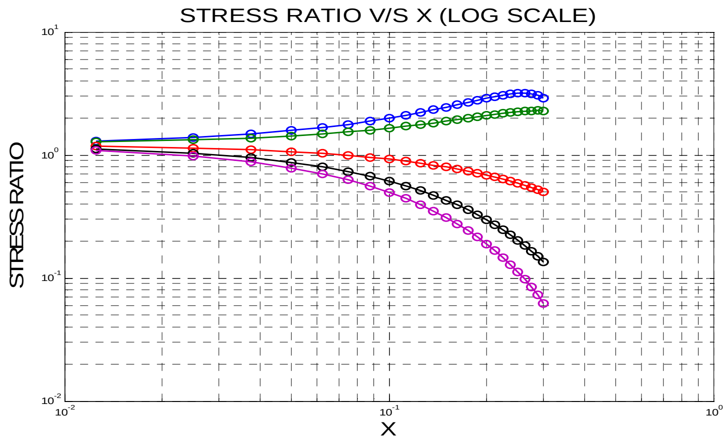


Fig. 8 Variation of stress ratio *v/s* distance for non-homogeneous case by taking $\alpha = 4, 3, 0, -1$ and $-2$

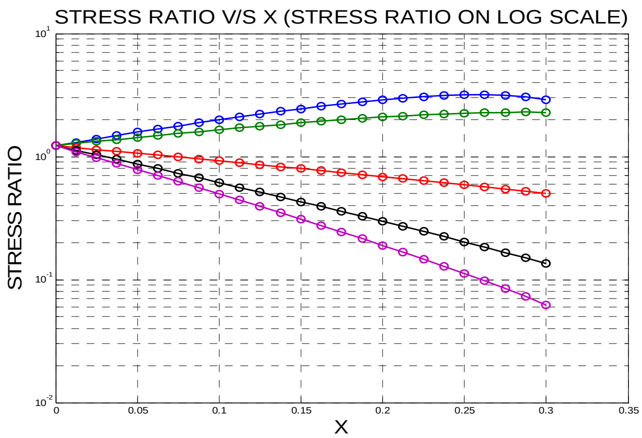


Fig. 9 Variation of stress ratio *v/s* distance for non-homogeneous case by taking $\alpha = 4, 3, 0, -1$ and $-2$

## 7. Conclusions

A numerical simulation procedure for predicting the behavior of harmonic wave propagation in non-homogeneous viscoelastic has been proposed in this study.

- When the density, rigidity and viscosity all are equal for the first material specimen, the sound speed is constant i.e., non-homogeneous has no effect on speed and phase of the wave is given $h(x) = \frac{x}{c_0}$. So it becomes the case of semi non-homogeneous medium (a medium when characteristics are space dependent while the speed is independent of space variable). The harmonic speed will be equal to $c = \sqrt{\frac{G_0}{\rho_0}}$

- When the density, rigidity and viscosity are not equal for the second material specimen, the speed of sound is given as $c_0 = \sqrt{\dfrac{G_{01}(1+\cos\alpha_2 x)}{\rho_0(1+\cos\alpha_1 x)}}$ .

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