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Torsional surface waves in a non-homogeneous isotropic layer over viscoelastic half-space

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Abstract. The aim of this paper is to study the propagation of torsional surface waves in non-homogeneous isotropic layer of finite thickness placed over a homogeneous viscoelastic half-space, when both density and rigidity of the non-homogeneous medium are assumed to vary exponentially with depth. The frequency equations are obtained by using simple method of separation of variables. Further, it is seen that when viscoelastic parameter and non-homogeneity parameter is neglected, the dispersion equation gives the dispersion equations of Love waves in homogeneous, elastic and isotropic layer placed over homogeneous viscoelastic medium. The problem has been solved numerically and the effects of various inhomogeneities of the medium on torsional waves have been illustrated graphically.

Keywords: torsional waves; viscoelastic media; inhomogeneities; internal friction

1. Introduction

The theory of viscoelasticity is useful in the field of solid mechanics, engineering, seismology, exploration and geophysics. Love waves, rayleigh waves, stoneley waves and torsional waves are a well-known feature of the wave theory. The amplitude of the surface torsional waves decays exponentially with depth as they propagate in free surface of the earth. These waves are also known as horizontal polarized waves and gave a twist to the medium in which it travels.

Much literature is available on torsional surface wave propagation in homogeneous elastic and viscoelastic media. Alfrey (1944) discussed non-homogeneous stress in viscoelastic media. Frank *et al.* (1959) presented their views on elastic wave propagation in homogeneous and Inhomogeneous Media. Achenbach and Reddy (1967) gave a note on the wave-propagation in linear viscoelastic media. Batra (1998) successfully applied this theory to wave-propagation in homogeneous elastic media. Pal (2000) presented a note on torsional body forces in a viscoelastic half space. Kumar (2010) studied wave motion in micro-polar transversely isotropic thermoelastic half space without energy dissipation. Dey *et al.* (1996, 2000, 2002, 2003) investigated the effect of torsional surface waves in non-homogeneous anisotropic medium, torsional surface waves in an elastic layer with void pores, torsional surface waves in an elastic layer with void pores over an elastic half space with void pores and effect of gravity and initial stress on torsional surface waves

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in dry sandy medium. Kumar and Gupta (2010) presented wave motion in micropolar transversely isotropic thermoelastic half space without energy dissipation. Davini *et al.* (2008) investigated torsional problem in thin rectangular domains with the help of asymptotic approach. The work done by Gupta *et al.* (2011, 2012) on torsional waves cannot be ignored. Akbarov *et al.* (2011) discussed dispersion of torsional wave in a finitely pre-strained hollow sandwich circular cylinder.

In this paper, we have studied the propagation of torsional surface wave in a non-homogeneous isotropic layer over a homogeneous viscoelastic half-space. The method of separation of variables is adopted to find dispersion relation. A numerical simulation procedure for predicting the behavior of torsional wave propagation in non-homogeneous viscoelastic media has been proposed in this study. The influence of viscoelastic parameter, non-homogeneity parameter, wave number, rigidity and time period on the phase velocity is shown graphically. The effects of non-homogeneities in terms of rigidity, density and internal friction can be employed to torsional surface waves for the future research of the present study.

2. Governing equations

The torsional surface waves are assumed to propagate in viscoelastic medium, if r and θ denotes radial and circumferential co-ordinates respectively, then the equations of motion for the torsional surface waves travelling along z-direction can be written as Biot (1965).

$$\frac{\partial s_{rr}}{\partial r}+\frac{1}{r}\frac{\partial s_{r\theta}}{\partial \theta}+\frac{\partial s_{rz}}{\partial z}+\frac{1}{r}(s_{rr}-s_{\theta\theta})+T_R=\rho\frac{\partial^2 u}{\partial t^2}, \tag{1.1}$$

$$\frac{\partial s_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial s_{\theta\theta}}{\partial \theta}+\frac{\partial s_{\theta z}}{\partial z}+\frac{2s_{r\theta}}{r}+T_\theta=\rho\frac{\partial^2 v}{\partial t^2}, \tag{1.2}$$

$$\frac{\partial s_{rz}}{\partial r}+\frac{1}{r}\frac{\partial s_{\theta z}}{\partial \theta}+\frac{\partial s_{zz}}{\partial z}+\frac{s_{rz}}{r}+T_Z=\rho\frac{\partial^2 w}{\partial t^2}. \tag{1.3}$$

where, $s_{rr}, s_{r\theta}, s_{rz}, s_{rr}.s_{\theta\theta}, s_{\theta z}, s_{zz}$ are the respective stress components, T_R, T_θ, T_Z are the respective body forces and u, v, w are the respective displacement components.

The stress-strain relations are

$$s_{rr}=\lambda\Omega+2\mu e_{rr} \tag{2.1}$$

$$s_{\theta\theta}=\lambda\Omega+2\mu e_{\theta\theta} \tag{2.2}$$

$$s_{zz}=\lambda\Omega+2\mu e_{zz} \tag{2.3}$$

$$s_{r\theta}=2\mu e_{r\theta} \tag{2.4}$$

$$s_{rz}=2\mu e_{rz} \tag{2.5}$$

$$s_{\theta z} = 2\mu e_{\theta z} \tag{2.6}$$

where, λ and μ are Lame's constant.
and
$\Omega = \left(\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} + \frac{\partial w}{\partial z} \right)$ denotes the dilatation.
The strain-displacement relations are

$$e_{rr} = \frac{1}{2}\frac{\partial u}{\partial r}, \tag{3.1}$$

$$e_{\theta\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \right), \tag{3.2}$$

$$e_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial v}{\partial z} \right), \tag{3.3}$$

$$e_{rz} = \frac{1}{2}\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right), \tag{3.4}$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\ \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right) \tag{3.5}$$

$$e_{zz} = \frac{1}{2}\frac{\partial w}{\partial z}. \tag{3.6}$$

The rotational components are

$$\Omega_r = \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z} \right), \tag{4.1}$$

$$\Omega_\theta = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right), \tag{4.2}$$

$$\Omega_z = \frac{1}{r}\left(\frac{\partial (rv)}{\partial r} - \frac{\partial u}{\partial \theta} \right). \tag{4.3}$$

If u , v and w are the displacement then

$$u = 0,\ w = 0,\ v = v(r, z, t). \tag{5}$$

From Eqs. (1)-(2) and Eq. (5), we get

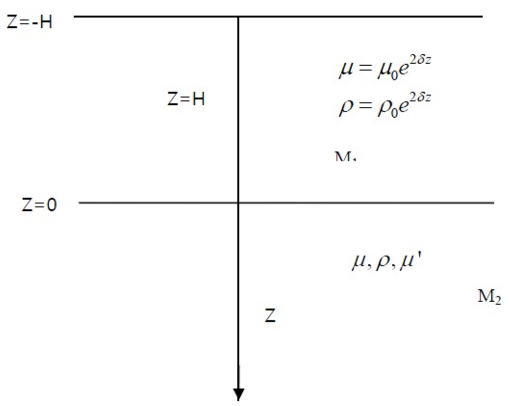

Fig. 1 Geometry of the problem

$$\frac{\partial s_{r\theta}}{\partial r} + \frac{\partial s_{z\theta}}{\partial z} + \frac{2s_{r\theta}}{r} = \rho(z)\,\frac{\partial^2 v}{\partial t^2} \tag{6}$$

Where, v (r, z, t) is the displacement along θ direction.
For an elastic medium, the stresses are related to displacement by

$$s_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \text{ and } s_{\theta z} = \mu\left(\frac{\partial v}{\partial z}\right) \tag{7}$$

3. Problem defination

Consider a non-homogeneous layer M_1 of finite thickness $Z = H$, over a viscoelastic homogeneous medium M_2. The cylindrical co-ordinate system (r, θ, z) is located at the interface separating the two layers at $z = 0$. The z-axis is acting downward. For medium M_1 the inhomogeneities are in rigidity and density. The rigidity and density are assumed to vary exponentially with depth and taken as $\mu = \mu_0 e^{2\delta z}$ and $\rho = \rho_0 e^{2\delta z}$ respectively. δ is the non-homogeneous parameter for medium M_1. Medium M_2 is viscoelastic homogeneous, therefore we have assumed rigidity, density and internal friction as μ, ρ and μ' respectively. μ' is also known as viscoelastic parameter. The problem is represented geometrically in Fig. 1.

4. Solution of the problem

Let the inhomogeneities in rigidity and density in medium M_1 are

$$\mu = \mu_0 e^{2\delta z} \text{ and } \rho = \rho_0 e^{2\delta z} \tag{8}$$

Eq. (6) takes the form

$$\mu_0\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)v+\frac{\partial}{\partial z}\left(\mu_0\frac{\partial v}{\partial z}\right)= \rho_0\frac{\partial^2 v}{\partial t^2} \tag{9}$$

We may assume the solution of Eq. (9) as

$$v=\zeta(z)J_1(kr)e^{i\omega t} \tag{10}$$

where ζ is the solution of the solution of following Eq.

$$\frac{d^2\zeta}{dz^2}+2\delta\frac{d\zeta}{dz}-K^2\left(1-\frac{c^2}{c_0{}^2}\right)\zeta=0 \tag{11}$$

Where, $c=\dfrac{\omega}{k}$, $c_0=\left(\dfrac{\mu_0}{\rho_0}\right)^{1/2}$ and $J_1(kr)$ is the Bessel function of first kind and first order.

Now, since the layer is homogeneous and isotropic, Eq. (11) reduces to

$$\frac{d^2\zeta}{dz^2}-K^2\left(1-\frac{c^2}{c_0{}^2}\right)\zeta=0 \tag{12}$$

The solution of Eq. (11) may be given as

$$\zeta=Ae^{(-\delta+m)z}+B\,e^{-(\delta+m)z} \tag{13}$$

where,

$m^2=\delta^2+K^2\left(1-\dfrac{c^2}{c_0{}^2}\right)$ and A and B are arbitrary constants.

Hence, the displacement in the upper non-homogeneous layer is

$$v_0=[Ae^{(-\delta+m)z}+Be^{-(\delta+m)z}]J_1(kr)\,e^{i\omega t}\,. \tag{14}$$

4. Solution of the lower half space

For lower half space which is homogeneous and we have assumed rigidity, density and internal friction as μ, ρ and μ' respectively. μ' is also known as viscoelastic parameter. Let us assume that the torsional surface wave propagates in radial direction only, such that all other mechanical properties are independent of θ. For torsional surface wave, $u=0,\ w=0,\ v=v(r,z,t)$. Therefore, the equation of motion for medium M_2 will be viscoelastic voigt type as Biot (1965)

$$\left(\mu+\mu'\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\right)=\rho\frac{\partial^2 v}{\partial t^2} \tag{15}$$

The solution of Eq. (15) for the torsional surface wave propagating along 'r' direction one may assume as

$$v=\zeta(z)J_1(kr)e^{i\omega t} \tag{16}$$

where ζ is the solution of following equation.

$$\frac{d^2\zeta}{dz^2}-L^2\left\{1-\frac{c_1^{\,2}}{c_2^{\,2}(1+iD)}\right\}\zeta(z)=0 \tag{17}$$

where, $c_1=\dfrac{\omega}{k}$, c$_2$ = $c_2=\left(\dfrac{\mu}{\rho}\right)^{1/2}$, $D=\dfrac{\omega\mu'}{\mu}$, $\omega=\dfrac{2\pi}{T}$.

The solution of Eq. (17) satisfying the condition $\lim_{z\to\infty}\zeta(z)=0$ is

$$\zeta=\left[F\cos(\alpha_1\alpha_3 z)-G\sin(\alpha_1\alpha_3 z)\right]e^{-\alpha_1\alpha_2 z} \tag{18}$$

Where

$$\alpha_1=\left[\frac{k^2\left[\left\{c_2^{\,2}(1+D^2)-c_1^{\,2}\right\}^2+\left(Dc_1^{\,2}\right)^2\right]^{1/2}}{c_2^{\,2}(1+D^2)}\right]^{1/2},\quad \alpha_2=\cos\left(\frac{\theta}{2}\right),\quad \alpha_3=\sin\left(\frac{\theta}{2}\right),$$

$$\theta=\tan^{-1}\frac{D}{\left(\dfrac{c_2^{\,2}}{c_1^{\,2}}+D\dfrac{c_2}{c_1}-1\right)}$$

Therefore the final solution of Eq. (15) may be written as

$$v=\left[F\cos(\alpha_1\alpha_3 z)-G\sin(\alpha_1\alpha_3 z)\right]J_1(kr)\,e^{i\omega t-\alpha_1\alpha_2 z} \tag{19}$$

4. Boundary conditions

The above problem satisfies the following boundary conditions
(1) For traction free surface $z = -H$,

$$\mu_0 \frac{\partial v_0}{\partial z} = 0 \tag{20.1}$$

(2) For interface z = 0, continuity of stress component gives

$$\mu_0 \frac{\partial v_0}{\partial z} = \left(\mu + \mu' \frac{\partial}{\partial t} \right) \left(\frac{\partial v_1}{\partial z} \right) \tag{20.2}$$

(3) Continuity of the displacement component at z = 0 gives,

$$v_0 = v_1 \tag{20.3}$$

(4) $\left(\mu + \mu' \frac{\partial}{\partial t} \right) \left(\frac{\partial v_1}{\partial r} - \frac{v_1}{r} \right) = 0$ at z = 0

Now, using Eq. (14), Eq. (20.1) becomes

$$A(-\delta + m)e^{-mH} - B(\delta + m)\, e^{mH} = 0. \tag{21.1}$$

From Eq. (14), Eq. (19) and Eq. (20.1), we have

$$[A(-\delta + m) - B(\delta + m)]\mu_0 = F(-\mu\alpha_1\alpha_2 - \mu'\alpha_1\alpha_2 i\omega) + G(-\mu\alpha_1\alpha_3 - \mu'\alpha_1\alpha_3 i\omega) \tag{21.2}$$

Similarly, we have from Eqs. (20.3) and (20.4)

$$A + B = F \tag{21.3}$$

$$\cot\left(\frac{\theta}{2}\right) F + G = 0 \tag{21.4}$$

Eliminating the arbitrary constants A, B, F and G, we have

$$\begin{vmatrix} (-\delta + m)\, e^{-mH} & (\delta + m)\, e^{mH} & 0 & 0 \\ \mu_0(-\delta + m) & \mu_0(\delta + m) & (\mu + \mu' i\omega)\alpha_1\alpha_2 & (\mu + \mu' i\omega)\alpha_1\alpha_3 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & \cot\left(\frac{\theta}{2}\right) & -1 \end{vmatrix} = 0 \tag{22}$$

On solving the above determinant, and equating the real part of above equation, we get

$$\tan\left(kH\sqrt{\left(\frac{c^2}{c_0{}^2}-1\right)-\left(\frac{\delta}{k}\right)^2}\right)=\left\{2\frac{\mu}{\mu_0}\frac{\left[\left\{\left(1+D^2\right)-\frac{c^2}{c_1{}^2}\right\}^2+\left(D\frac{c^2}{c_1{}^2}\right)^2\right]^{1/4}\cot\left(\frac{\theta}{2}\right)}{\sqrt{1+D^2}\ \left(\frac{c^2}{c_0{}^2}-1\right)}\right\}\left[\left(\frac{\delta}{k}\right)\tan\left(kH\sqrt{\left(\frac{c^2}{c_0{}^2}-1\right)-r^2}\right)+\left(\sqrt{\left(\frac{c^2}{c_0{}^2}-1\right)-\left(\frac{\delta}{k}\right)^2}\right)\right] \quad (23)$$

If we assume the non-homogeneity parameter $e=\left(\frac{\delta}{k}\right)$, then Eq. (23) can be written as

$$\tan\left(kH\sqrt{\left(\frac{c^2}{c_0{}^2}-1\right)-e^2}\right)=\left\{2\frac{\mu}{\mu_0}\frac{\left[\left\{\left(1+D^2\right)-\frac{c^2}{c_1{}^2}\right\}^2+\left(D\frac{c^2}{c_1{}^2}\right)^2\right]^{1/4}\cot\left(\frac{\theta}{2}\right)}{\sqrt{1+D^2}\ \left(\frac{c^2}{c_0{}^2}-1\right)}\right\}\left[e\tan\left(kH\sqrt{\left(\frac{c^2}{c_0{}^2}-1\right)-r^2}\right)+\left(\sqrt{\left(\frac{c^2}{c_0{}^2}-1\right)-e^2}\right)\right] \quad (24)$$

Eq. (24) is the dispersion relation for torsional waves in a non-homogeneous isotropic layer over a viscoelastic medium.

4.1 Torsional wave propagation for the homogeneous case

When $e=0$, $\mu'=0$, then the Eq. (24) reduces to

$$\tan\left(kH\sqrt{\frac{c^2}{c_0{}^2}-1}\right)=\left(\frac{\mu}{\mu_0}\right)\left(\sqrt{1-\frac{c^2}{c_1{}^2}}\Big/\sqrt{\frac{c^2}{c_0{}^2}-1}\right) \quad (25)$$

Eq. (25) the dispersion equation for torsional surface waves in elastic, homogeneous and isotropic medium M_2. Eq. (25) matches with the dispersion equation of Love waves in a homogeneous layer over an elastic homogeneous medium.

5. Numerical analysis

In order to see the effect of non-homogeneity parameter (e) and viscoelastic parameter (μ') on the obtained dispersion Eq. (24) for torsional surface waves propagating in non-homogeneous isotropic layer of finite thickness placed over a homogeneous viscoelastic half-space, we have used the Gubbin's (1990) numerical data. The phase velocity depends on wave number, time period, internal friction and rigidity see Eq. (24). The phase velocity (dimensionless) is kept at 1.2 for all the plotted graphs. The description of various graphs is explained below:

- Fig. 2; is plotted between dimensionless phase velocity *v/s* dimensionless wave number for *T*=0.15 sec, $\mu/\mu_0=0.4$, $e=0.002$, $\mu/\mu'=10,50,100$.
- Fig. 3; shows the variation of dimensionless phase velocity *v/s* dimensionless wave nu mber for *T*=0.15 sec, $\mu/\mu_0=0.4$, $e=0.02$, $\mu/\mu'=10,50,100$.
- Fig. 4; is drawn for dimensionless phase velocity *v/s* dimensionless wave number for *T*=0.15 sec, $\mu/\mu_0=0.4$, $e=0.1$, $\mu/\mu'=10,50,100$.

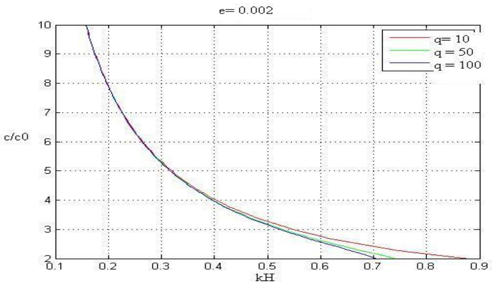

Fig. 2

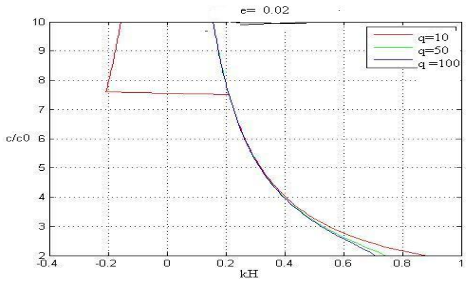

Fig. 3

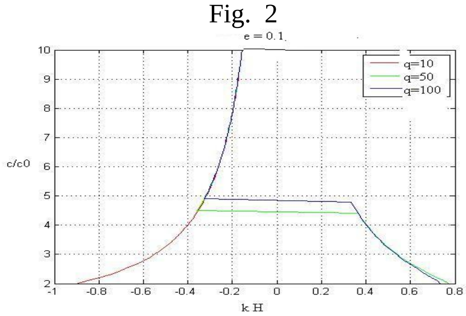

Fig. 4

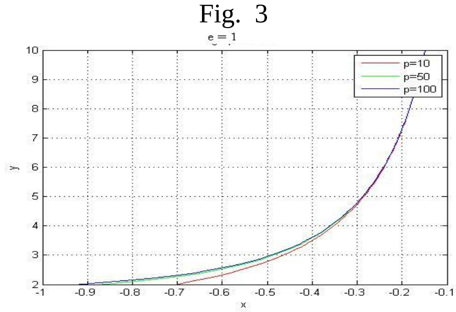

Fig. 5

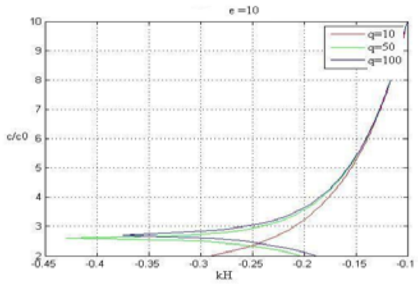

Fig. 6

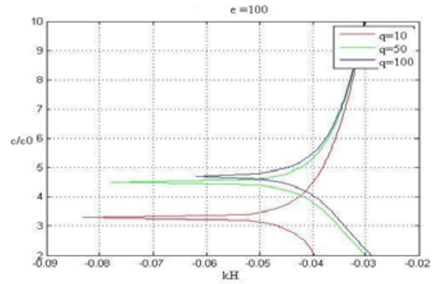

Fig. 7

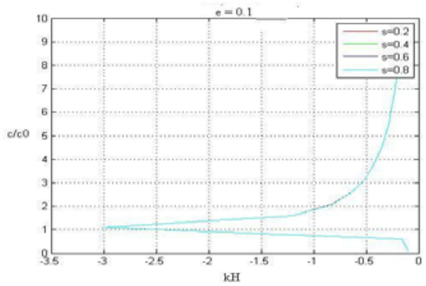

Fig. 8

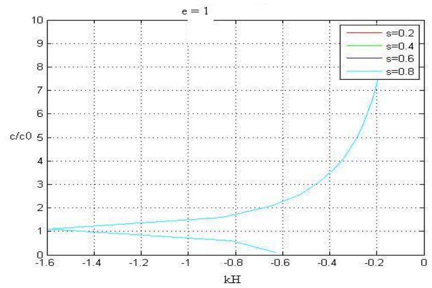

Fig. 9

- Fig. 5; gives dimensionless phase velocity *v/s* dimensionless wave number for *T*=0.15 sec, $\mu/\mu_0 = 0.4$, $e = 1$, $\mu/\mu' = 10, 50, 100$.
- Fig. 6; is plotted between dimensionless phase velocity and dimensionless wave numbe r for *T*=0.15 sec, $\mu/\mu_0 = 0.4$, $e = 10$, $\mu/\mu' = 10, 50, 100$.

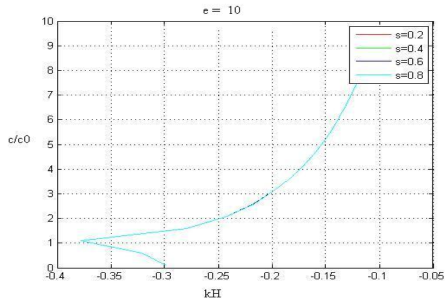

Fig. 10

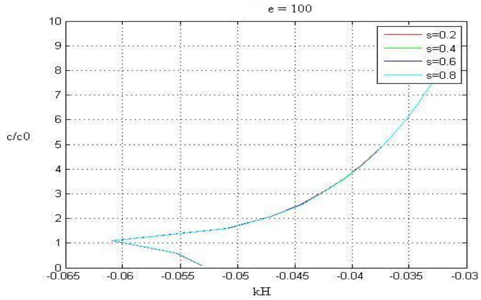

Fig. 11

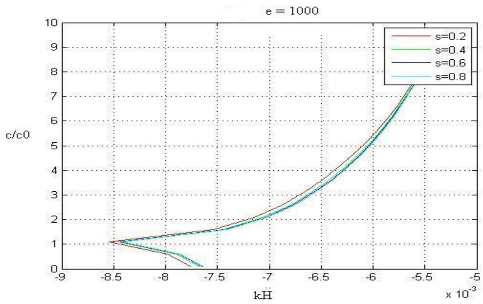

Fig. 12

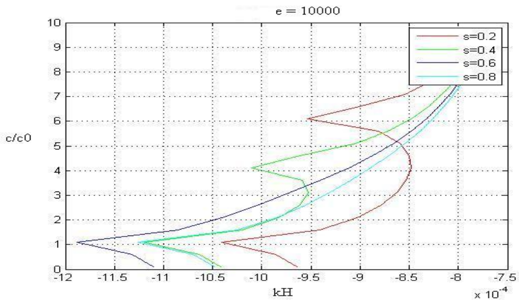

Fig. 13

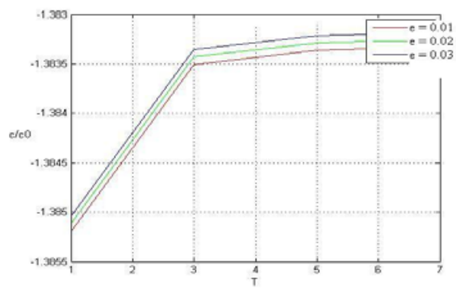

Fig. 14

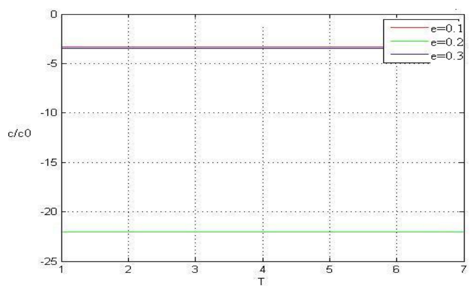

Fig. 15

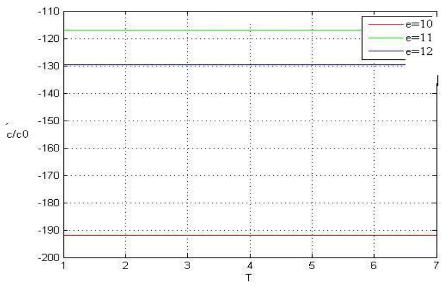

Fig. 16

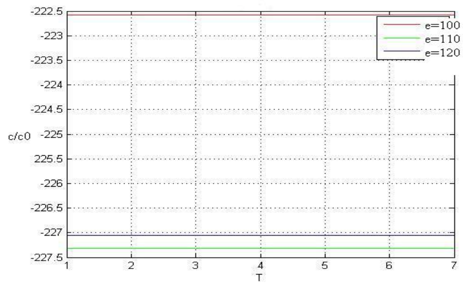

Fig. 17

● Fig. 7; represents dimensionless phase velocity *v/s* dimensionless wave number for *T*= 0.15 sec, $\mu/\mu_0 = 0.4$, $e = 100$, $\mu/\mu' = 10, 50, 100$.

● Fig. 8; shows dimensionless phase velocity *v/s* dimensionless wave number for *T*=0.15 sec, $\mu/\mu_0 = 0.4$, $e = 0.01$, $\mu/\mu' = 0.2, 0.4, 0.6, 0.8$.

● Fig. 9; represents dimensionless phase velocity *v/s* dimensionless wave number for *T*= 0.15 sec, $\mu/\mu_0 = 0.4$, $e = 1$, $\mu/\mu' = 0.2, 0.4, 0.6, 0.8$.

● Fig. 10; shows dimensionless phase velocity *v/s* dimensionless wave number for *T*=0. 15 sec, $\mu/\mu_0 = 0.4$, $e = 10$, $\mu/\mu' = 0.2, 0.4, 0.6, 0.8$.

● Fig. 11; represents dimensionless phase velocity *v/s* dimensionless wave number for *T* =0.15 sec, $\mu/\mu_0 = 0.4$, $e = 0.100$, $\mu/\mu' = 0.2, 0.4, 0.6, 0.8$.

● Fig. 12; shows dimensionless phase velocity *v/s* dimensionless wave number for *T*=0. 15 sec, $\mu/\mu_0 = 0.4$, $e = 1000$, $\mu/\mu' = 0.2, 0.4, 0.6, 0.8$.

● Fig. 13; shows dimensionless phase velocity *v/s* dimensionless wave number for *T*=0. 15 sec, $\mu/\mu_0 = 0.4$, $e = 10000$, $\mu/\mu' = 0.2, 0.4, 0.6, 0.8$.

● Fig. 14; shows dimensionless phase velocity *v/s* Time period, for $\mu/\mu_0 = 0.5$, $kH = 0.5$, $e = 0.01, 0.02, 0.03$, $\mu/\mu' = 70$.

● Fig. 15; shows dimensionless phase velocity *v/s* Time period, for $\mu/\mu_0 = 0.5$, $kH = 0.5$, $e = 0.1, 0.2, 0.3$, $\mu/\mu' = 70$.

● Fig. 16; shows dimensionless phase velocity *v/s* Time period, for $\mu/\mu_0 = 0.5$, $kH = 0.5$, $e = 10, 11, 12$, $\mu/\mu' = 70$.

● Fig. 17; shows dimensionless phase velocity *v/s* Time period, for $\mu/\mu_0 = 0.5$, $kH = 0.5$, $e = 100, 110, 120$, $\mu/\mu' = 70$.

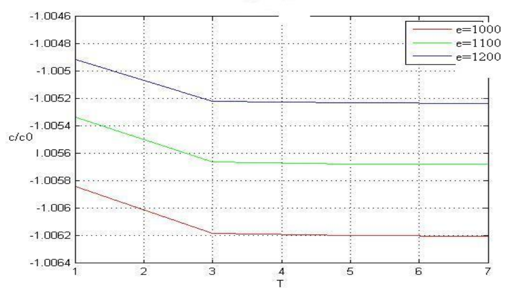

Fig. 18

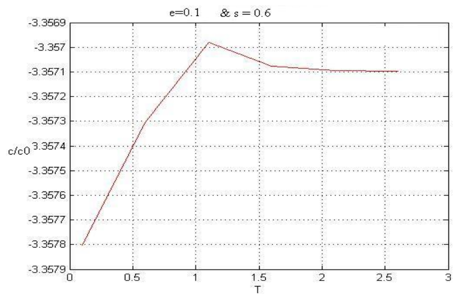

Fig. 19

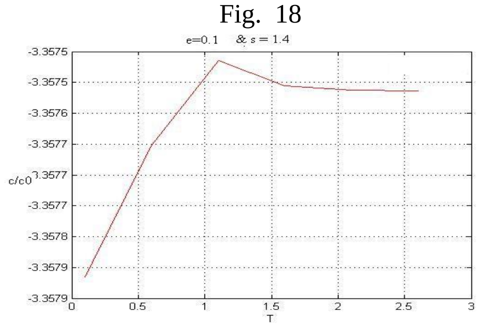

Fig. 20

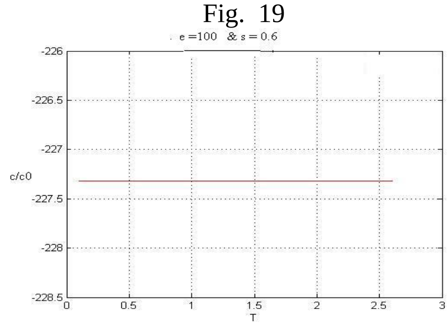

Fig. 21

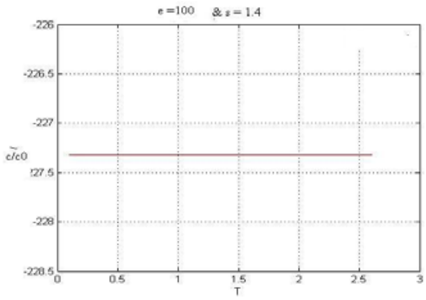

Fig. 22

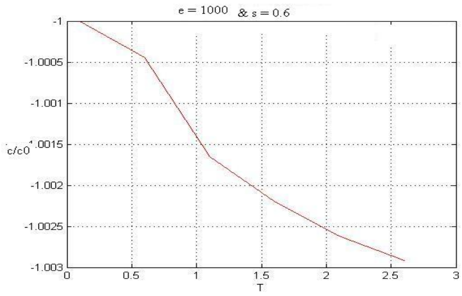

Fig. 23

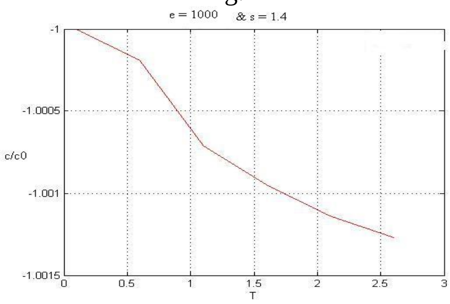

Fig. 24

- Fig. 18; shows dimensionless phase velocity *v/s* Time period, for $\mu/\mu_0 = 0.5$, $kH = 0.5,\ e = 1000,1100.1200$, $\mu/\mu' = 70$.
- Fig. 19; shows dimensionless phase velocity *v/s* Time period, for $\mu/\mu_0 = 0.6$, $kH = 0.5,\ e = 0.1$, $\mu/\mu' = 0.5$.
- Fig. 20; represents dimensionless phase velocity *v/s* Time period, for $\mu/\mu_0 = 1.4$, $kH = 0.5,\ e = 0.1$, $\mu/\mu' = 0.5$.
- Fig. 21; shows dimensionless phase velocity *v/s* Time period, for $\mu/\mu_0 = 0.6$, $kH = 0.5,\ e = 100$, $\mu/\mu' = 0.5$.
- Fig. 22; represents dimensionless phase velocity *v/s* Time period, for $\mu/\mu_0 = 1.4$, $kH = 0.5,\ e = 100$, $\mu/\mu' = 0.5$.
- Fig. 21; shows dimensionless phase velocity *v/s* Time period, for $\mu/\mu_0 = 0.6$, $kH = 0.5,\ e = 1000$, $\mu/\mu' = 0.5$.

Fig. 22; represents dimensionless phase velocity *v/s* Time period, for $\mu/\mu_0 = 1.4$, $kH = 0.5,$ $e = 1000$, $\mu/\mu' = 0.5$.

6. Conclusions

- It has been noticed that the phase velocity of torsional surface waves decreases as the rigidity ratio increases which means non-homogeneity parameter has significant role on phase velocity.

- Also, from various curves, it is observed that the phase velocity of torsional surface waves decreases as the wave number increases.
- The dispersion equation of torsional surface waves in a non-homogeneous isotropic layer over viscoelastic half-space depends on the non-homogeneity, phase velocity, wave number, rigidity and internal friction of the media.
- When effect of non-homogeneities and internal friction is neglected from the dispersion relation of the torsional waves in non-homogeneous media kept over homogeneous media, dispersion equation of Love waves in a homogeneous layer over an elastic homogeneous medium is obtained i.e. the results obtained matches with the classical results.

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