Interaction and Multiscale Mechanics, Vol. 5, No. 4 (2012) 385-397
DOI: http://dx.doi.org/10.12989/imm.2013.5.4.385

Dynamic stiffness matrix method for axially moving micro-beam

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(Received October 8, 2012, Revised November 11, 2012, Accepted December 5, 2012)

Abstract. In this paper the dynamic stiffness matrix method was used for the free vibration analysis of axially moving micro beam with constant velocity. The extended Hamilton's principle was employed to derive the governing differential equation of the problem using the modified couple stress theory. The dynamic stiffness matrix of the moving micro beam was evaluated using appropriate expressions of the shear force and bending moment according to the Euler-Bernoulli beam theory. The effects of the beam size and axial velocity on the dynamic characteristic of the moving beam were investigated. The natural frequencies and critical velocity of the axially moving micro beam were also computed for two different end conditions.

Keywords: dynamic stiffness matrix method; free vibration; axially moving micro-beam; modified couple stress theory.

1. Introduction

The dynamic stiffness matrix method has been widely used for the free vibration analysis of many structural members such as beams on an elastic foundation (Williams and Kennedy 1987), laminated composite beams (Eisenberger *et al*. 1995), twisted Timoshenko beam (Banerjee 2004), axially loaded Timoshenko beams (Capron and Williams 1988, Viola *et al*. 2007), sandwich beams (Banerjee 2003), beams carrying spring-mass systems (Banerjee 2012), axially moving beam (Banerjee and Gunawardana 2007) and other structures.

The axially moving beam problem is an important research area in the engineering science. Several numerical methods have been so far applied to the dynamic analysis of the moving beam problem using the classical theory of elasticity such as the finite element method (Sreeram and Sivaneri 1998), Galerkin method (Murphy and Zhang 2000), spectral element method (e.g. Lee *et al*. 2004, Lee and Oh 2005, Lee and Jang 2007) and dynamic stiffness matrix method by Banerjee and Gunawardana (2007).

The size-dependence behaviour of micro-scale structures has been studied in many recent researches. The results of experimental tests on the wide range of materials such as metals (Fleck *et al*. 1994, Poole *et al*. 1996) and polymers (Chong and Lam 1999a, 1999b), show that the classical

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theory of elasticity is not suitable for the modeling of size-dependence deformation behaviors. This is due to the fact that classical theory cannot capture the size-dependence in the elastic fields of defects such as dislocations, lattice defects and voids (Ke *et al*. 2012). In this regards, several non-classical elasticity theories such as the classical couple stress (Koiter 1964) and Eringen non-local theory (Eringen 1983, Lim *et al*. 2009) have been developed to consider the material size effect by defining appropriate material length scale parameters.

The modified couple stress is one of the non-classical theories of elasticity in which the material size effect is captured only by one additional material parameter. This parameter can be determined from torsion tests of slim cylinders (Ma *et al*. 2008, Chong *et al*. 2001) or bending tests (Ma *et al*. 2008, Park and Gao 2006) of thin beams in micron scale. This theory was firstly proposed by Yang *et al*. (2002) and has been recently developed for the dynamic analysis of the Euler-Bernoulli beams (Kong *et al*. 2008, Simsek 2010, Asghari *et al*. 2010, Wang *et al*. 2009).

In this study, a dynamic stiffness matrix method is employed for the free vibration analysis of axially moving micro beam problem using the Euler Bernoulli theory based on the modified couple stress theory .The layout of the paper is formed of five sections. In the next Section, the model used in this research is described. In Section 3, the dynamic stiffness matrix of the moving micro beam for two different end conditions is evaluated. The results obtained using the micro beam models are presented in Section 4. The effects of beam size and the axial velocity on the dynamic characteristics of the moving micro beam are also addressed in this section. Finally, in Section 5, we shall summarize the conclusions.

2. Beam model

In the modified couple stress theory, the constitutive relations of a deformable isotopic body with coefficients λ and μ as Lame constants, and the material length scale parameter l are given as Yang *et al*. (2002)

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \quad i,j = 1,2,3 \tag{1}$$

$$m_{ij} = 2\mu l^2\chi_{ij}, \quad i,j = 1,2,3 \tag{2}$$

where σ_{ij} and m_{ij} are the tensors of standard stress and the deviatoric couple stress, respectively. The additional parameter l is mathematically calculated as the square of the ratio of the modulus of curvature to the modulus of shear (Ma *et al*. 2008). The property measuring the effect of couple stress is the physical interpretation of this parameter (Ma *et al*. 2008). As can be seen in constitutive relations Eqs. (1) and (2), only one length scale parameter is added to the conventional Lame constants. This means that the problem of finding two independent higher order material length scale parameters in the coupled stress theory (Koiter 1964), is reduced to just one in the modified theory (Yang *et al*. 2002).

The strain tensor, ε_{ij}, and the symmetric curvature tensor, χ_{ij}, are also defined as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3}$$

$$\chi_{ij} = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}) \tag{4}$$

In these equations, u and θ are the components of the displacement and rotation vectors, respectively. Moreover, the governing differential equation of the axially moving micro-beam can be derived using the extended Hamilton's principle for the deformable bodies with continuing mass transport as follows (See Lee and Jang 2007)

$$\int_0^t (\delta K - \delta U + \delta W_{mass\ transport})dt = 0 \tag{5}$$

where K and U are the kinetic energy and the elastic energy, respectively and $\delta W_{mass\ transport}$ is the virtual momentum of transported mass across the boundaries of the moving beam of length L. The influence of the axial velocity, v, and transverse translational motion of beam w, with mass per unit length ρA, is considered in the kinetic energy as

$$K = \frac{1}{2}\rho A \int_0^L v^2 + (\dot{w} + vw')^2 dx \tag{6}$$

The virtual momentum of transported mass $\delta W_{mass\ transport}$, can be written from (Lee and Jang 2007), as follows

$$\delta W_{mass\ transport} = [\rho A v(v + w' + vw')]_0^L \tag{7}$$

Moreover, the elastic energy of the moving micro beam using the modified couple stress theory is expressed as follows

$$U = \frac{1}{2}\int_V (\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij})dV \tag{8}$$

According to the Euler–Bernoulli beam theory, the tensors ε_{ij} and χ_{ij} at any point of beam material are assumed as (Simsek 2010)

$$\varepsilon = \begin{bmatrix} -yw_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \chi = \begin{bmatrix} 0 & -1/2w_{xx} & 0 \\ -1/2w_{xx} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9}$$

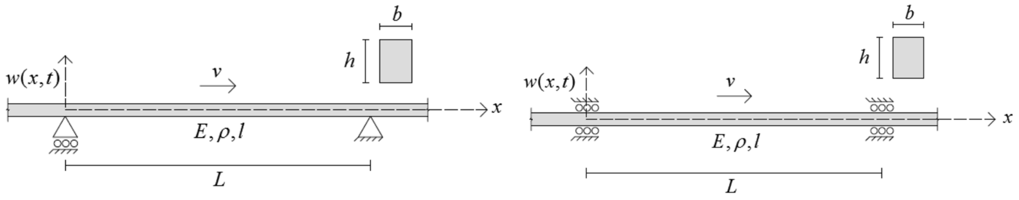
Substituting Eq. (9) in Eqs. (1), (2) and (8) leads to following form of the strain energy in terms of the transverse displacement, w

$$U = \frac{1}{2}\int_0^L ((\lambda + 2\mu)I + \mu l^2 A)(w'')^2 dx \tag{10}$$

where I and A are the moment of inertia and area of the moving beam cross section, respectively. Moreover, by inserting Eqs. (6), (7) and (10) in Eq. (5) the differential equation of transverse vibration for the axially moving micro-beam can be derived as

$$((\lambda + 2\mu)I + \mu l^2 A)\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} + 2\rho A v\frac{\partial^2 w}{\partial x \partial t} + \rho A v^2\frac{\partial^2 w}{\partial x^2} = 0 \tag{11}$$

where w is the unknown transverse displacement of the moving micro-beam within spatial coordinate x and time t. The relations for the evaluation of shear force, $S(x, t)$, and bending moment, $M(x, t)$, at any point of the moving micro-beam are as (Banerjee and Gunawardana 2007, Lee and Jang 2007)



(a) (b)

Fig. 1 Schematic representation of the micro moving beam with: (a) pined-pined and (b) fixed-fixed end conditions

$$S(x,t) = (\lambda + 2\mu)Iw'''' + \rho A v(\dot{w} + vw') \tag{12}$$

$$M(x,t) = (\lambda + 2\mu)Iw'' \tag{13}$$

Schematic representation of the micro moving beam with pined-pined and fixed-fixed end conditions is shown in Fig. 1.

3. Dynamic stiffness matrix method

The harmonic oscillation solution is assumed for the axially moving micro-beam equation with the circular frequency ω, and transverse displacement amplitude $f(x)$, as follows

$$w(x,t) = f(x)e^{i\omega t} \tag{14}$$

Substituting Eq. (14) in Eq. (11) results in an ordinary differential equation as

$$((\lambda + 2\mu)I + \mu l^2 A)\frac{d^4 f(x)}{dx^4} - \rho A\omega^2 f(x) + 2i\rho A v\omega\frac{df(x)}{dx} + \rho A v^2\frac{d^2 f(x)}{dx^2} = 0 \tag{15}$$

The general solution of the above differential equation can be assumed as $f(x) = e^{rx}$ where r can be determined using the algebraic equation shown below

$$((\lambda + 2\mu)I + \mu l^2 A)r^4 - \rho A\omega^2 + 2i\rho A v\omega r + \rho A v^2 r^2 = 0 \tag{16}$$

Considering Eq. (16) four roots may be found for parameter r in terms of ω as seen in Eqs. (17) to (20)

$$r_1 = \frac{1}{2}(iv\alpha - \sqrt{-4\omega\alpha - v^2\alpha^2}) \tag{17}$$

$$r_2 = \frac{1}{2}(iv\alpha + \sqrt{-4\omega\alpha - v^2\alpha^2}) \tag{18}$$

$$r_3 = \frac{1}{2}(-iv\alpha - \sqrt{-4\omega\alpha - v^2\alpha^2}) \tag{19}$$

$$r_4 = \frac{1}{2}(-iv\alpha + \sqrt{4\omega\alpha - v^2\alpha^2}) \tag{20}$$

where

$$\alpha = \sqrt{\frac{\rho A}{(\lambda + 2\mu)I + \mu l^2 A}} \tag{21}$$

Considering these equations, the solution to Eq. (15) may be written as follows

$$f(x) = \sum_{n=1}^{4} C_n e^{r_n x} \tag{22}$$

In the above relation, the unknown coefficients C_n are to be found from the end conditions of the moving beam. In the dynamic stiffness matrix method these coefficients must be related to the end displacements and rotations of beam (W_1, ϕ_1, W_2, ϕ_2). In this regards, Eq. (22) is firstly used to evaluate the expressions of the bending rotation, $\phi(x)$, in the frequency domain as

$$\phi(x) = \sum_{n=1}^{4} C_n r_n e^{r_n x} \tag{23}$$

and then the following relation can be written in the matrix form

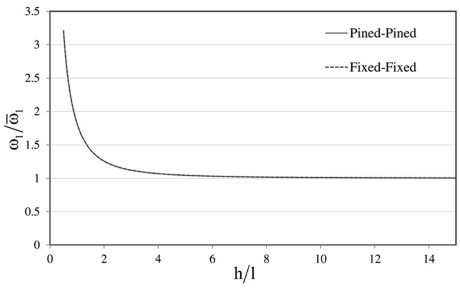
$$\begin{bmatrix} W_1 \\ \phi_1 \\ W_2 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 \\ e^{r_1 L} & e^{r_2 L} & e^{r_3 L} & e^{r_4 L} \\ r_1 e^{r_1 L} & r_2 e^{r_2 L} & r_3 e^{r_3 L} & r_4 e^{r_4 L} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{24}$$

In the next step, Eq. (14) is substituted into Eqs. (12) and (13) to evaluate the end shear force and bending moment, (S_1, M_1, S_2, M_2) in terms of C_n considering the appropriate sign convention from (Banerjee and Gunawardana 2007) as

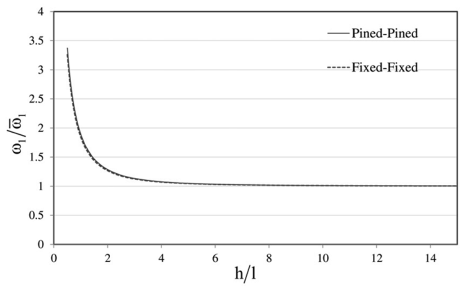
$$\begin{bmatrix} S_1 \\ M_1 \\ S_2 \\ M_2 \end{bmatrix} = (\lambda + 2\mu)I \begin{bmatrix} r_1^3 & r_2^3 & r_3^3 & r_4^3 \\ r_1^2 & r_2^2 & r_3^2 & r_4^2 \\ -r_1^3 e^{r_1 L} & -r_2^3 e^{r_2 L} & -r_3^3 e^{r_3 L} & -r_4^3 e^{r_4 L} \\ -r_1^2 e^{r_1 L} & -r_2^2 e^{r_2 L} & -r_3^2 e^{r_3 L} & -r_4^2 e^{r_4 L} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$+ \rho A v \begin{bmatrix} i\omega + vr_1 & i\omega + vr_2 & i\omega + vr_3 & i\omega + vr_4 \\ 0 & 0 & 0 & 0 \\ -(i\omega + vr_1)e^{r_1 L} & -(i\omega + vr_2)e^{r_2 L} & -(i\omega + vr_3)e^{r_3 L} & -(i\omega + vr_4)e^{r_4 L} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{25}$$

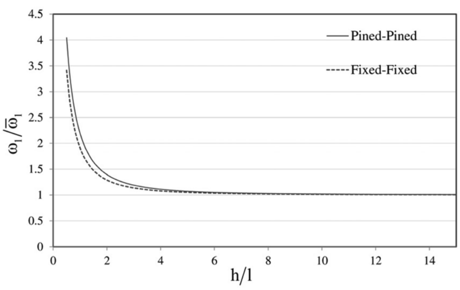
Finally, the dynamic stiffness matrix of the axially moving micro beam, $\mathbf{K}(\omega)$, can be determined using Eqs. (24) and (25)



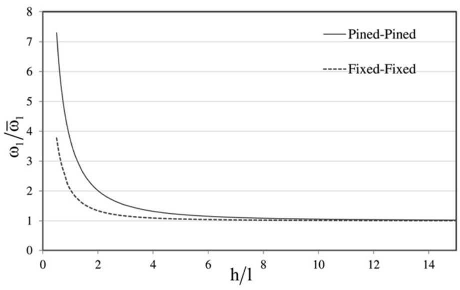
(a)



(b)



(c)



(d)

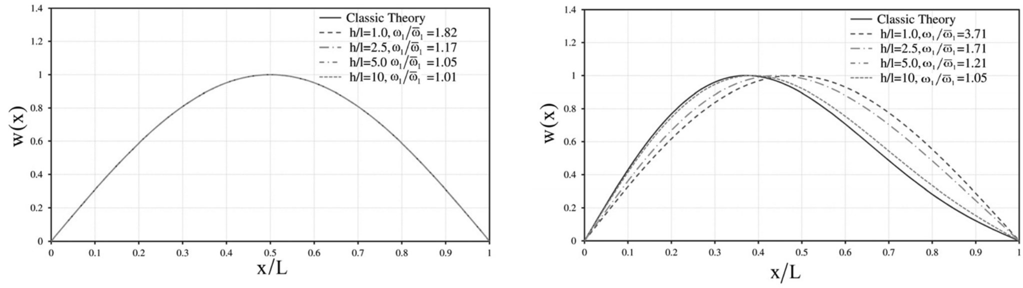
Fig. 2 Variations of $\omega_1/\overline{\omega}_1$ for the rectangular micro beam with $L = 20h$ and $b = 2h$ against h/l for: (a) $v = 0$, (b) $v = 20$ m/s, (c) $v = 40$ m/s and (d) $v = 60$ m/s

$$\mathbf{K}(\omega)\mathbf{U} = \mathbf{F}$$

where **U** and **F** are two vectors containing $(W_1, \phi_1, W_2, \phi_2)$ and (S_1, M_1, S_2, M_2), respectively. The components of $\mathbf{K}(\omega)$ are given explicitly in Appendix A. For the free vibration analysis of the problem of interest, the determinant of $\mathbf{K}(\omega)$ is set to zero after imposing the boundary conditions. This leads to a non-standard eigenvalue problem, which should be solved numerically to evaluate the natural frequencies, ω and fundamental modes. We shall employ this dynamic stiffness matrix in the numerical examples presented in next Section for different boundary conditions.

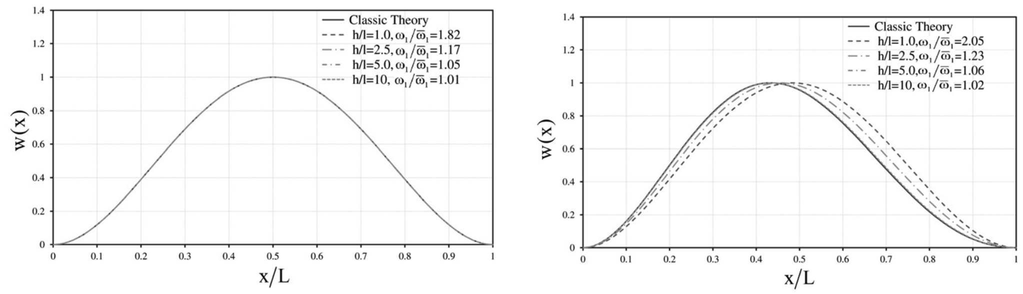
4. Numerical results

The introduced dynamic stiffness matrix was applied to the free vibration analysis of the pined-pined and fixed-fixed end conditions of the moving micro beam. In this regard, a micro-beam with a rectangular cross section and the same material properties from (Simsek 2010) as $\rho = 1220$ kg m^{-3}, $\lambda = 1.65217$ GPa, $\mu = 0.521739$ GPa and $l = 17.6$ μm is considered. The dynamic characteristics of the moving micro beam are computed for various axial velocity and different ratios of the beam thickness (h) to the material length scale parameter (l). In Fig. 2, the variation of the normalized first natural frequency of the moving micro beam with two end conditions against h/l is shown for



(a) (b)

Fig. 3 Variations of the first fundamental mode for the pined-pined rectangular micro beam with $L = 20h$ and $b = 2h$ for different h/l: (a) $v = 0$ and (b) $v = 60$ m/s



(a) (b)

Fig. 4 Variations of the first fundamental mode for the fixed-fixed rectangular micro beam with $L = 20h$ and $b = 2h$ for different h/l: (a) $v = 0$ and (b) $v = 60$ m/s

different axial velocities. In this Figure, $\overline{\omega}_1$ indicates the first natural frequency of the moving and is calculated using the classical elasticity theory ($l = 0$). From Fig. 2, it can be concluded that the first natural frequency of resting beam ($v = 0$) is decreased whereas *h*/*l* increases in a similar trend for both two end conditions. On the other hand, the effect of the moving micro beam size in high axial velocities, by simple supports is more than one with fixed end conditions. It can be also observed that for the *h*/*l* ratios greater than 10, the micro beam becomes size independently. The similar result has also obtained for the natural frequency of Bernoulli-Euler micro-beams at rest (Kong *et al*. 2008).

The size of the micro beam can also change the fundamental modes of the moving beam gradually. Figs. 3 and 4 depict the variation of the first normalized fundamental mode of the beam by pined-pined and fixed-fixed end conditions for different ratios of *h*/*l* and axial velocities.

The natural frequencies of the moving beam decreased due to increasing the axial velocity and therefore the beam may become dynamically unstable (Banerjee and Gunawardana 2007). In this regard, the critical values of axial velocity of the moving beam are computed for different *h*/*l* and as shown in Table 1.

Table 1 The critical axial velocity of the moving micro beam v_{cr} (m/s), with $L = 20h$ and $b = 2h$ and by fixed-fixed and simply supported end conditions for different h/l

h/l	Fixed-fixed	Simply supported
$h/l = 0.50$	432.4398	216.2199
$h/l = 0.75$	305.3015	152.6507
$h/l = 1.00$	245.7248	122.8622
$h/l = 2.00$	169.4839	84.7419
$h/l = 5.00$	140.9295	70.4647
$h/l = 10.0$	136.3631	68.1816

Table 2 The first three normalized natural frequencies ($\hat{\omega}_i = \frac{\omega_i}{L}\sqrt{(\lambda + 2\mu)/\rho}$), of the fixed-fixed and simply supported axially travelling micro beam with $L = 20h$ and $b = 2h$

		Fixed-fixed			Simply supported		
v (m/s)	h/l	$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{\omega}_3$	$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{\omega}_3$
0.00	1.00	0.58864	1.62260	3.18094	0.25967	1.03867	2.33700
	2.00	0.40600	1.11916	2.19399	0.17910	0.71640	1.61190
	5.00	0.33760	0.93060	1.82435	0.14893	0.59570	1.34033
15.00	1.00	0.58698	1.62141	3.17980	0.25735	1.03709	2.33563
	2.00	0.40360	1.11743	3.62518	0.17573	0.71411	1.60990
	5.00	0.33471	0.92852	1.82236	0.14487	0.59294	1.33793
30.00	1.00	0.58202	1.61782	3.17636	0.25034	1.03233	2.33149
	2.00	0.39641	1.11223	2.18735	0.16545	0.70718	1.60390
	5.00	0.32606	0.92226	1.81636	0.13236	0.58460	1.33070
45.00	1.00	0.57375	1.61184	3.17063	0.23846	1.02435	2.32458
	2.00	0.38440	1.10352	2.17904	0.14755	0.69552	1.59386
	5.00	0.31160	0.91175	1.80636	0.10997	0.57045	1.31861
60.00	1.00	0.56215	1.60344	3.16261	0.22129	1.01311	2.31489
	2.00	0.36754	1.09124	2.16739	0.12016	0.67891	1.57975
	5.00	0.29126	0.89689	1.79234	0.07238	0.55015	1.30157

Finally, in Table 2 the first three natural frequencies of the pined-pined and fixed-fixed axially travelling micro-beam are presented for different sets of axially velocity and h/l.

6. Conclusions

We presented a dynamic stiffness matrix method for the axially moving micro beam problem based on the modified couple stress theory. The displacement field of the moving beam was considered using the Euler-Bernoulli beam theory. The extended Hamilton's principle was employed to derive the governing differential equation of the moving micro beam with the constant axially velocity. The influences of the beam size and the axial velocity on the dynamic characteristic of the moving micro beam were investigated. The critical axial velocity of the moving beam is computed for both fixed-fixed and pined-pined end conditions. The numerical results show that how the size of the beam can change the natural frequency and fundamental modes of the moving micro beam.

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Appendix A

The components of the dynamic stiffness matrix $\mathbf{K}(\omega)$, introduced in Section 3 is given here explicitly.

$$\begin{aligned}\mathbf{K}_{11} = (&-d_1(\omega)(r_1r_2(r_1+r_2)\eta - i\rho Av\omega) + d_2(\omega)(r_1r_3(r_1+r_3)\eta - i\rho Av\omega)\\ &- d_3(\omega)(r_2r_3(r_2+r_3)\eta - i\rho Av\omega) - d_4(\omega)(r_1r_4(r_1+r_4)\eta - i\rho Av\omega)\\ &+ d_5(\omega)(r_2r_4(r_2+r_4)\eta - i\rho Av\omega) - d_1(\omega)(r_3r_4(r_3+r_4)\eta - i\rho Av\omega))/D(\omega)\end{aligned} \tag{A-1}$$

$$\begin{aligned}\mathbf{K}_{12} = (&+d_1(\omega)((r_1^2+r_1r_2+r_2^2)\eta + \rho Av^2) - d_2(\omega)((r_1^2+r_1r_3+r_3^2)\eta + \rho Av^2)\\ &+ d_3(\omega)((r_2^2+r_2r_3+r_3^2)\eta + \rho Av^2) + d_4(\omega)((r_1^2+r_1r_4+r_4^2)\eta + \rho Av^2)\\ &- d_5(\omega)((r_2^2+r_2r_4+r_4^2)\eta + \rho Av^2) + d_6(\omega)((r_3^2+r_3r_4+r_4^2)\eta + \rho Av^2))/D(\omega)\end{aligned} \tag{A-2}$$

$$\mathbf{K}_{13} = \eta(+g_1(\omega)r_1 - g_2(\omega)r_2 + g_3(\omega)r_3 - g_4(\omega)r_4)/D(\omega) \tag{A-3}$$

$$\mathbf{K}_{14} = \eta(+g_1(\omega) - g_2(\omega) + g_3(\omega) - g_4(\omega))/D(\omega) \tag{A-4}$$

$$\mathbf{K}_{21} = \eta(+d_1(\omega)r_1r_2 - d_2(\omega)r_1r_3 + d_3(\omega)r_2r_3 + d_4(\omega)r_1r_4 - d_5(\omega)r_2r_4 + d_6(\omega)r_3r_4)/D(\omega) \tag{A-5}$$

$$\begin{aligned}\mathbf{K}_{22} = \eta(&-d_1(\omega)(r_1+r_2) + d_2(\omega)(r_1+r_3) - d_3(\omega)(r_2+r_3)\\ &- d_4(\omega)(r_1+r_4) + d_5(\omega)(r_2+r_4) - d_6(\omega)(r_3+r_4))/D(\omega)\end{aligned} \tag{A-6}$$

$$\mathbf{K}_{23} = \eta(-g_1(\omega)r_1 + g_2(\omega)r_2 - g_3(\omega)r_3 + g_4(\omega)r_4)/D(\omega) \tag{A-7}$$

$$\mathbf{K}_{24} = \eta(+g_1(\omega) - g_2(\omega) + g_3(\omega) - g_4(\omega))/D(\omega) \tag{A-8}$$

$$\begin{aligned}\mathbf{K}_{31} = (&+r_2(r_1-r_3)(r_1-r_4)(r_3-r_4)(r_1+r_3+r_4)\eta e^{(r_1+r_3+r_4)L}\\ &- r_3(r_1-r_4)((r_1-r_2)(r_2-r_4)(r_1+r_2+r_4)\eta - (1-i)\rho Av\omega)e^{(r_1+r_2+r_4)L}\\ &+ r_4(r_1-r_3)((r_1-r_2)(r_2-r_3)(r_1+r_2+r_3)\eta - (1-i)\rho Av\omega)e^{(r_1+r_2+r_3)L}\\ &- r_1(r_3-r_4)((r_2-r_3)(r_2-r_4)(r_2+r_3+r_4)\eta + (1-i)\rho Av\omega)e^{(r_2+r_3+r_4)L})/D(\omega)\end{aligned} \tag{A-9}$$

$$\begin{aligned}\mathbf{K}_{32} = (&-(r_1-r_3)(r_1-r_4)(r_3-r_4)(r_1+r_3+r_4)\eta e^{(r_1+r_3+r_4)L}\\ &+ (r_1-r_4)((r_1-r_2)(r_2-r_4)(r_1+r_2+r_4)\eta - (1-i)\rho Av\omega)e^{(r_1+r_2+r_4)L}\\ &- (r_1-r_3)((r_1-r_2)(r_2-r_3)(r_1+r_2+r_3)\eta - (1-i)\rho Av\omega)e^{(r_1+r_2+r_3)L}\\ &+ (r_3-r_4)((r_2-r_3)(r_2-r_4)(r_2+r_3+r_4)\eta + (1-i)\rho Av\omega)e^{(r_2+r_3+r_4)L})/D(\omega)\end{aligned} \tag{A-10}$$

$$\begin{aligned}\mathbf{K}_{33} = (&+d_1(\omega)(r_3r_4(r_3+r_4)\eta - i\rho Av\omega) - d_2(\omega)(r_2r_4(r_2+r_4)\eta - i\rho Av\omega)\\ &+ d_3(\omega)(r_1r_4(r_1+r_4)\eta - i\rho Av\omega) + d_4(\omega)(r_2r_3(r_2+r_3)\eta - i\rho Av\omega)\\ &- d_5(\omega)(r_1r_3(r_1+r_3)\eta - i\rho Av\omega) + d_6(\omega)(r_1r_2(r_1+r_2)\eta - i\rho Av\omega))/D(\omega)\end{aligned} \tag{A-11}$$

$$\begin{aligned}\mathbf{K}_{34} = &(-d_1(\omega)((r_3^2 + r_3 r_4 + r_4^2)\eta + \rho A v^2) - d_3(\omega)((r_1^2 + r_1 r_4 + r_4^2)\eta + \rho A v^2) \\ &+ d_2(\omega)((r_2^2 + r_2 r_4 + r_4^2)\eta + \rho A v^2 + (1-i)\rho A v\omega/(r_2 - r_4)) \\ &- d_4(\omega)((r_2^2 + r_2 r_3 + r_3^2)\eta + \rho A v^2 + (1-i)\rho A v\omega/(r_2 - r_3)) \\ &- d_6(\omega)((r_1^2 + r_1 r_2 + r_2^2)\eta + \rho A v^2 + (1-i)\rho A v\omega/(r_1 - r_2)) \\ &+ d_5(\omega)((r_1^2 + r_1 r_3 + r_3^2)\eta + \rho A v^2)/D(\omega)\end{aligned} \tag{A-12}$$

$$\begin{aligned}\mathbf{K}_{41} = &\eta(-r_2(r_1 - r_3)(r_1 - r_4)(r_3 - r_4)(r_1 + r_3 + r_4)e^{(r_1+r_3+r_4)L} \\ &+ r_3(r_1 - r_4)((r_1 - r_2)(r_2 - r_4)(r_1 + r_2 + r_4)e^{(r_1+r_2+r_4)L} \\ &- r_4(r_1 - r_3)((r_1 - r_2)(r_2 - r_3)(r_1 + r_2 + r_3)e^{(r_1+r_2+r_3)L} \\ &+ r_1(r_3 - r_4)((r_2 - r_3)(r_2 - r_4)(r_2 + r_3 + r_4)e^{(r_2+r_3+r_4)L})/D(\omega)\end{aligned} \tag{A-13}$$

$$\begin{aligned}\mathbf{K}_{42} = &\eta(+(r_1 - r_3)(r_1 - r_4)(r_3 - r_4)(r_1 + r_3 + r_4)e^{(r_1+r_3+r_4)L} \\ &- (r_1 - r_4)((r_1 - r_2)(r_2 - r_4)(r_1 + r_2 + r_4)e^{(r_1+r_2+r_4)L} \\ &+ (r_1 - r_3)((r_1 - r_2)(r_2 - r_3)(r_1 + r_2 + r_3)e^{(r_1+r_2+r_3)L} \\ &- (r_3 - r_4)((r_2 - r_3)(r_2 - r_4)(r_2 + r_3 + r_4)e^{(r_2+r_3+r_4)L})/D(\omega)\end{aligned} \tag{A-14}$$

$$\mathbf{K}_{43} = \eta(-d_1(\omega)r_3 r_4 + d_2(\omega)r_2 r_4 - d_3(\omega)r_1 r_4 - d_4(\omega)r_2 r_3 - d_5(\omega)r_1 r_3 - d_6(\omega)r_1 r_2)/D(\omega) \tag{A-15}$$

$$\begin{aligned}\mathbf{K}_{44} = &\eta(+d_1(\omega)(r_3 + r_4) - d_2(\omega)(r_2 + r_4) + d_3(\omega)(r_1 + r_4) \\ &+ d_4(\omega)(r_2 + r_3) - d_5(\omega)(r_1 + r_3) + d_6(\omega)(r_1 + r_2))/D(\omega)\end{aligned} \tag{A-16}$$

In the above relations, $\eta = (\lambda + 2\mu) I$ and the auxiliary functions $d_i(\omega)$, $i = 1, \cdots, 6$, $g_i(\omega)$, $i = 1, \cdots, 4$ and $D(\omega)$ are defined as

$$d_1(\omega) = e^{L(r_2+r_3)}(r_2 - r_3)(r_1 - r_4) \tag{A-17}$$

$$d_2(\omega) = e^{L(r_1+r_4)}(r_2 - r_3)(r_1 - r_4) \tag{A-18}$$

$$d_3(\omega) = e^{L(r_1+r_3)}(r_1 - r_3)(r_2 - r_4) \tag{A-19}$$

$$d_4(\omega) = e^{L(r_2+r_4)}(r_1 - r_3)(r_2 - r_4) \tag{A-20}$$

$$d_5(\omega) = e^{L(r_1+r_2)}(r_1 - r_2)(r_3 - r_4) \tag{A-21}$$

$$d_6(\omega) = e^{L(r_3+r_4)}(r_1 - r_2)(r_3 - r_4) \tag{A-22}$$

and

$$g_1(\omega) = e^{L r_1} r_1(r_2 - r_3)(r_2 - r_4)(r_3 - r_4)(r_2 + r_3 + r_4) \tag{A-23}$$

$$g_2(\omega) = e^{Lr_2} r_2 (r_1 - r_2)(r_1 - r_4)(r_2 - r_4)(r_1 + r_2 + r_4) \tag{A-24}$$

$$g_3(\omega) = e^{Lr_3} r_3 (r_1 - r_2)(r_1 - r_4)(r_2 - r_4)(r_1 + r_2 + r_4) \tag{A-25}$$

$$g_4(\omega) = e^{Lr_4} r_4 (r_1 - r_2)(r_1 - r_3)(r_2 - r_3)(r_1 + r_2 + r_3) \tag{A-26}$$

and finally,

$$D(\omega) = d_1(\omega) + d_2(\omega) - d_3(\omega) - d_4(\omega) + d_5(\omega) + d_6(\omega) \tag{A-27}$$