Interaction and Multiscale Mechanics, Vol. 1, No. 3 (2008) 329-337
DOI: http://dx.doi.org/10.12989/imm.2008.1.3.329

Weak forms of generalized governing equations in theory of elasticity

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(Received February 21, 2008, Accepted July 8, 2008)

Abstract. This paper presents the derivation of the generalized governing equations in theory of elasticity, their weak forms and the some applications in the numerical analysis of structural mechanics. Unlike the differential equations in classical elasticity theory, the generalized equations of the equilibrium and compatibility equations presented here take the form of integral equations, and the generalized equilibrium equations contain the classical differential equations and the boundary conditions in a single equation. By using appropriate test functions, the weak forms of these generalized governing equations can be established. It can be shown that various variational principles in structural analysis are merely the special cases of these weak forms of generalized governing equations in elasticity. The present weak forms of elasticity equations extend greatly the choices of the trial functions for approximate solutions in the numerical analysis of various engineering problems. Therefore, the weak forms of generalized governing equations in elasticity provide a powerful modeling tool in the computational structural mechanics.

Keywords: elasticity; generalized governing equations; weak form; variational principles; finite element method; computational mechanics.

1. Introduction

The governing equations in theory of elasticity are a group of equations to be satisfied pointwise within the domain or on the boundaries of the body in consideration. As a result, very few engineering problems can be analytically solved by theory of elasticity directly. Instead of using these differential equations, engineers resort to various methods to seek approximate solutions, such as weighted residuals and various variational principles. In the computational structural mechanics, none of the popular computer-based numerical methods, such as the finite element method, the boundary element method and the meshless method, is directly based on the differential equations in elasticity. But, all these numerical methods are based on various variational principles, or weighted residuals, or other variations of the classical governing equations (Hughes 1987, Atluri 2005).

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Although these variations of the classical governing equations have some advantages compared with the classical elasticity equations, they still possesses some unnecessary restricts on the choices of the trail functions for the approximate solutions. For instance, in the thin plate bending analysis, the minimum potential energy principles reduce the differential order from the forth-order in the equilibrium equation to the second-order in the strain energy, but it still requires a C^1-continuity displacement across the element common boundaries. Such a restrict results in many difficulties in the element formulations of plate bending analysis. On the other hand, some non-conforming plate elements are quite reliable and accurate. This means that both the theory of elasticity and the present variational principles should be modified to provide a more powerful theoretical foundation for the computational structural mechanics (Tang *et al* 2001, Atluri 2005).

The objective of this paper is to present the generalized governing equations in elasticity, their weak forms as well as the some examples of applications in the numerical analysis of structural mechanics. The study indicates that the new theory of elasticity presented here can broaden greatly the choices of trial functions in seeking approximate solutions. Therefore, the weak forms of generalized governing equations in theory of elasticity would provide a more powerful modeling tool in the computational structural mechanics.

2. Governing equations in theory of elasticity

This study confines to the linear elasticity. If let x_i (i = 1,2,3) denotes the rectangular Cartesian coordinates, then the displacement, strain and stress fields as well body forces can be expressed, respectively, as u_i, ε_{ij}, σ_{ij} and f_i with i, j = 1,2,3. For a boundary value problem, theory of elasticity gives the governing equations of a body with domain V and boundary $\overline{S}$ depicted in Fig. 1 as

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 \qquad \text{in } V \tag{1}$$

$$\sigma_{ij}\, n_j = \bar{t}_i \quad \text{on } \overline{S}_\sigma \tag{2}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \text{in } V \tag{3}$$

$$u_i = \bar{u}_i \qquad \text{on } \overline{S}_u \tag{4}$$

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} \text{ or } \varepsilon_{ij} = \frac{\partial W}{\partial \sigma_{ij}} \qquad \text{in } V \tag{5}$$

Where $\overline{S}_\sigma$ and $\overline{S}_u$ represent, respectively, the force boundary and the displacement boundary on $\overline{S}$ respectively; $\bar{t}_i$ are the given tractions on $\overline{S}_\sigma$, n_j (j = 1,2,3) are the direction cosines of a point on $\overline{S}_\sigma$; $\bar{u}_i$ are the given displacements on $\overline{S}_u$; $U(\varepsilon_{ij})$ denotes the strain energy density in terms of strains ε_{ij}, and $W(\sigma_{ij})$ denotes the complementary energy density in terms of stresses σ_{ij}. Although the strain energy U and complementary energy W are identical in the case of linear deformations, they have different independent variables and the choice of the independent variables is very useful in the multi-field finite element formulations.

It is worthwhile to point out that Eqs. (1-5) are required to be satisfied pointwise in V or on $\overline{S}_\sigma$ and $\overline{S}_u$; and the equilibrium equations in the domain V and the traction conditions on the boundary

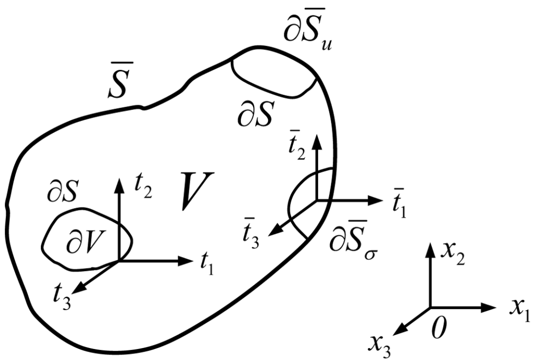

Fig. 1 A typical continuum

$\bar{S}_\sigma$ have separate equations as indicated by Eq. (1) and Eq. (2). The same are true for the displacement compatibility conditions given in Eqs. (3) and (4).

3. Generalized governing equations

3.1 Generalized equilibrium equations

Let us consider the static equilibrium of a subdomain ∂V with surface ∂S taken from inside of the domain V shown in Fig. 1. The forces associated with ∂V are stresses σ_{ij} and body force f_i in ∂V and the tractions t_i on ∂S. The tractions t_i at a point on ∂S are of the form:

$$t_i = \sigma_{ij}\, n_j, \qquad (i, j = 1,2,3) \tag{6}$$

In stead of the equilibrium equations to be satisfied pointwise given in Eq. (1), now let us consider the equilibrium over the subdomain ∂V as a whole, which is called as generalized equilibrium here. For instance, the equilibrium of ∂V in the direction-x_1 is of the form

$$\oint\!\!\int_{\partial S} t_1 ds + \iiint_{\partial V} f_1 dv = 0 \tag{7}$$

in which the second integral is a volume integration in subdomain ∂V, and the first one is the surface integration over ∂S, the surface of ∂V. The generalized equilibrium above is in an integral form as opposed to the differential form of equilibrium equations in theory of elasticity defined in Eq. (1).

When a part of the surface ∂S of subdomain in ∂V belongs to the traction boundary $\bar{S}_\sigma$ as shown in Fig. 1, represented by $\partial\bar{S}_\sigma$, Eq. (7) should be modified as

$$\oint\!\!\int_{\partial S} t_1 ds + \iint_{\partial\bar{S}_\sigma} \bar{t}_1 ds + \iiint_{\partial V} f_1 dv = 0 \tag{8}$$

where $\bar{t}_1$ is the given traction in the direction-x_1 on $\bar{S}_\sigma$. Eq. (7) and Eq. (8) can be written into a unified equation as

$$\oint\!\!\int_{\partial S} t_1 ds + \iint_{\partial \bar{S}_\sigma} (\bar{t}_1 - t_1) ds + \iiint_{\partial V} f_1 dv = 0 \tag{9}$$

Eq (9) is valid for the whole domain of V. Using Green theorem, then Eq. (9) can be recast as

$$\iiint_V \left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 \right) dv + \iint_{\partial \bar{S}_\sigma} (\bar{t}_1 - t_1) ds = 0 \tag{10}$$

By considering the equilibrium in the direction-x_2 and direction-x_3 in a similar way, we will have two more equations similar to Eq. (10). These three equilibrium equations over the whole domain of V can be written as

$$\iiint_V (\sigma_{ij,j} + f_i) dv + \iint_{\bar{S}_\sigma} (\bar{t}_1 - t_1) ds = 0, \qquad (i, j = 1,2,3) \tag{11}$$

As a special case, the quantities in each integral above are zero at every point, then the first term of Eq. (11) leads to the classical equilibrium equations defined by Eq. (1) and the second part of Eq. (11) gives

$$t_i = \bar{t}_1 \quad \text{on } \bar{S}_\sigma \qquad (i = 1,2,3) \tag{12}$$

Obviously, Eq. (12) is the boundary conditions on tractions defined by Eq. (2). Therefore, the present generalized equilibriums are not only the integral form of the classical equilibrium equations, but also capable of combining the equilibriums and the traction boundary conditions in a unified equation. As a result, the surface integration in Eq. (12) would results in the so-called natural boundary automatically for all loading cases.

3.2 Generalized deformation and compatibility equations

Eq. (3) is the strain and displacement relations that are required to be satisfied pointwise in V. However in the case of one-dimensional structures, the integration of the axial strain along the axis of a segment gives the axial deformation of the segment. As a result, the strain and displacement relation in one-dimensional problem can be enforced for a segment as a whole rather than at every point along the segment as presented by Shi and Atluri (1988). By a similar way, the strain and displacement relations in Eq. (3) can be enforced over a domain for the three-dimensional deformation. For example, along the direction-x_1 of a subdomain ∂V not containing any part of the displacement boundary $\bar{S}_u$, we have

$$\iiint_{\partial V} \varepsilon_{11} dv = \iiint_{\partial V} \frac{\partial u_1}{\partial x_1} dV = \oint\!\!\int_{\partial S} u_1 n_1 ds \tag{13}$$

The Green theorem is used in the equation above. The surface integration over ∂S in Eq. (13) gives the generalized deformation in the direction-x_1 in ∂V. When a part of ∂S belongs to $\bar{S}_u$ with $\bar{u}_i$, denoted by $\partial \bar{S}_u$, Eq. (13) takes the form

$$\iiint_{\partial V} \varepsilon_{11} dv = \oint\!\!\int_{\partial S} u_1 n_1 ds + \iint_{\partial \bar{S}_u} (\bar{u}_1 - u_1) n_1 ds \tag{14}$$

Utilizing the Green theorem for the first term at the right-hand side of Eq. (14) and rearranging the results, we have

$$\iiint_{\partial V} \left(\frac{\partial u_1}{\partial x_1} - \varepsilon_{11} \right) dv + \iint_{\partial \bar{S}_u} (\bar{u}_1 - u_1) n_1 ds = 0 \tag{15}$$

Eq. (15) is valid for any subdomain ∂V of a continuum V. Considering the generalized displacements associated with other strain components over the whole domain V, we have

$$\iiint_V \left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - 2\varepsilon_{ij}\right] dv + \iint_{\bar{S}_u} [(\bar{u}_i - u_i)n_i + (\bar{u}_j - n_j)n_j] ds = 0$$

$$(i, j{=}1,2,3 \textit{ and no summation}) \tag{16}$$

Eq 16 contains the pointwise strain-displacement relations and boundary conditions as a special case when the quantities in each integral in Eq. (16) are zero at every point, in which the first term of Eq. (16) gives the strain-displacement relation defined in Eq. (3), and the second term leads to the displacement boundary conditions in Eq. (4). Hence, the generalized compatibility equation defined in Eq. (16) combines the strain-displacement relations and the displacement boundary conditions in a unified equation.

4. Weak form of generalized governing equations

The generalized equilibrium equations in Eq. (11) change the equilibrium from pointwise to over a domain as a whole. But it still requires the stresses to be differentiable, which results in stronger restricts on the choices of trial functions for approximate solutions. Therefore, the continuity requirements on the trial functions should be reduced by using test functions to the generalized governing equations. Let q_i (i = 1,2,3) be the test functions, Eq. (11) becomes

$$\iiint_V (\sigma_{ij,j} + f_i) q_i dv + \iint_{\bar{S}_\sigma} (\bar{t}_1 - t_1) q_i ds = 0 \qquad (i, j = 1,2,3 \ \& \textit{ no summation on } i) \tag{17}$$

Utilizing the Green Theorem, the equation above can be rewritten as

$$\oint\!\!\int_S t_1 q_i ds + \iint_{\bar{S}_\sigma} (\bar{t}_1 - t_1) q_i ds + \iiint_V f_i q_i dv - \iiint_V \sigma_{ij} q_{i,j} dv_i = 0$$

$$(i = 1,2,3 \ \& \textit{ no summation}) \tag{18}$$

The equation above is called as the weak form of the generalized equilibrium equations. The wording of weak form here has two meanings, one is that the generalized equilibrium equations are satisfied under a weighting function, the other is that the continuity requirement on the trial function σ_{ij} are reduced by using the Green theorem.

Using test functions p_{ij} (i,j = 1,2,3), Eq. (16) yields

$$\iiint_V \left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - 2\varepsilon_{ij})\right] p_{ij} dv + \iint_{\bar{S}_u} [(\bar{u}_i - u_i)n_i + (\bar{u}_j - n_j)n_j] p_{ij} ds = 0$$

$$(i, j = 1,2,3, \ \& \textit{ no summation}) \tag{19}$$

Integrating by parts, Eq. (19) leads to

$$\oint\!\!\int_S u_i n_i p_{ii} ds - \iiint_V [u_i p_{ii,i} + \varepsilon_{ii} p_{ii}] dv + \iint_{\bar{S}_u} (\bar{u}_i - u_i) n_i p_{ii} ds = 0$$

$$(i = 1,2,3 \ \& \textit{ no summation}) \tag{20}$$

$$\oint\!\!\int_S (u_i n_j + u_j n_i) p_{ij} ds - \iiint_V [(u_i p_{ij,j} + u_j p_{ij,i}) + 2\varepsilon_{ij} p_{ij}] dv + \iint_{\bar{S}_u} [(\bar{u}_i - u_i) n_j + (\bar{u}_j - u_j) n_i] p_{ij} ds = 0$$

$$(i, j = 1,2,3,\ i \neq j\ \&\ no\ summation) \quad (21)$$

Eq. (17), (19) or Eqs. (18), (20) and (21) are the weak forms of the generalized governing equations. These equations together with Eq. (5) would lead to more choices for the trial functions of the approximate solutions in computational structural mechanics, and their features and some applications will be discussed in the sections follows.

5. Variational principles and the weak forms of generalized governing equations

The generalized equilibrium equations in Eq. (18) have three components in the direction-x_1 with $i = 1,2,3$ respectively. If choosing the variations of displacements, δu_i, as the test functions, i.e.

$$q_i = \delta u_i \qquad (i = 1,2,3) \tag{22}$$

then the three generalized equilibrium equations have the form of energy, and is able to be added into a single equation as

$$\iiint_V \sigma_{ij} (\delta u_i)_{,j} dv - \iiint_V f_i \delta u_i dv - \iint_{\bar{S}_\sigma} \bar{t}_1 \delta u_i ds = 0 \qquad (i, j = 1,2,3) \tag{23}$$

in which $\delta u_i = 0$ on $\bar{S}_u$ is used. Utilizing the following relationship

$$\frac{\partial(\delta u_i)}{\partial x_j} + \frac{\partial(\delta u_j)}{\partial x_i} = \delta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 2\delta\varepsilon_{ij}$$

then Eq. (23) gives

$$\iiint_V \sigma_{ij} \delta\varepsilon_{ij} dv - \iiint_V f_i \delta u_i dV - \iint_{\bar{S}_\sigma} \bar{t}_1 \delta u_i ds = 0 \qquad (i, j = 1,2,3) \tag{24}$$

Eq. (24) has the same statement as the virtual work principle. Therefore, the generalized equilibrium equations given by Eq. (18) include the virtual work principle as a special case when the test functions defined in Eq. (22) are used.

Similarly, the test functions p_{ij} for the generalized compatibility conditions can be chosen as

$$p_{ij} = \delta\sigma_{ij} \tag{25}$$

Then, the quantities in Eq. (20) and (21) are also of the energy form. Adding together these quantities gives

$$\iiint_V \varepsilon_{ij} \delta\sigma_{ij} dv - \iint_{\bar{S}_u} \bar{u}_i \delta t_i ds = 0 \qquad (i, j = 1,2,3) \tag{26}$$

which is the virtual complementary principle. Hence, the generalized compatibility equations given by Eqs. (20) (21) contain the virtual complementary principle as a special case when the test functions in Eq. (25) are used.

Any approximate solutions have to satisfy the weak forms of the generalized governing equations given by Eqs. (18), (20) and (21). But if one of these equations is satisfied *a priori*, then only one condition needs to be enforced. The displacement field satisfying the strain-displacement relation in Eq. (3) is called as the admissible displacements. Taking the admissible displacements u_i as the independent variables for the trial function, and letting the constitutive equation in Eq. (5) be satisfied *a priori*, then Eq. (24) gives

$$\iiint_V [\sigma_{ij}\delta\,\varepsilon_{ij} - f_i\delta u_i]dv - \iint_{\bar{S}_\sigma} \bar{t}_1\, \delta u_i ds \;=\; \iiint_V \left[\frac{\partial U}{\partial \varepsilon_{ij}}\delta\varepsilon_{ij} - f_i\delta u_i\right]dv - \iint_{\bar{S}_\sigma} \bar{t}_1\, \delta u_i ds \;=\; 0 \qquad (i, j = 1,2,3) \qquad (27)$$

Since the strain-displacement relation in Eq. (3) is satisfied exactly, the strain energy density U in the equation above can be written in terms of displacements u_i. As a result, Eq. (24) finally leads to

$$\delta\left\{\iiint_V [(U(u_i) - f_i u_i)]dv - \iint_{\bar{S}_\sigma} \bar{t}_1\, u_i ds\right\} \;=\; \delta\,\Pi_p \;=\; 0 \qquad (i = 1,2,3) \qquad (28)$$

The quantity Π_p in Eq. (28) is the potential energy of a system. Then Eq. (27) means that the generalized equilibrium equations given by Eq. (18) include the minimum potential energy principle as a special case when the test functions defined in Eq. (22) are used.

Taking the stress field σ_{ij} as the independent trial function and making use of Eq. (5), then Eq. (26) yields

$$\iiint_V \frac{\partial W}{\partial \sigma_{ij}}\partial\sigma_{ij} dv - \iint_{\bar{S}_u} \bar{u}_i\delta t_i ds \;=\; \delta\left[\iiint_V W(\sigma_{ij})dv - \iint_{\bar{S}_u} \bar{u}_i t_i ds\right] \;=\; \delta\,\Pi_c \;=\; 0 \qquad (i, j = 1,2,3) \qquad (29)$$

where Π_c is the complementary energy. Therefore, Eq. (26) is the same as the minimum complementary principle.

6. Some examples of applications in the finite element method

In both the minimum potential energy principle and the minimum complementary principles, the trial functions have to be so-called admissible in order that only one of the weak forms of the generalized governing equations needs to be enforced. The admissible trial displacement filed in the case of the minimum potential principle is the one satisfying the strain-displacement relation in Eq. (3) exactly. The admissible trial stress field in the minimum complementary principle is the one satisfying the equilibrium equation in Eq. (1) exactly. Such a requirement makes the computation simplified, but it restricts the choices of the trial functions.

The finite element method is a powerful numerical method seeking approximate solutions in engineering. In the finite element analysis, one way to expand the admissible trial functions is using the multi-field formulation (Tang *et al.* 1980, 1983). If choosing both the displacements u_i and independent strains $\tilde{\varepsilon}_{ij}$ as the trial functions, the strain-displacement relation in Eq. (3) is not satisfied *a priori*. Then the weak forms of both the generalized equilibrium and the generalized compatibility have to be enforced. If using the test functions given in Eqs. (22) and (25), we have

$$\delta\left\{\iiint_V [U(\tilde{\varepsilon}_{ij}) - f_i u_i] dv - \iint_{\bar{S}_\sigma} \bar{t}_1 u_i ds\right\} = 0 \tag{30}$$

$$\iiint_V \left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - 2\tilde{\varepsilon}_{ij}\right] \delta\sigma_{ij} dv + \iint_{\bar{S}_u} [(\bar{u}_i - u_i) n_i + (\bar{u}_j - u_j) n_j] \delta\sigma_{ij} ds = 0 \tag{31}$$

It should be noticed that the strain energy density U in Eq. (30) is in terms of independent trial strains $\tilde{\varepsilon}_{ij}$. The concept shown above leads to the assumed strain elements and also called as the quasi-conforming element technique (Tang *et al.* 1980), and they are the same as the Hu-Washizu variational principle (Tang *et al.* 1980). A number of assumed strain elements based on this formulation have been developed (e.g. Tang *et al.* (1980), (1983), Shi and Voyiadjis 1991); the numerical examples shows that these assumed strain elements could not only solve the conforming difficulties in plate elements, but also very accurate and efficient.

If choosing the stress field $\tilde{\sigma}_{ij}$ as the independent variable, but the equilibrium is not satisfied *a priori*, the generalized governing equations should be

$$\delta\Pi_c = \delta\left[\iiint_V W(\tilde{\sigma}_{ij}) dv - \iint_{\bar{S}_u} \bar{u}_i t_i ds\right] = 0 \tag{32}$$

$$\iiint_V (\tilde{\sigma}_{ij,j} + f_i) \delta u_i dv + \iint_{\bar{S}_\sigma} (\bar{t}_i - t_i) \delta u_i ds = 0 \tag{33}$$

The two equations above lead to the assumed stress elements. Some examples of the assumed stress elements based on these two generalized governing equations can be found from the papers given by Shi and Atluri (1988) as well by Shi and Tong (1996).

7. Conclusions

The generalized governing equations in theory of elasticity and their weak forms are presented in this paper. These equations have the following features.

(1) The present generalized equilibrium equations not only take the equilibrium in an integral form over a domain, but also contain the force boundary conditions in a unified equation. The same is true for the generalized compatibility equations.

(2) The weak forms of the generalized governing equations are more general than all the current variations of the governing equations in theory of elasticity, as they include the minimum potential energy principle, the minimum complementary principle and the Hu-Washizu variational principle as special cases.

(3) By using Green theorem, the differential order in the weak forms of the generalized governing equations can be reduced further, and the continuity requirement for the trial functions is reduced accordingly. Hence the weak forms of generalized governing equations lead to more choices of the trial functions for approximate solutions than those in the classical theories of elasticity and variational principles.

(4) Some of the previously developed assumed strain elements and assumed stress elements are the direct solutions of the generalized governing equations in elasticity, and more solutions can be expected to be obtained by the new theory.

Therefore, the present weak forms of generalized governing equations in theory of elasticity can provide a more powerful modeling tool in the numerical analysis of structural mechanics.

Acknowledgement

The financial support of 08JCYBJC01100 by TSTC, Tianjin is gratefully acknowledged.

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