Coupled Systems Mechanics, *Vol. 3, No. 3 (2014) 247-265*
DOI: http://dx.doi.org/10.12989/csm.2014.3.3.247

A new approach to modeling the dynamic response of Bernoulli-Euler beam under moving load

J.T. Maximov*

Technical University of Gabrovo, Department of Applied Mechanicas, 5300 Gabrovo, Bulgaria

(Received February 25, 2014, Revised September 10, 2014, Accepted September 14, 2014)

Abstract. This article discusses the dynamic response of Bernoulli-Euler straight beam with angular elastic supports subjected to moving load with variable velocity. A new engineering approach for determination of the dynamic effect from the moving load on the stressed and strained state of the beam has been developed. A dynamic coefficient, a ratio of the dynamic to the static deflection of the beam, has been defined on the base of an infinite geometrical absolutely summable series. Generalization of the R. Willis' equation has been carried out: generalized boundary conditions have been introduced; the generalized elastic curve's equation on the base of infinite trigonometric series method has been obtained; the forces of inertia from normal and Coriolis accelerations and reduced beam mass have been taken into account. The influence of the boundary conditions and kinematic characteristics of the moving load on the dynamic coefficient has been investigated. As a result, the dynamic stressed and strained state has been obtained as a multiplication of the static one with the dynamic coefficient. The developed approach has been compared with a finite element one for a concrete engineering case and thus its authenticity has been proved.

Keywords: Bernoulli-Euler beam; moving load; dynamic stress; dynamic deflection; elastic supports; FE analysis

1. Introduction

The load moving on a beam causes bending vibrations and as a result of which bigger stresses arise in the beam in comparison with the case of static action of the same load. Vibrations of this kind are found in many engineering objects, such as bridges, railroad rails, the principal beams of bridge cranes, etc. The occurrence of this engineering problem is connected with the construction and exploitation of railroad installations. The first mathematical model of the elastic curve of Bernoulli – Euler beam, subjected to a load, moving with a constant horizontal velocity v, is obtained by Willis (1849) for the case of freely supported beam. The beam mass has been neglected and the system "beam-moving load" is modeled to such with one degree of freedom, having a constant mass and changeable elasticity.

The opposite model of behaviour of the elastic curve of a freely supported beam under the influence of a moving load with a constant velocity is when its mass is neglected, respectively only the distributed beam mass is taken into account. An analytical solution of this problem is

*Corresponding author, Professor, E-mail: maximov@tugab.bg

http://www.techno-press.org/?journal=csm&subpage=7 ISSN: 2234-2184 (Print), 2234-2192 (Online)

obtained by Krilov (1905) for the case of a force, constant in magnitude. An analytical solution for the case of a harmonic exciting force is proposed by Timoshenko (1922). Based on that, Timoshenko (1972) suggests an approximated solution to the problem, when the mass of the moving load is accounted too: initially the task with the beam's forced vibrations under the influence of a constant force, equal to the beam's weight and moving with a constant velocity, is solved; based on the obtained displacements of the points from the beam elastic curve, an expression is composed for the load's force of inertia, which is added to the static weight. However, such analytical approach is not applicable to beams with other boundary conditions.

Afterwards, three fundamental works devoted to this problem have been published: by Inglis (1934), Hillerborg (1951) proposed an analytical solution through Fourier's method of simple supported beams; by Friba (1999).

During the last few decades many solutions have been made for beam models, subjected to a moving load, in different engineering applications: beam on elastic foundation – linear-elastic foundation (Kien and Ha 2011, Chonan 1978, Amiri and Onyango 2010, Awodola 2007), viscoelastic foundation (Zehsaz *et al*. 2009, Sun and Luo 2008, Khorramabadi and Nezamabadi 2012), nonlinear elastic foundation (Ding *et al*. 2012, Hryniewicz 2011), elastic foundation, modeled through springs with different stiffness (Thambiratnam and Zhuge 1996); freely supported beam (Yang *et al*. 1997, Nikkhoo and Amankhani 2012, Michaltos 2002, Michaltos *et al*. 1996); inclined beam (Wu 2005); complex beam (Yau 2004); continuous beam (Save and Prager 1963, Zheng *et al*. 1998, Kerr 1972); beam on elastic supports (Mehril *et al*. 2009; Piccardo and Tubino 2012); beam with generalized boundary conditions (Hilal and Mohsen 2000); contilever beam (Lin and Chang 2006). In view of the model of stressed and strained state, Bernoulli – Euler beams prevail (Xia *et al*. 2006, Hilal and Zibden 2000, Javanmard *et al*. 2013, Liu *et al*. 2013) over beams of Timoshenko (Azam *et al*. 2013, Chonan 1975).

The analytical approach for solution prevails over the finite element one (Lin and Trethewey 1990, Lin and Trethewey 1993). Some of the models refer to previously stressed through axial compressive load beams (Omolofe 2013, Zibden and Rackwitz 1995).

The models in which the mass of the moving load is not accounted are prevailing. The effect of inertia from the passing load is numerically obtained by Yang *et al*. (1997) through Newmark-β method, and is afterwards included in the force function of the load. In (Michaltos 2002) are accounted the mass and the moment of inertia of the moving load, which is moving with constant horizontal acceleration, as well as the effect of inertia from the rotation of the beam's sections. However, the effect of inertia from the load mass directly on the beam transverse vibrations is not accounted, but only through the weight force and the force of inertia from the horizontal acceleration of the load.

In the majority of publications the case of the load's constant velocity is examined. A study of the influence of the acceleration has been made by Michaltos (2002) and Hilal and Mohsen (2000). The models with constant magnitude of the moving load are prevailing. In (Hilal and Mohsen 2000) a model with moving force, altering by magnitude by a harmonic law has been made. In (Awodola 2007) the moving force changes exponentially by magnitude and a numerical approach is applied – the finite difference method. In some of the models a dynamic absorber moving on the beam's axis has been implemented (Samani and Pellicano 2009, Soares *et al*. 2010).

Models of more complicated objects have been made – a bridge system, examined in a resonance state by Xia *et al*. (2006), and a vehicle model by Esmailzadeh and Jalili (2003).

It is noteworthy that the models of two-supported beams mainly refer to a freely supported beam, which has a logical explanation: for this case the frequency equation is most simple and therefore the general solution in infinite sums can be obtained (Timoshenko 1972).

In the engineering practice the technical solutions for fixture of the ends of a beam, subjected to bending, lead to a model with elastic angular supports. These supports restrict the rotations of the end cross-sections for beam bending, depending on the stiffness of the supports. For instance, in some constructional solutions the principal beam of a bridge crane is connected in both its ends for the vertical internal faces of the front beams through plates and coupling flanges with fitted bolt connections. The elasticity of the angular supports in a vertical plane is a function of the front beams torsion stiffness. In terms only of bending, the beam is double statically indeterminate: hyperstatic quantities are the elastic moments in the two additional angular supports.

The researchers' efforts are directed towards modeling the beam transverse vibrations under the influence of a passing load and the emphasis is placed on examining the mathematical model's behaviour. This approach has its irreplaceable advantages, for example in studying the resonance states of work, beams on elastic foundation, etc. Simultaneously the obtained solutions are complicated enough, so as for the final consumer, i.e., the engineer, not to need computing machinery and corresponding software. The finite element solutions have all the advantages and disadvantages of the numerical ones. For the engineer in many cases of two-supported beams a more simple formula is necessary, which would offer a solution in first approximation. In this aspect, a possible approach for solution is a generalization of R. Willis equation. A few arguments exist in favor of this idea.

- In the engineering applications the elastic curve of a two-supported beam corresponds to its basic eigentone under free vibrations, and the presumption is that the working regime is outside the resonance. Therefore it is advisable for the R.Willis approach to be applied for obtaining the dynamic deflection;
- The practice shows that the bending stresses in a two-supported beam are biggest when the load is equally distant from both supports. In the model could be included the beam's reduced mass for this position of the load;
- Due to the bending of the beam, the load moves on a curve, because of which in the model could be included the force of inertia not only from the transverse displacements, but also from the normal and Coriolis accelerations. The latter arises due to the rotations of the beam's sections. Moreover, the angle between the vectors of the transfer angular velocity and the load's relative velocity is 90^0;
- In addition, this approach allows easier investigation of the effect of the load's horizontal acceleration.

The main aim of the article is through an engineering approach to obtain a simplified mathematical model of the dynamic effect, caused by load motion on two-supported Bernoulli – Euler beam with elastic angular supports.

2. Willis' formulation

A simple beam with length ℓ and bending stiffness EJ is considered. A load with weight Q moves on the beam with constant velocity v. The beam mass is ignored. The beam deflection w under the load is proportional to the pressure P which the load exerts on the beam

$$w(x) = \frac{Px^2(\ell - x)^2}{3\ell EJ} \tag{1}$$

where

$$P = Q\left(1 - \frac{v^2}{g} w''(x)\right),$$

x and $(\ell - x)$ are distances from the load to the beam supports; g is the acceleration of gravity.

Eq. (1) has first been obtained by Willis (1849) with the presumption for: constant horizontal load velocity; massless beam; load's force of inertia, caused only by the beam vertical displacements. A full solution of (1), initially in the form of series, and later in closed form, is obtained by Stokes (1849). A numerical approach for solution of (1), based on the method of Runge, is first applied by Petrov (1903). Information for the history of the problem and detailed data for approximated solutions are given by Clebsch (1883).

3. Generalization of the R. Willis equation

The dynamic deflection $w(x,t)$ of the beam's elastic curve (Fig. 1) under the load is presented as

$$w(x,t) = P(t)y(x) \tag{2}$$

where $y(x)$ is deflection under the influence of a force with magnitude equal to 1, and $P(t)$ is equivalent dynamic load. The latter includes: normal reaction from the load towards the beam; force of inertia from the reduced mass of the beam

$$-\frac{G_{red}}{g}\frac{\partial^2 w(x,t)}{\partial t^2}$$

and a force from the reduced weight G_{red} of the beam.

The normal reaction from the load towards the beam is a sum of: the static weight Q of the load; force of inertia from vertical displacement (deflection) of the beam

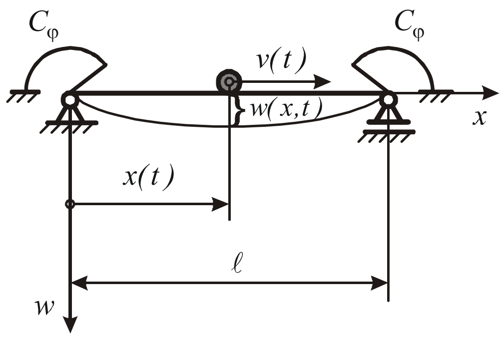

Fig. 1 Model of the beam elastic curve with moving load

$$-\frac{Q}{g}\frac{\partial^2 w(x,t)}{\partial t^2}$$

force of inertia from the normal acceleration a_{nor} due to the load motion on a curve line (the curvature is negative for the adopted coordinate system depicted in Fig. 1)

$$\frac{Q}{g}a_{nor} = -\frac{Qv^2}{g}\frac{\partial^2 w(x,t)}{\partial x^2}$$

force of inertia from the Coriolis acceleration of the load due to a rotation of the beam cross-sections

$$-2\frac{Q}{g}\frac{\partial^2 w(x,t)}{\partial x \partial t}v\,.$$

For the dynamic deflection $w(x,t)$ follows

$$w(x,t) = \left[(Q+G_{red}) - \frac{(Q+G_{red})}{g}\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{Qv^2}{g}\frac{\partial^2 w(x,t)}{\partial x^2} - 2\frac{Q}{g}\frac{\partial^2 w(x,t)}{\partial x \partial t}v\right]y(x) \tag{3}$$

Let us consider $v = v(t)$. Then after switching the order of differentiation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\frac{\partial x}{\partial x}\right) = \frac{\partial}{\partial t}\left[v(t)\frac{\partial w}{\partial x}\right] = \dot{v}(t)\frac{\partial w}{\partial x} + v(t)\frac{\partial^2 w}{\partial x \partial t}\frac{\partial x}{\partial x} = \dot{v}(t)\frac{\partial w}{\partial x} + v^2(t)\frac{\partial^2 w}{\partial x^2} \tag{4}$$

On the other hand

$$\frac{\partial^2 w(x,t)}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial t}\frac{\partial x}{\partial x}\right) = v(t)\frac{\partial^2 w}{\partial x^2} \tag{5}$$

After substitution of (4) and (5) in (3)

$$w(x,t) = \left[(Q+G_{red}) - \frac{(4Q+G_{red})v^2}{g}\frac{\partial^2 w(x,t)}{\partial x^2} - \frac{(Q+G_{red})\dot{v}(t)}{g}\frac{\partial w(x,t)}{\partial x}\right]y(x) \tag{6}$$

Eq. (6) is a generalization of the R. Willis equation.

4. Nature of the engineering approach

We introduce the notion "dynamic coefficient" k_d, which shows how many times the static load on the beam is increased due to the load motion

$$k_d = \frac{w(x,t)}{w(x)} \tag{7}$$

where $w(x)$ is the beam deflection from the static load $Q + G_{red}$

$$w(x) = (Q+G_{red})y(x) \tag{8}$$

After substitution of (6) and (8) in (7)

$$k_d = 1 - \left[\frac{(4Q + G_{red})v^2}{(Q + G_{red})g} \frac{\partial^2 w(x,t)}{\partial x^2} + \frac{\dot{v}(t)}{g} \frac{\partial w(x,t)}{\partial x} \right] \tag{9}$$

Eq. (9) is presented as

$$k_d = 1 + \alpha_d \tag{10}$$

where

$$\alpha_d = -\left[C(t) \frac{\partial^2 w(x,t)}{\partial x^2} + \frac{\dot{v}(t)}{g} \frac{\partial w(x,t)}{\partial x} \right] \tag{11}$$

$$C(t) = \frac{(4Q + G_{red})v^2(t)}{(Q + G_{red})g} \tag{12}$$

The determination of k_d is conducted through consecutive approximations

• In the approximation is set

$$P^{(1)}(t) = Q + G_{red}$$

whence

$$w^{(1)}(x,t) = (Q + G_{red})y(x)$$

and after substitution in (11)

$$\alpha_d^{(1)} = \alpha = -\left[C(t)y''(x) + \frac{\dot{v}(t)}{g} y'(x) \right](Q + G_{red})$$

For the dynamic coefficient it follows

$$k_d^{(1)} = 1 + \alpha$$

where $0 < \alpha < 1$.

• In the second iteration is set

$$P^{(2)}(t) = k_d^{(1)}(Q + G_{red})$$

It is found

$$\alpha_d^{(2)} = -\left[C(t)y''(x) + \frac{\dot{v}(t)}{g} y'(x) \right] k_d^{(1)}(Q + G_{red})$$

or

$$\alpha_d^{(2)} = \alpha k_d^{(1)}$$

For $k_d^{(2)}$ follows

$$k_d^{(2)} = 1 + \alpha_d^{(2)} = 1 + \alpha k_d^{(1)} = 1 + \alpha(1+\alpha) = 1 + \alpha + \alpha^2$$

- In the third iteration is set

$$P^{(3)}(t) = k_d^{(2)}(Q + G_{red})$$

whence

$$\alpha_d^{(3)} = -\left[C(t)y''(x) + \frac{\dot{v}(t)}{g}y'(x)\right]k_d^{(2)}(Q + G_{red})$$

or

$$\alpha_d^{(3)} = \alpha k_d^{(2)} = \alpha\left(1 + \alpha + \alpha^2\right)$$

For $k_d^{(3)}$ follows

$$k_d^{(3)} = 1 + \alpha + \alpha^2 + \alpha^3$$

- In the n – th iteration

$$k_d^{(n)} = 1 + \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^n$$

Since always $0 < \alpha < 1$, then k_d is presented as an infinite geometrical absolutely summable series. For k_d follows

$$k_d = \lim_{n\to\infty} k_d^{(n)} = \frac{1}{1-\alpha} \tag{13}$$

where

$$\alpha = -\left[C(t)y''(x) + \frac{\dot{v}(t)}{g}y'(x)\right](Q + G_{red}) \tag{14}$$

5. Application of trigonometric series method for modeling the beam elastic curve

5.1 Generalized model of the beam elastic curve

The trigonometric series method for investigation of beams and plates bending is developed by Timoshenko (1972). The expression for the deflection of the beam elastic curve can always be presented in the form of infinite trigonometric series.

Timoshenko (1972) examines in detail the cases of Bernoulli-Euler beam for two cases: (i) – freely supported beam; (ii) – restrained beam, for which he draws approximated formulas for the deflection of the beam's symmetry section for particular cases of loading.

If it is accepted that the solutions, based on integration of the differential equation of the beam elastic curve are accurate from a mathematical point of view, then the method of series is approximate since after taking a finite number of series members, the physical equivalent is an exchange of an elastic system with infinite degrees of freedom with an elastic system with finite degrees of freedom.

Such approximation of the exact behaviour of the elastic curve could be useful from an engineering point of view for the case of Bernoulli-Euler two-supported beam with elastic angular supports.

A straight beam is studied with elastic angular supports and stiffness c_φ, restricting the rotations of the end cross-sections bending. The beam elastic curve lies in the xw plane (Fig. 1). The deflection $w(x)$ needs to satisfy the conditions $w(0)=w(\ell)=0$, but $w'(0)\neq 0$, $w'(\ell)\neq 0$, $w''(0)\neq 0$, $w''(\ell)\neq 0$, in which between $w'(0)$ and $w''(0)$, respectively $w'(\ell)$ and $w''(\ell)$, a correlation exists: of a concrete angle of the beam end cross-section rotation exists a specifically set elastic bending moment. The general expression for the deflection $w(x)$ of the elastic curve is presented in the form

$$w(x)=\sum_{n=1,3,5...}^{n=\infty} A_n\left(1-\cos\frac{2n\pi x}{\ell}\right)+\sum_{n=1,3,5,...}^{n=\infty} B_n \sin\frac{n\pi x}{\ell} \tag{15}$$

Each of the functions under the sums obviously satisfies the first group of boundary conditions:

$$w(0)=w(\ell)=0$$

The derivatives to second order of (15) are

$$w'=\frac{2\pi}{\ell}\sum_{n=1,3,5,...}^{n=\infty} nA_n \sin\frac{2n\pi x}{\ell}+\frac{\pi}{\ell}\sum_{n=1,3,5,...}^{n=\infty} nB_n \cos\frac{n\pi x}{\ell}$$

$$w''=\frac{4\pi^2}{\ell^2}\sum_{n=1,3,5,...}^{n=\infty} n^2 A_n \cos\frac{2n\pi x}{\ell}-\frac{\pi^2}{\ell^2}\sum_{n=1,3,5,...}^{n=\infty} n^2 B_n \sin\frac{n\pi x}{\ell}$$

The dependence between the A_n coefficients on one side, and the B_n coefficients on the other, is found through the second group of boundary conditions

$$w_n'(0)=\frac{n\pi B_n}{\ell};\quad w_n'(\ell)=-\frac{n\pi B_n}{\ell} \tag{16}$$

The end cross-sections rotations, dependences (16), cause elastic bending moments

$$M(0)=-EJw_n''(0);\quad M(\ell)=-EJw_n''(\ell) \tag{17}$$

where EJ is beam bending stiffness and

$$w_n''(0)=\frac{4n^2\pi^2}{\ell^2}A_n;\quad w_n''(\ell)=\frac{4n^2\pi^2}{\ell^2}A_n \tag{18}$$

The elastic moments are defined as

$$M(0)=c_\varphi w_n'(0);\quad M(\ell)=c_\varphi w_n'(\ell) \tag{19}$$

After substitution of (16)-(18) in (19), taking into account that $M(0)$ and $M(\ell)$ are opposite and solving toward B_n

$$B_n=\frac{kn\pi}{4}A_n \tag{20}$$

where

$$k = \frac{16EJ}{\ell c_\varphi} \tag{21}$$

In view of (20), the deflection and its derivatives to second order obtain the form

$$w(x) = \sum_{n=1,3,5,...}^{n=\infty} A_n\left(1 - cos\frac{2n\pi x}{\ell}\right) + \frac{k\pi}{4}\sum_{n=1,3,5,...}^{n=\infty} nA_n\, sin\frac{n\pi x}{\ell} \tag{22}$$

$$w' = \frac{2\pi}{\ell}\sum_{n=1,3,5,...}^{n=\infty} nA_n\, sin\frac{2n\pi x}{\ell} + \frac{k\pi^2}{4\ell}\sum_{n=1,3,5,...}^{n=\infty} n^2 A_n\, cos\frac{n\pi x}{\ell} \tag{23}$$

$$w'' = \frac{4\pi^2}{\ell^2}\sum_{n=1,3,5,...}^{n=\infty} n^2 A_n\, cos\frac{2n\pi x}{\ell} - \frac{k\pi^3}{4\ell^2}\sum_{n=1,3,5,...}^{n=\infty} n^3 A_n\, sin\frac{n\pi x}{\ell} \tag{24}$$

The unknown A_n coefficients could for example be determined through the principle of virtual displacements for equilibrium state of a previously set load.

5.2 Defining the function $y = y(x)$

According to Eq. (2), if $P(t) = 1$, then $y(x) = w(x)$. Let a bending force $P = 1$, at a ξ distance from the left end, act on the beam. The A_n coefficients are defined by the expression for virtual work

$$\frac{\partial U}{\partial A_n}\delta A_n = P\delta w_P \tag{25}$$

where

$$U = U_b + U_s \tag{26}$$

is potential energy of the elastic system, U_b is potential energy due to the beam bending, U_s is potential energy of the elastic angular supports, δA_n is an increase of the A_n coefficient, and δw_P is a virtual displacement of the applied point of the force P.

For the U_b and U_s components (from both supports), it follows

$$U_b = \frac{EJ}{2}\int_0^\ell w''^2(x)dx = \frac{EJ\pi^4}{\ell^3}\left[4\sum_{n=1,3,5,...}^{n=\infty} n^4 A_n^2 + k\sum_{n=1,3,5,...}^{n=\infty}\left(\frac{2}{3}n^4 + \frac{\pi^2 k^2}{64}n^6\right)A_n^2\right] \tag{27}$$

$$U_s = c_\varphi w'^2(0) = c_\varphi \frac{k^2\pi^4}{16\ell^2}\sum_{n=1,3,5,...}^{n=\infty} n^4 A_n^2$$

or, if (21) is taken into account

$$U_s = \frac{EJk\pi^4}{\ell^3}\sum_{n=1,3,5,...}^{n=\infty} n^4 A_n^2 \tag{28}$$

For an increase δA_n of the coefficient A_n, the increase of the potential energy U is

$$\frac{\partial U}{\partial A_n}\delta A_n = \frac{2EJ\pi^4}{\ell_3}\left(4n^4 + \frac{5}{3}kn^4 + \frac{\pi^2 k^2}{64}n^6\right)A_n\delta A_n \tag{29}$$

The virtual displacement δw_P for δA_n increase is

$$\delta w_P = \delta A_n\left[\left(1 - cos\frac{2n\pi\xi}{\ell}\right) + \frac{nk\pi}{4}sin\frac{n\pi\xi}{\ell}\right] \tag{30}$$

After substitution of (29) and (30) in (25), for the A_n coefficients is obtained

$$A_n = \frac{P\ell^3\left(1-\cos\frac{2n\pi\xi}{\ell}+\frac{nk\pi}{4}\sin\frac{n\pi\xi}{\ell}\right)}{2EJ\pi^4\left(4n^4+\frac{5}{3}kn^4+\frac{\pi^2k^2n^6}{64}\right)} \tag{31}$$

After substitution of (31) in Eqs. (22) – (24) and setting $P = 1$, for the function $y = y(x)$ and its derivatives to second order is obtained

$$\begin{aligned} y(x) = {} & \frac{\ell^3}{2EJ\pi^4}\sum_{n=1,3,5,\ldots}^{n=\infty}\frac{1-\cos\frac{2n\pi\xi}{\ell}+\frac{nk\pi}{4}\sin\frac{n\pi\xi}{\ell}}{4n^4+\frac{5}{3}kn^4+\frac{\pi^2k^2}{64}n^6}\left(1-\cos\frac{2n\pi x}{\ell}\right)+ \\ & +\frac{\ell^3 k}{8EJ\pi^3}\sum_{n=1,3,5,\ldots}^{n=\infty}\frac{1-\cos\frac{2n\pi\xi}{\ell}+\frac{nk\pi}{4}\sin\frac{n\pi\xi}{\ell}}{4n^3+\frac{5}{3}kn^3+\frac{\pi^2k^2}{64}n^5}\sin\frac{n\pi x}{\ell} \end{aligned} \tag{32}$$

$$\begin{aligned} y'(x) = {} & \frac{\ell^2}{EJ\pi^3}\sum_{n=1,3,5,\ldots}^{n=\infty}\frac{1-\cos\frac{2n\pi\xi}{\ell}+\frac{nk\pi}{4}\sin\frac{n\pi\xi}{\ell}}{4n^3+\frac{5}{3}kn^3+\frac{\pi^2k^2}{64}n^5}\sin\frac{2n\pi x}{\ell}+ \\ & +\frac{\ell^2 k}{8EJ\pi^2}\sum_{n=1,3,5,\ldots}^{n=\infty}\frac{1-\cos\frac{2n\pi\xi}{\ell}+\frac{nk\pi}{4}\sin\frac{n\pi\xi}{\ell}}{4n^2+\frac{5}{3}kn^2+\frac{\pi^2k^2}{64}n^4}\cos\frac{n\pi x}{\ell} \end{aligned} \tag{33}$$

$$\begin{aligned} y''(x) = {} & \frac{2\ell}{EJ\pi^2}\sum_{n=1,3,5,\ldots}^{n=\infty}\frac{1-\cos\frac{2n\pi\xi}{\ell}+\frac{nk\pi}{4}\sin\frac{n\pi\xi}{\ell}}{4n^2+\frac{5}{3}kn^2+\frac{\pi^2k^2}{64}n^4}\cos\frac{2n\pi x}{\ell}- \\ & -\frac{\ell k}{8EJ\pi}\sum_{n=1,3,5,\ldots}^{n=\infty}\frac{1-\cos\frac{2n\pi\xi}{\ell}+\frac{nk\pi}{4}\sin\frac{n\pi\xi}{\ell}}{4n+\frac{5}{3}kn+\frac{\pi^2k^2}{64}n^3}\sin\frac{n\pi x}{\ell} \end{aligned} \tag{34}$$

6. Dynamic coefficient study

The most unfavorable loading of the beam is obtained when the load is situated in the beam's middle. Moreover the most essential contribution in the sums of Eqs. (32) – (34) have the first members. After substitution of $\xi = \frac{\ell}{2}$ and $n = 1$, Eqs. (32) – (34) get the following form

$$y(x) = \frac{\ell^3 f_1(k)}{2EJ\pi^4}\left(1 - cos\frac{2\pi x}{\ell} + \frac{k\pi}{4} sin\frac{\pi x}{\ell}\right) \tag{35}$$

$$y'(x) = \frac{\ell^2 f_1(k)}{EJ\pi^3}\left(sin\frac{2\pi x}{\ell} + \frac{k\pi}{8} cos\frac{\pi x}{\ell}\right) \tag{36}$$

$$y''(x) = \frac{2\ell f_1(k)}{EJ\pi^2}\left(cos\frac{2\pi x}{\ell} - \frac{k\pi}{16} sin\frac{\pi x}{\ell}\right) \tag{37}$$

where

$$f_1(k) = \frac{2 + \frac{\pi k}{4}}{4 + \frac{5k}{3} + \frac{\pi^2 k^2}{64}}$$

6.1 The load moves with constant velocity

The study is carried out for the position of the load, when it is in the beam's middle. Then, after substitution of $x = \frac{\ell}{2}$ in (37)

$$y''\left(\frac{\ell}{2}\right) = -\frac{2\ell}{EJ\pi^2} f_2(k) \tag{38}$$

where

$$f_2(k) = \frac{384 + 72\pi k + 3\pi^2 k^2}{768 + 320k + 3\pi^2 k^2}.$$

The function $f_2(k)$ is a generalization of the beam boundary conditions. For a restrained beam ($k = 0$): $f_2(k) = \frac{1}{2}$. For freely supported beam ($k \to \infty$): $f_2(k) = 1$.

Taking into account (12) and (38), equation (14) gets the following form

$$\alpha = \frac{2(4Q + G_{red})v^2}{g} \frac{\ell}{EJ\pi^2} f_2(k) \tag{39}$$

and the dynamic coefficient (13) is

$$k_d = \frac{1}{1 - \frac{2(4Q + G_{red})v^2}{g} \frac{\ell}{EJ\pi^2} f_2(k)}$$

where

$$G_{red} = \beta_{red} G ,$$

$$\beta_{red} = \frac{1}{\ell} \int_0^{\ell} \left(\frac{y(x)}{y(x)\big|_{x=\frac{\ell}{2}}} \right)^2 dx = \frac{144 + 128k + 3\pi^2 k^2}{384 + 96\pi k + 6\pi^2 k^2} ,$$

is a reduction coefficient of the beam weight, and G is the beam weight.

Obviously β_{red} non-linearly depends on k, respectively on the stiffness of the angular connections. For a restrained beam ($k \to 0$): $\min \beta_{red} = \frac{3}{8}$. For a freely supported beam ($k \to \infty$): $max\, \beta_{red} = \frac{1}{2}$. Apparently the way of restriction of the end cross-sections' rotations does not essentially affect the reduced weight.

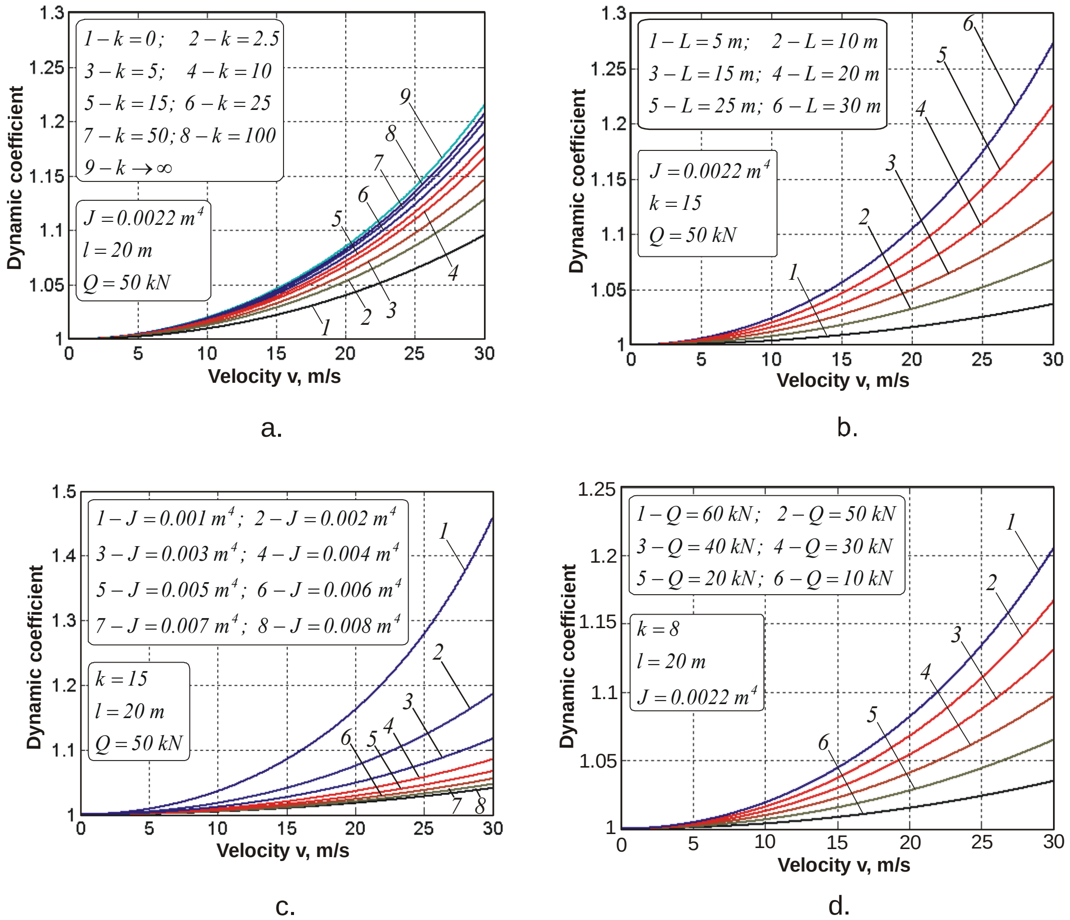

Fig. 2 Dependence of the dynamic coefficient α on the load velocity

Fig. 2 shows the dependence of the dynamic coefficient k_d from the load's velocity. Obviously, at high velocities, k_d increases significantly. The stiffness of the elastic angular supports influences substantially on k_d - for instance, at $v = 30\ m/s$, k_d increases with almost 25% for freely supported beam, in comparison with the case of ideally fixed beam (Fig. 2(a)). With the augmentation of the beam length, all other conditions being equal, k_d increases (Fig. 2(b)), and this tendency is more significant at high velocities. With the decrease of the beam bending stiffness, k_d increases significantly (Fig. 2(c)). The unlimited augmentation of the bending stiffness through an increase of the moment of inertia J leads to $\alpha \to 0$, and from there to $k_d \to 1$ on the right. The increment of the force Q, at constant other parameters, leads to an increase of k_d (Fig. 2(d)).

6.2 Influence of the acceleration on the dynamic coefficient

Dependence (14) could be presented as

$$\alpha = \alpha_v + \alpha_a$$

where

$$\alpha_v = -\frac{(4Q + G_{red})v^2}{g} y''(x) \tag{40}$$

$$\alpha_a = -\frac{(Q + G_{red})a}{g} y'(x) \tag{41}$$

are components, accounting the influence, respectively of the load's velocity and acceleration on the dynamic coefficient.

As seen from the comparison between (36) and (37), the extremums of $y'(x)$ and $y''(x)$ have different phases: in the section of symmetry, where the deflection is biggest and the component α_v has extremum (maximum), and the component α_a becomes 0.

Fig. 3 displays the dependence of the coefficient α and its components α_v and α_a in a function of the x-coordinate, accounting the position of the load, when the velocity and acceleration are equal to 1. The maximum value of α is shifted from the section of symmetry towards the increase of x and it occurs for $x \approx \frac{5}{8}\ell$. If a negative acceleration $(a = -1)$ was accepted, α_a changes its sign, and the maximum value of α stays the same, but is in section with abscissa $x \approx \frac{3}{8}\ell$. The acceleration itself influences the dynamic coefficient to the same degree, as the velocity does. However, that influence of the acceleration does not refer to the section of symmetry of the beam, i.e., to the critical section. Depending on the law for the load motion, it is theoretically possible to prove that it is not the section of symmetry of the beam that is critical.

A practical significance has the case, when $a = const$, i.e.

$$v = v_0 + at$$

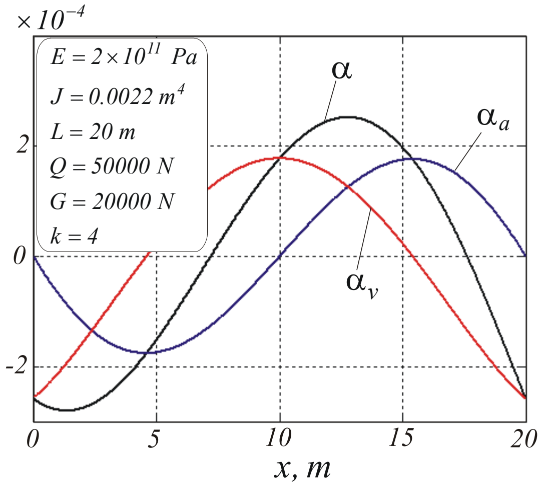

Fig. 3 Dependence of the coefficient α and its components α_v and α_a in a function of the x-coordinate

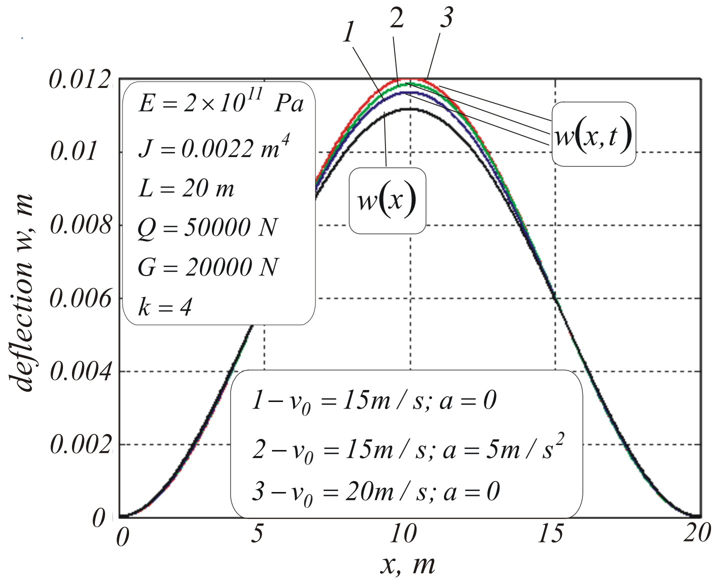

Fig. 4 A comparison between the static and dynamic deflections for different kinematic parameters v_0 and a for load motion

where v_0 is the velocity of the load for $x = 0$. The transformation

$$\int_0^v v dv = a \int_0^x dx$$

is advisable, whence

$$v = \sqrt{2ax + v_0^2} \tag{42}$$

From (2), (7) and (8) for the dynamic deflection follows

$$w(x,t) = \frac{1}{1-\alpha}\left(Q + G_{red}\right)y(x) \tag{43}$$

where the component α_v of α (look dependence(40)), taking into account (42), gets the following form

$$\alpha_v = -\frac{\left(4Q + G_{red}\right)\left(2ax + v_0^2\right)y''(x)}{g}$$

Fig. 4 shows a comparison between the static and dynamic deflections, dependences (8) and (43), for different kinematic parameters v_0 and a for load motion. The section of symmetry of the beam is critical, and the dynamic effect from load motion in comparison with the static deflection is under 10%.

7. Comparison with a finite element solution

The matter of beam vibration with elastic angular supports under the influence of a moving on it load does not have a full analytical solution, since the equation, from which the natural frequencies are defined, is transcendental and could only be solved numerically. For that reason, the comparison is carried out with a finite element (FE) solution. The numerical input data are the following: beam length $\ell = 20\,m$, the beam cross-section is box with overall sizes $0.3 \times 0.6\,m$ and with walls thickness, respectively $0.01m$ and $0.02m$; Young's modulus $E = 2 \times 10^{11}\,Pa$, density $\rho = 7850kg/m^3$; the stiffness of the angular supports is $c_\varphi = 2.6046 \times 10^7\,Nm/rad$, respectively $k = 8$; weight and constant velocity of the moving load, respectively $Q = 50kN$ and $v = 20m/s$.

Dynamic implicit analysis using ABAQUS/Standard has been fulfilled. The general direct integration method, called the Hilber-Hughes-Taylor operator has been used. The principal advantage of this operator is that it is unconditionally stable for linear systems.

The governing equation of the FE method can be written as

$$[M]\{\ddot{w}(t)\} + [C]\{\dot{w}(t)\} + [K]\{w(t)\} = \{P(t)\} \tag{44}$$

where $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrix respectively; $\{\ddot{w}\}$, $\{\dot{w}\}$ and $\{w\}$ denote acceleration, velocity and displacement vectors; $\{P(t)\}$ is the vector of the external nodal forces. The latter is defined as

$$[P(t)]^T = [0\; P_2(t)\, P_3(t) \ldots P_i(t) \ldots P_N(t)\; 0]^T$$

where t is the real time; $i = 1, 2, \ldots, N$; N is the number of the beam FEs, having identical lengths; $P_i(t)=0$ when $t\in\left\langle 0,\frac{\ell(i-2)}{Nv}\right\rangle$ and $t\in\left\langle \frac{\ell i}{Nv},\frac{\ell}{v}\right\rangle$, $P_i(t)\neq 0$ when $t\in\left(\frac{\ell(i-2)}{Nv},\frac{\ell i}{Nv}\right)$, $\max P_i = P(t)\big|_{t=\frac{\ell(i-1)}{Nv}}$.

Thus, when $N\to\infty$, a lightly passing of the force P on the beam is assured. Taking into account the mass of the passing load, $[M]$ is a variable matrix. In order to alleviate the finite element analysis and given the specificity of ABAQUS/Standard, in stead of simulating a variable mass of the system "beam – moving load", it is accepted for $[M]$ to be a constant matrix, and the load's mass is placed in the middle of the beam. Then

$$max\, P_i = P(t)\big|_{t=\frac{\ell(i-1)}{Nv}} = Q$$

The FE model consists of 2000 linear line FEs type B21. In order to make an accurate comparison with the analytical solution, the damping has not been included in the FE solution.

For a critical point of the section of symmetry is obtained a normal stress $\max\sigma_x^d = 44.065MPa$. For a static loading in the section of symmetry with force $P=Q$ in the same point, the normal stress is $\max\sigma_x^s = 40.23MPa$. The dynamic coefficient is $k_d = \frac{\max\sigma_x^d}{\max\sigma_x^s} = 1.095$.

With the same numerical values with which a finite element solution has been made, the dynamic coefficient (13) has been calculated, where α is defined by (39). It is obtained $k_d = 1.1367$. As could be expected, the dynamic coefficient, computed with the proposed method, is slightly bigger, since the forces of inertia from the normal and Coriolis accelerations from load motion, have been accounted.

8. Conclusions

A new engineering approach has been developed for determination of the dynamic effect from a passing load on the stressed state of two-supported Bernoulli-Euler beam with elastic angular supports. A dynamic coefficient has been defined as a ratio of the dynamic deflection toward the static one. A generalization of the R.Willis equation has been made. Generalized boundary conditions have been established for this purpose. The forces of inertia of the normal and Coriolis' accelerations of the load have been accounted, as well as the beam mass. It has been studied the influence of the boundary conditions and the kinematic characteristics of the moving load on the dynamic coefficient k_d. It is shown that at contemporary velocities, the dynamic effect is under 15%.

References

Abu-Hilal, M. and Zibden, H.S. (2000), "Vibration analysis of beams with general boundary conditions traversed by a moving force", *J. Sound Vib.*, **229**(2), 377-388.

Amiri, S.N. and Onyango, M. (2010), "Simply supported beam response on elastic foundation carrying repeated rolling concentrated loads", *J. Eng. Sci. Technol.*, **5**(1) 52- 66.

Awodola, T.O. (2007), "Variable velocity influence on the vibration of simply supported Bernoulli-Euler beam under exponentially varying magnitude moving load", *J. Math. Stat.*, **3**(4), 228-232.
Azam, E., Mofid, M. and Khoraskani, R.A. (2013), "Dynamic response of Timoshenko beam under moving mass", *Scientia Iranica*, **20** (1), 50-56.
Chonan, S. (1975), "The elastically supported Timoshenko beam subjected to an axial force and a moving load", *Int. J. Mech. Sci.*, **17**(9), 573-581.
Chonan, S. (1978), "Moving harmonic load on an elastically supported Timoshenko beam", *J. Appl. Math. Mech*., **58** (1), 9-15.
Clebsch, A. (1883), *Theorie de l'elasticite des corps solides, Traduite par Barre de Saint-Venant et A. Flamant*, Dunodm, Paris.
Ding, H., Chen, L.Q. and Yang, S.P. (2012), "Convergence of Galerkin truncation for dynamic response of finite beams on nonlinear foundations under a moving load", *J. Sound Vib.*, **331**, 2426-2442.
Esmailzadeh, E. and Jalili, N. (2003), "Vehicle–passenger–structure interaction of uniform bridges traversed by moving vehicles", *J. Sound Vib.*, **260**, 611-635.
Friba, L. (1999), *Vibration of solid and structures under moving loads*, Thomas Telford, London.
Hilal, M.A. and Mohsen, M. (2000), "Vibration of beams with general boundary conditions due to a moving harmonic load", *J. Sound Vib.*, **232**(4), 703-717.
Hillerborg, A. (1951), *Dynamic influence of smoothly running loads of simple supported girders*, Kungliga Tekniska Hogskolan, Stockholm.
Hryniewicz, Z. (2011), "Dynamics of Rayleigh beam on nonlinear foundation due to moving load using Adomian decomposition and coiflet expansion", *Soil Dyn. Earthq. Eng.*, **31**, 1123-1131.
Inglis, C.E. (1934), *A mathematical treatise on vibration in railway bridges*, Cambridge University Press, Cambridge.
Javanmard, M., Bayat, M. and Ardakani, A. (2013), "Nonlinear vibration of Euler-Bernoulli beams resting on linear elastic foundation", *Steel Compos. Struct.*, **15**(4), 439-449.
Karami-Khorramabadi, M. and Nezamabadi, A.R. (2012), "Dynamic analysis of infinite composite beam subjected to a moving load located on a viscoelastic foundation based on the third order shear deformation theory", *J. Basic Appl. Sci. Res.*, **2**(8), 8378-8381.
Kerr, A.D. (1972), "The continuously supported rail subjected to an axial force and a moving load", *Int. J. Mech. Sci.*, **14**(1), 71-78.
Kien, N.D. and Ha, L.T. (2011), "Dynamic characteristics of elastically supported beam subjected to a compressive axial force and a moving load", *Vietnam J. Mech.*, **33**(2), 113-131.
Krilov, A.N. (1905), "Uber die erzwungenen schwingungen von gleichformigen elastischen staben", *Mathematishce Annalen*, **61**, 211.
Lin, H.P. and Chang, S.C. (2006), "Forced responses of cracked contilever beam subjected to a concentrated moving load", *Int. J. Mech. Sci*., **48**(12), 1456-1463.
Lin, Y.H. and Trethewey, M.W. (1990), "Finite element analysis of elastic beams subjected to moving dynamic loads", *J. Sound Vib.*, **136**(2), 323-342.
Lin, Y.H. and Trethewey, M.W. (1993), "Active vibration suppression of beam structures subjected to moving loads: a feasibility study using finite elements", *J. Sound Vib.*, **166** (3), 383-395.
Liu, Z., Yin, Y., Wang, F., Zhao, Y. and Cai, L. (2013), "Study on modified differential transform method for free vibration analysis of uniform Euler-Bernoulli beam", *Struct. Eng. Mech.*, **48** (5) 697-709.
Mehri1, B., Davar, A. and Rahmani, O. (2009), "Dynamic Green function solution of beams under a moving load with different boundary conditions", *J. Sharif Univ. of Technol., Transaction B: Mech.Eng.*, **16**(3), 273-279.
Michaltos, G.T. (2002), "Dynamic behaviour of a single-span beam subjected to loads moving with variable speeds", *J. Sound Vib.*, **258**(2), 359-372.
Michaltos, G.T., Sophianopulos, D. and Kounadis, A.N. (1996), "The effect of a moving mass and other parameters on the dynamic response of a simply supported beam", *J. Sound Vib.*, **191**(3), 357-362.
Nikkhoo, A. and Amankhani, M. (2012), "Dynamic behaviour of functionally graded beams traversed by a moving random load", *Indian J. Sci. Technol.*, **5**(12), 3727-3731.

Omolofe, B. (2013), “Deflection profile analysis of beams on two-parameter elastic subgrade”, *Latin American J. Solids Struct.*, **10**, 263-282.

Petrov, N.P. (1903), “Influence of the translational velocity of the wheel on the rail stress”, *Reports of the Imperial Russian Technological Society,* **37**(2), 27-115 (in Russian).

Piccardo, G. and Tubino, F. (2012), “Dynamic response of Euler-Bernoulli beams to resonant harmonic moving loads”, *Struct. Eng. Mech.*, **44** (5) 681-704.

Prager, W. and Save, M. (1963), “Minimum-weight design of beams subjected to fixed and moving loads”. *J. Mech. Physics Solids*, **11**, 255-267.

Samani, F. and Pellicano, F. (2009), “Vibration reduction on beams subjected to moving loads using linear and nonlinear dynamic absorbers”, *J. Sound Vib.*, **325**, 742-754.

Soares, R.M., del Prado, Z.J.G.N. and Goncales, P.B. (2010), “On the vibration of beams using a moving absorber and subjected to moving loads”, *Mecanica Comput.*, (29) 1829-1840.

Stokes, G.G. (1849), “Discussion of a differential equation relating to the breaking of railway bridges”, *Transactions of the Cambridge Philosophical Society*, **85**(5), 707-735.

Sun, L. and Luo, F. (2008), “Steady-state dynamic response of a Bernoulli-Euler beam on a viscoelastic foundation subjected to a platoon of moving dynamic load”, *J. Vib. Acoust*., **130**, 051002-1 – 051002-19.

Thambiratnam, D. and Zhuge, Y. (1996), “Dynamic analysis of beams on an elastic foundation subjected to moving loads”, *J. Sound Vib.*, **198** (2), 149-169.

Timoshenko, S.P. (1922), “On the forced vibrations of bridges”, *Philosophical Magazine Series 6*, **43**(257), 1018-1019.

Timoshenko , S.P. (1972), *Theory of elasticity*, Naukova Dumka, Kiev (In Russian)

Willis, R. (1849), *Report of the commissioners appointed to inquire into the application of iron to railway structures*, William Clowes & Sons, London.

Wu, J.J. (2005), “Dynamic analysis of inclined beam due to moving load”, *J. Sound Vib.*, **288**(1-2), 107-131.

Xia, H., Zhang, N. and Guo, W.W. (2006), “Analysis of resonance mechanism and conditions of train–bridge system”, *J. Sound Vib.*, **297,** 810-822.

Yang, Y.B., Yau, J.D. and Hsu, L.C. (1997), “Vibration of simple beams due to trains moving at high speeds”, *Eng. Struct.*, **19**(11), 936-944.

Yau, J.D. (2004), “Vibration of simply supported compound beams to moving loads”, *J. Marine Sci. Technol.*, **12**(4), 319-328.

Zehsaz, M., Sadeghi, M.H. and Asl, A.Z. (2009), “Dynamic Response of railway under a moving load”, *J. Appl. Sci*., **9**(8), 1474-1481.

Zheng, D.Y., Cheung, Y.K., Au, F.T.K. and Cheng, Y.S. (1998), “Vibration of multi-span non-uniform beams under moving loads by using modified beam vibration functions”, *J. Sound Vib*., **212**(3), 455-467.

Zibdeh, H.S. and Rackwitz, R. (1995), “Response moments of an elastic beam subjected to Poissonian moving loads”, *J. Sound Vib.*, **188** (4), 479-495.

Appendix

Notation

a	acceleration of the moving load	v	velocity of the moving load
a_{nor}	normal acceleration	v_0	initial velocity of the moving load
A_n	coefficients of the trigonometric series	w	beam deflection
B_n	coefficients of the trigonometric series	x	axial coordinate
c_{φ}	angular stiffness	$y(x)$	deflection due to force equal to 1
E	Young's modulus	$[C]$	damping matrix
EJ	bending stiffness	$[K]$	stiffness matrix
f_j	functions of k, $j = 1,2$	$[M]$	mass matrix
g	acceleration of gravity		
G	beam weight	*Greek symbols*	
G_{red}	reduced beam weight	α	coefficient
i	a serial number of a finite element	β_{red}	reduction coefficient of the beam weight
J	axial moment of inertia	ρ	density
k	coefficient	σ	normal stress
k_d	dynamic coefficient	ξ	distance from the left beam end
ℓ	beam length		
M	bending moment		
n	a serial number of the trigonometric members	*Subscripts*	
N	number of the finite elements	d	dynamic
$P(t)$	equivalent dynamic load	n	a serial number
Q	weight of the moving load	nor	normal
t	real time	red	reduced
U	potential energy of the elastic system	φ	angular
		0	initial