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Approximate formulation for bifurcation buckling loads of axially compressed cylindrical shells with an elastic core

Motohiro Sato* and Kenta Shimazaki

Faculty of Engineering, Hokkaido University, Sapporo 060-8628, Japan

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Abstract. This paper proposes an approximate formulation to estimate the bifurcation buckling loads of cylindrical shells with soft elastic cores under the conditions of axial compression. In general, thin-walled, axially compressed cylindrical shells buckle into a diamond pattern in the elastic range. However, buckling symmetrical with respect to the axis of the cylinder may occur when the cylindrical shell is supported by an elastic medium. By considering this characteristic, we introduce the simplified approximate formulation that can give sufficiently accurate results for the bifurcation buckling loads of cylindrical shells. Moreover the results are compared with the exact buckling loads in order to confirm the accuracy of the proposed approximate formulation.

Keywords: cylindrical shell; bifurcation buckling; elastic medium.

1. Introduction

Thin-walled cylindrical shell structures are widely used in many engineering fields, for example, in offshore, civil, mechanical and aircraft structures, and so on. Moreover in many engineering applications, thin- walled cylindrical shells are often supported by an inner elastic core or outer elastic medium; Practical examples include from “huge” cylindrical tunnel liners (Croll 2001), pipe-in-pipe systems (Sato and Patel 2007, Sato *et al.* 2008, Arjomandi and Taheri 2010, 2011a, 2011b) to “quite small” carbon nanotubes (Ru 2001, He *et al.* 2005, Sato and Shima 2009). The analytical investigations on the elastic buckling of a thin cylindrical shell filled with an elastic core have been carried out for a variety of loading configurations (Yao 1962, Seide 1962, Yabuta 1980, Karam and Gibson 1995, Hutchinson and He 2000, Dawson and Gibson 2007). In the practical structural design of such structures, one of the most important issues is to estimate the buckling pressure precisely. Hence, the effects of interaction between the shells and the inner elastic core material (or surrounding elastic mediums) on the buckling behavior are of great interest. In this case, however, the structural behavior of the inner core is governed by three-dimensional theory of elasticity; hence the interactive buckling behavior and its governing equation to describe such a phenomenon based on the thin cylindrical shell theory are quite complex.

From this point of view, we offer a simplified approximate formulation for obtaining the axially

* Corresponding author, Dr., Eng., E-mail: tayu@eng.hokudai.ac.jp

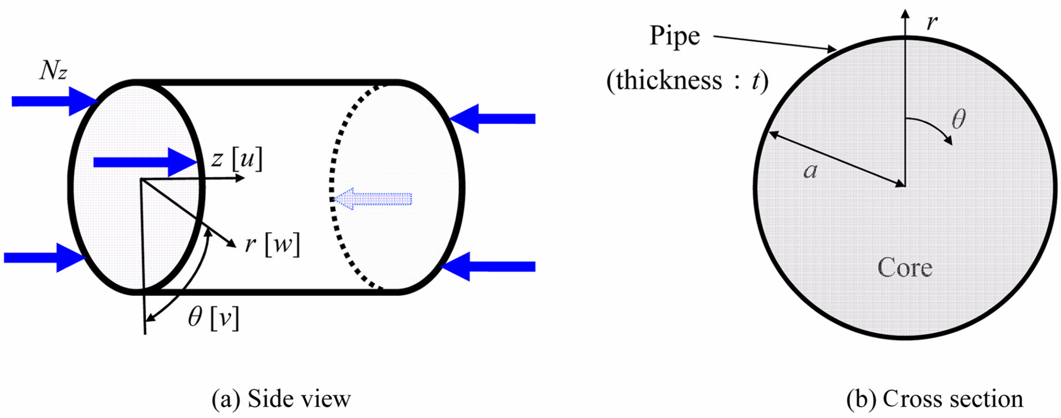

Fig. 1 Analytical model

compressed buckling pressure in this study. The simplified critical elastic buckling pressure obtained here is shown to give sufficiently accurate results compared with the exact values obtained by the formulation introduced by Yao (1962). It can be calculated without much difficulty and is thus very effective and convenient for most practical purposes in determining the critical elastic buckling pressure of axially compressed cylindrical shells.

2. Analytical model

Fig. 1 shows the analytical model considered in this study. The system is idealized as a thin cylindrical shell with Young's modulus E_p, Poisson's ratio ν_p, shell radius a and thickness h, under the axial compression, in contact with an homogeneous and isotropic elastic core material having Young's modulus E_c, Poisson's ratio ν_c. Here we consider the cylindrical shell under the axial compression N_z.

3. Exact formulation

Here we briefly introduce the exact formulation by Yao (1962). The axially compressed buckling load has been solved by using the differential equations for equilibrium of the shell. Moreover, the stress function has been used to account for the spring constant of the compliant core. Thus, the displacements and stress components of a point in the elastic core can be determined as follows (Yao 1962)

$$\begin{aligned} 2G_c \cdot u &= -n\cos nz \sin m\theta \{ c_1 I_m(nr) + c_2 K_m(nr) \\ &\left. + \frac{r}{a}[c_3 I_{m+1}(nr) + c_4 K_{m+1}(nr) + c_5 I_{m-1}(nr) + c_6 K_{m-1}(nr)] \right\} \end{aligned} \tag{1a}$$

$$2G_c \cdot v = -\sin nz \cos m\theta \left\{ \frac{m}{r}[c_1 I_m(nr) + c_2 K_m(nr)] \right.$$

$$\left. + \frac{m+2\varsigma}{a}[c_3 I_{m+1}(nr) + c_4 K_{m+1}(nr)] + \frac{m-2\varsigma}{a}[c_5 I_{m-1}(nr) + c_6 K_{m-1}(nr)] \right\} \tag{1b}$$

$$2G_c \cdot w = -\sin nz \sin m\theta \{ c_1 I_m'(nr) + c_2 K_m'(nr)$$

$$+ \frac{r}{a}[c_3 I_{m+1}'(nr) + c_4 K_{m+1}'(nr) + c_5 I_{m-1}'(nr) + c_6 K_{m-1}'(nr)]$$

$$\left. - \frac{3-4\nu_c}{a}[c_3 I_{m+1}(nr) + c_4 K_{m+1}(nr) + c_5 I_{m-1}(nr) + c_6 K_{m-1}(nr)] \right\} \tag{1c}$$

$$\tau_{rz} = -n \cos nz \sin m\theta \{ c_1 I_m'(nr) + c_2 K_m'(nr)$$

$$+ \frac{r}{a}[c_3 I_{m+1}'(nr) + c_4 K_{m+1}'(nr) + c_5 I_{m-1}'(nr) + c_6 K_{m-1}'(nr)]$$

$$\left. - \frac{\varsigma-1}{a}[c_3 I_{m+1}(nr) + c_4 K_{m+1}(nr) + c_5 I_{m-1}(nr) + c_6 K_{m-1}(nr)] \right\} \tag{1d}$$

$$\tau_{r\theta} = -\sin nz \cos m\theta \left\{ \frac{m}{r}[c_1 I_m'(nr) + c_2 K_m'(nr)] + \frac{m+\varsigma}{a}[c_3 I_{m+1}'(nr) + c_4 K_{m+1}'(nr)] \right.$$

$$+ \frac{m-\varsigma}{a}[c_5 I_{m-1}'(nr) + c_6 K_{m-1}'(nr)] - \frac{m}{r^2}[c_1 I_m(nr) + c_2 K_m(nr)]$$

$$\left. - \frac{\varsigma(m+1)}{r \cdot a}[c_3 I_{m+1}(nr) + c_4 K_{m+1}(nr)] - \frac{\varsigma(m-1)}{r \cdot a}[c_5 I_{m-1}(nr) + c_6 K_{m-1}(nr)] \right\} \tag{1e}$$

$$\sigma_r = \sin nz \sin m\theta \left\{ c_1 \left[-\left(n^2 + \frac{m^2}{r^2}\right) I_m(nr) + \frac{1}{r} I_m'(nr) \right] + c_2 \left[-\left(n^2 + \frac{m^2}{r^2}\right) K_m(nr) + \frac{1}{r} K_m'(nr) \right] \right.$$

$$+ \frac{1+\varsigma}{a}[c_3 I_{m+1}'(nr) + c_4 K_{m+1}'(nr) + c_5 I_{m-1}'(nr) + c_6 K_{m-1}'(nr)]$$

$$- \frac{n^2 r^2 + (m+1)^2 - (2-\varsigma)(m+1)}{r \cdot a}[c_3 I_{m+1}(nr) + c_4 K_{m+1}(nr)]$$

$$\left. - \frac{n^2 r^2 + (m-1)^2 + (2-\varsigma)(m-1)}{r \cdot a}[c_5 I_{m-1}(nr) + c_6 K_{m-1}(nr)] \right\} \tag{1f}$$

in which u, v, and w are the axial, circumferential and outward normal displacements, $\varsigma = 2(1-\nu_\varsigma)$, $G_c = E_c/2(1+\nu_c)$, $n = k\pi/L$ (L is the length of the cylinder), $\alpha = 2(1-\nu_c)$, $h = t/1-\nu_p^2$, m is the number of buckling waves in the circumferential direction, k is the number of half buckling waves in the z direction, I_m and K_m are the modified Bessel functions of first and second kind of order m, respectively.

We suppose that the governing equations for the deformation of thin cylindrical shells are (Yao 1962)

$$\left[\frac{\partial^2 u}{\partial z^2} + \frac{(1-\nu_p)(1+\lambda)}{2a^2}\frac{\partial^2 u}{\partial \theta^2}\right] + \frac{1+\nu_p}{2a}\frac{\partial^2 v}{\partial\theta\partial z} + \frac{\nu_p}{a}\frac{\partial w}{\partial z} + \frac{f_z}{K_p t} = 0 \tag{2a}$$

$$\left(\frac{1+\nu_p}{2a} + \frac{N_z}{K_p t a}\right)\frac{\partial^2 u}{\partial\theta\partial z} + \left[\frac{1}{a^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{1-\nu_p}{2}\frac{\partial^2 v}{\partial z^2}\right] + \left[\frac{1}{a^2}\frac{\partial w}{\partial\theta} - \lambda\frac{\partial^3 w}{\partial\theta\partial z^2}\right] + \frac{f_\theta}{K_p t} = 0 \tag{2b}$$

$$\frac{\nu_p}{a}\frac{\partial u}{\partial z} + \frac{1}{a^2}\frac{\partial v}{\partial\theta} + \left[\lambda a^2\frac{\partial^4 w}{\partial z^4} + 2\lambda\frac{\partial^4 w}{\partial\theta^2\partial z^2} + \frac{\lambda}{a^2}\frac{\partial^4 w}{\partial\theta^4} + \frac{2\lambda}{a^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{N_z}{K_p t}\frac{\partial^2 w}{\partial z^2} + \frac{1+\lambda}{a^2}w\right] - \frac{f_r}{K_p t} = 0 \tag{2c}$$

where $K_p = E_p/(1-\nu_p^2)$, $\lambda = t^2/12a^2$ and f_z, f_θ, f_r are the restoring forces which can be determined from the stresses in the core derived in Eqs. (1a)-(1f).

4. Simplified formulation

4.1 Evaluation of the core stiffness

The fact that the displacement in the core at $r = 0$ is equal to 0 leads

$$c_2 = c_4 = c_6 = 0 \tag{3}$$

Thus, the relationships between the shell displacements (1) can be rewritten as

$$\begin{Bmatrix} u(a)/\cos nz\sin m\theta \\ v(a)/\sin nz\cos m\theta \\ w(a)/\sin nz\sin m\theta \end{Bmatrix} = (X(a))\cdot\begin{Bmatrix} c_1 \\ c_3 \\ c_5 \end{Bmatrix} \tag{4a}$$

$$\begin{Bmatrix} \tau_{rz}/\cos nz\sin m\theta \\ \tau_{r\theta}/\sin nz\cos m\theta \\ \sigma_r/\sin nz\sin m\theta \end{Bmatrix} = (Y(a))\cdot\begin{Bmatrix} c_1 \\ c_3 \\ c_5 \end{Bmatrix} \tag{4b}$$

Moreover, when we consider the "Winkler foundation model", only the core rigidity in the radial direction should be modeled, namely

$$\begin{Bmatrix} u(a)/\cos nz\sin m\theta \\ v(a)/\sin nz\cos m\theta \\ w(a)/\sin nz\sin m\theta \end{Bmatrix} = (X(a))\cdot\begin{Bmatrix} c_1 \\ c_3 \\ c_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{5}$$

By solving Eq. (5) and substituting the solutions into Eq. (4b) gives the following for the core stress

$$S = \sigma_r(a) = \frac{-2G_c[(1+\alpha)naI_0(na)^2 - 2naI_1(na)^2 + I_0(na)\{2(1-\alpha)I_1(na) + (1+\alpha)naI_2(na)\}]}{a[naI_0(na)^2 - 2naI_1(na)^2 + I_0(na)\{2(1-2\alpha)I_1(na) + naI_2(na)\}]} \times \sin nz \sin m\theta \tag{6}$$

Clearly, from Eq. (6) the modulus of the "Winkler foundation" due to the elastic core can be obtained.

4.2 Simplified buckling pressure

From the exact analysis mentioned above, we can find that axisymmetric buckling mode ($m = 0$) occurs when a cylindrical shell is supported by a relatively soft elastic core. Thus, we assume that the cylindrical shell displacements can be expressed as follows

$$u = 0$$

$$v = 0$$

$$w = A \sin m\theta \sin nz \tag{7}$$

Substituting Eq. (7) and the restoring force in the radial direction into Eq. (2c) gives

$$n^4 + \frac{12}{a^2t^2} + \frac{1}{a^4} - \frac{12n^2N_{cr}}{E_pht^2}$$
$$\frac{-24G_c[(1+\alpha)naI_0(na)^2 - 2naI_1(na)^2 + I_0(na)\{2(1-\alpha)I_1(na) + (1+\alpha)naI_2(na)\}]}{aE_pht^2[naI_0(na)^2 - 2naI_1(na)^2 + I_0(na)\{2(1-2\alpha)I_1(na) + naI_2(na)\}]} = 0 \tag{8}$$

By solving the above equation under the assumption that a is much larger than t, the buckling load can be derived as

$$N_{cr} = \frac{E_pht^2}{12n^2}\left[-n^4 - \frac{12}{a^2t^2} + \frac{24G_c[(1+\alpha)naI_0(na)^2 - 2naI_1(na)^2 + I_0(na)\{2(1-\alpha)I_1(na) + (1+\alpha)naI_2(na)\}]}{aE_pht^2[naI_0(na)^2 - 2naI_1(na)^2 + I_0(na)\{2(1-2\alpha)I_1(na) + naI_2(na)\}]}\right] \tag{9}$$

where

$$I_2(na) = I_0(na) - \frac{2}{na}I_1(na) \tag{10a}$$

and the assumption that the wave number k is large gives the following simplifications

$$I_1(na)/I_0(na) \to 1 \tag{10b}$$

$$I_2(na)/I_0(na) \to 1 - \frac{2}{na} \tag{10c}$$

From Eqs. (9) and (10), we obtain the normalized form as

$$\frac{N_{cr}}{K_p t} = \frac{1}{n^2 a^2} - \frac{2G_c}{K_p n^2 a} + \frac{G_c}{K_p n} + \frac{n^2 t^2}{12}$$

$$= \frac{1}{\pi^2 k^2 (a/L)^2} - \frac{2G_c}{\pi^2 K_p k^2 (t/a)(a/L)^2} + \frac{G_c}{\pi K_p k (t/a)(a/L)} + \frac{\pi^2 k^2 (t/a)^2 (a/L)^2}{12} \tag{11}$$

Eq. (11) has a minimum value when

$$\frac{\partial N_{cr}}{\partial k} = 0 \tag{12}$$

Thus we have the equations below

$$24G_c - 12K_p(t/a) - 6\pi G_c(a/L)k + \pi^4 K_p (a/L)^4 (t/a)^3 k^4 = 0 \tag{13}$$

From Eqs. (13) and (11), the wave number associated with the lowest buckling load and the corresponding normalized simplified critical load can be obtained by ignoring the first term of Eq. (13), which results in the following explicit form

$$k_{cr} = \frac{3^{1/4}\sqrt{2}(-2a^3 G_c + a^2 K_p t)^{1/4}}{\pi K_p^{1/4}(a/L)t^{3/4}} \tag{14}$$

$$\frac{N_{cr}}{K_p t} = \frac{3^{1/4}\sqrt{2}a^{3/2}G_c(-2aG_c + K_p t)^{1/4}t^{1/2} - 4aG_c K_p^{1/4} t^{5/4} + 2K_p^{5/4} t^{9/4}}{2\sqrt{3}K_p^{3/4} t^{3/4}(-2aG_c + K_p t)^{1/2} a}$$

$$= \frac{3^{1/4}\sqrt{2}a^{7/4}G_c K_p^{1/4}(-2(G_c/K_p) + (t/a))^{1/4} t^{1/2} - 4aG_c K_p^{1/4} t^{5/4} + 2K_p^{5/4} t^{9/4}}{2\sqrt{3}K_p^{5/4} t^{3/4}(-2(G_c/K_p) + (t/a))^{1/2} a^{3/2}}$$

$$= \frac{(3/4)^{1/4}(t/a)^{-1/4}(G_c/K_p)(-2(G_c/K_p) + (t/a))^{1/4} - 2(G_c/K_p)(t/a)^{1/2} + (t/a)^{3/2}}{\sqrt{3}(-2(G_c/K_p) + (t/a))^{1/2}} \tag{15}$$

5. Accuracy of the simplified formulation

Fig. 2 shows the comparison of the ratio of the simplified buckling load to the exact one. The nondimensional length parameters are set as $b/a = 0.5$ and $a/L = 0.1$; the Poisson's ratio for the core ν_C and the pipe ν_P take the values 0.4 and 0.3, respectively. We can see from this figure that the exact and simplified buckling pressures agree well with each other. Especially when the shell is relatively thick and/or the core material is relatively soft, the simplified Eq. (15) has high accuracy. This is because the formulation shown here is based on the simplification of the core stiffness.

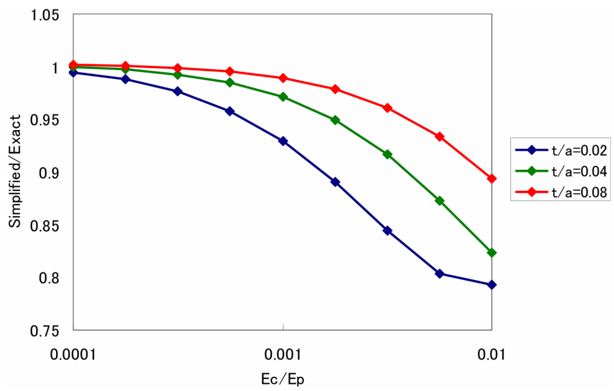

Fig. 2 Comparison between the exact and simplified buckling pressures

These comparisons confirm the validity of the proposed simplified approximations for the axially compressed buckling pressure.

6. Conclusions

This paper presents a theoretical formulation for the bifurcation buckling loads of cylindrical shells with an elastic core under the axial compression. In this investigation, a simplified formula for estimating the elastic buckling load was derived. The simplified formula obtained from this study is shown to give sufficiently accurate results compared with the exact solution. It can be calculated without much difficulty and is thus quite convenient for most practical purposes in determining the critical elastic buckling loads for cylindrical shells with an elastic core.

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