Coupled Systems Mechanics, *Vol. 3, No. 1 (2014) 89-110*
DOI: http://dx.doi.org/10.12989/csm.2014.3.1.089

Nonlinear instability problems including localized plastic failure and large deformations for extreme thermo-mechanical loads

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(Received January 11, 2014, Revised March 16, 2014, Accepted March 30, 2014)

Abstract. In this work we provide the theoretical formulation, discrete approximation and solution algorithm for instability problems combing geometric instability at large displacements and material instability due to softening under combined thermo-mechanical extreme loads. While the proposed approach and its implementation are sufficiently general to apply to vast majority of structural mechanics models, more detailed developments are provided for truss-bar model. Several numerical simulations are presented in order to illustrate a very satisfying performance of the proposed methodology.

Keywords: nonlinear instability; localized failure; large deformations; thermo-mechanical coupling

1. Introduction

In our recent work (see Ngo *et al*. 2013), we have studied the thermo-plasticity problem for small deformation along with the material instability phenomena related to softening under fully combined thermo-mechanic extreme conditions. However, in controlling the behavior of steel truss-bar (or cable) structure, the large deformations have to be considered as well; especially in the case when we need to determine the 'ultimate' resistance of the structure. The main goal of this paper is to provide a thermo-mechanical model, its discrete approximation, as well as the numerical solution procedure for such a problem for fully nonlinear case with geometric instability due to large deformations and material instability due to softening. The main novelty of this work is precisely such a general framework to be studied. Namely, there are a number of works on large deformation plasticity (e.g., Eterovic and Bathe 1990 or Ibrahimbegovic and Chorfi 2000), or those on extending these development further tothermomechanical coupling (e.g., Ibrahimbegovic and Chorfi 2002, Ibrahimbegovic *et al*. 2001, Simo and Miehe1992), but none of them considers material instability problems. Similarly, the works considering geometric instabilities (e.g., Hozjan *et al*. 2008, or Ibrahimbegovic *et al*. 1996, Ibrahimbegovic *et al*. 2013), do not in general include material instability phenomena either.

The paper is presented in four main sections. The first section covers the continuum behavior of the structure with large deformation. A model development to estimate the thermo-mechanical

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http://www.techno-press.org/?journal=csm&subpage=7 ISSN: 2234-2184 (Print), 2234-2192 (Online)

response of the structure when localized failure happens will be carried out in the second section. Section three focus on the numerical solution of the problem by using the 'adiabatic' operator split procedure. Several numerical examples will be presented in section four, followed by the final section with conclusions.

2. Continuum thermo-plasticity model for large deformation problem

2.1 Thermodynamic model and system of local balance equation

In the finite deformation 1D problem, the position vector in the deformed configuration is defined as (see Fig. 1)

$$\varphi(x,t) = x + u(x,t) \tag{1}$$

where *u(x,t)* is the corresponding displacement of the particle initially placed at position *x*. The intrinsic finite deformation measure can here be defined in terms of the stretch (e.g., Ibrahimbegovic 2009)

$$\lambda := \frac{d\varphi}{dx} = 1 + \frac{du}{dx} = 1 + \bar{\varepsilon} \tag{2}$$

Note that in the limit of small strain we recover the small strain deformation measure, $\lim_{\bar{\varepsilon}\to 0} \frac{\ln(1+\bar{\varepsilon})}{\bar{\varepsilon}} = 1$. For a number of theoretical formulations, it is convenient to employ the finite strain measure in terms of the logarithmic strain

$$\bar{\varepsilon} \approx \ln(1+\bar{\varepsilon}) = \ln(\lambda) \tag{3}$$

For example, in elasto-plastic material response, the stretch is considered to be the multiplicative decomposition of the 'elastic' stretch (the one that is recovered after complete unloading) and the 'plastic' stretch

$$\lambda := \frac{d\varphi}{dx} = \frac{l^{\varphi}}{l} = \frac{l^{\varphi}}{l^{p}}\frac{l^{p}}{l} = \lambda^{e}\lambda^{p} \tag{4}$$

where l^p is the residual length of the bar after complete unloading; the choice of the logarithmic strain, denoting $\bar{\varepsilon} := ln\lambda$; $\bar{\varepsilon}^e := ln\lambda^e$; $\bar{\varepsilon}^p := ln\lambda^p$, will than lead to the corresponding additive decomposition of the 'strain' that is equivalent to the small strain case

$$\bar{\varepsilon} = \bar{\varepsilon}^e + \bar{\varepsilon}^p \tag{5}$$

The thermo-plasticity free energy for large deformation problem is the composition of mechanical part, thermal part and thermo-mechanical interaction part (see Ibrahimbegovic *et al*. 2001, Ibrahimbegovic and Chorfi 2002, Ngo *et al*. 2013).

$$\psi\left(\bar{\varepsilon}^e, \bar{\xi}, \vartheta\right) = \frac{1}{2}E(\bar{\varepsilon} - \bar{\varepsilon}^p)^2 - \bar{q}\bar{\xi}^2 - \alpha E(\bar{\varepsilon} - \bar{\varepsilon}^p)(\vartheta - \vartheta_0) - \rho c[(\vartheta - \vartheta_0) - \vartheta \ln\left(\frac{\vartheta}{\vartheta_0}\right) \tag{6}$$

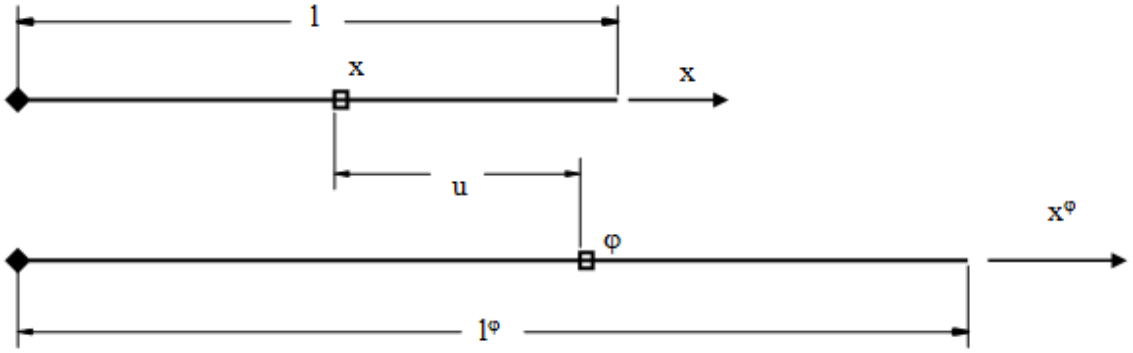


Fig. 1 Initial configuration and deformed configuration for 1D bar in finite deformation

Here ρc denotes the heat capacity coefficient, ϑ denotes the temperature, ϑ_0 denotes the reference temperature, E denotes the Young modulus and α denotes the thermal expansion coefficient.

With this kind of choice for thermo-mechanical free energy, we can establish the state equations for the Kirchhoff stress τ and the 'elastic' part of entropy $\bar{\eta}^e$

$$\tau = \frac{\partial \psi}{\partial \varepsilon^e} = E(\bar{\varepsilon} - \bar{\varepsilon}^p) - \alpha E(\vartheta - \vartheta_0) \tag{7}$$

$$\bar{\eta}^e = -\frac{\partial \psi}{\partial \vartheta} = \alpha E(\bar{\varepsilon} - \bar{\varepsilon}^p) + \rho c ln\left(\frac{\vartheta}{\vartheta_0}\right) \tag{8}$$

We note that the total entropy is also decomposed additively the 'elastic' entropy and the 'plastic' entropy.

The total dissipation is obtained from the second principle of thermodynamics (e.g., Ibrahimbegovic 2009) resulting with the additive decomposition of the mechanical and thermal parts

$$D^p = \underbrace{\tau \dot{\bar{\varepsilon}}^p + \bar{q}\dot{\bar{\xi}}}_{\bar{D}_{mech}} + \underbrace{\vartheta \dot{\bar{\eta}}^p}_{\bar{D}_{ther}} \tag{9}$$

Finally, from the principle of maximum plastic dissipation, we obtain the flow equations specifying the evolution of the internal variables, the plastic strain $\dot{\bar{\varepsilon}}^p$, the hardening variable $\dot{\bar{\xi}}$ and the plastic entropy part $\dot{\bar{\eta}}^p$

$$\dot{\bar{\varepsilon}}^p = \dot{\bar{\gamma}} \frac{\partial \bar{\phi}}{\partial \tau} = \dot{\bar{\gamma}} sign(\tau) \tag{10}$$

$$\dot{\bar{\xi}} = \dot{\bar{\gamma}} \frac{\partial \bar{\phi}}{\partial q} = \dot{\bar{\gamma}} \tag{11}$$

$$\dot{\bar{\eta}}^p = \dot{\bar{\gamma}} \frac{\partial \bar{\phi}}{\partial \vartheta} \tag{12}$$

Here, $\bar{\phi}$ is the yield criterion for the thermoplasticity problem, which can be presented in the simplest form as

$$0 \geq \bar{\phi}(\tau, \bar{q}, \vartheta) \coloneqq |\tau| - (\tau_y - \bar{q}) \tag{13}$$

where τ_y denotes the initial yield value of the Kirchhoff stress, which is very close to the true or Cauchy stress for plastically incompressible materials undergoing small elastic strain. With this kind of yield criterion, we can re-write the mechanical dissipation as a linear function of the plastic multiplier

$$D_{mech} = \dot{\gamma}\tau_y \tag{14}$$

The energy balance equation can now be re-written in the local form

$$\dot{e}\left(\bar{\varepsilon}^e, \bar{\xi}, \eta^e\right) + \left(\rho\dot{v} - \frac{dP}{dx} - b\right)v = P\dot{\lambda} + r - \frac{dQ}{dx} \tag{15}$$

where e is the internal energy density, v is the velocity field, b is the external distributed loading applied in the initial configuration, P is the first Piola-Kirchhoff stress as the natural replacement of the true stress in the initial configuration, r is the internal heat supply and Q is the first Piola-Kirchhoff type of heat flow representing the 'real' heat flow in the initial configuration.

The internal energy can be related to the free energy by using the Lagrange transformation

$$\bar{e}\left(\bar{\varepsilon}^e, \bar{\xi}, \eta^e\right) = \bar{\psi}\left(\bar{\varepsilon}^e, \bar{\xi}, \vartheta\right) + \bar{\eta}^e\vartheta \tag{16}$$

Combining these results leads to a final set of local balance equations

$$\begin{cases} \dfrac{dP}{dx} + b = 0 \\ \vartheta\dot{\bar{\eta}}^e = -\dfrac{dQ}{dx} + r + \bar{D}_{mech} \end{cases} \tag{17}$$

where $= \frac{d\bar{\psi}}{d\lambda} = \frac{d\bar{\psi}}{d\bar{\varepsilon}}\frac{d\bar{\varepsilon}}{d\lambda}$. By considering further that

$$P = \frac{d\bar{\psi}}{d\lambda} = \frac{d\bar{\psi}}{d\bar{\varepsilon}}\frac{d\bar{\varepsilon}}{d\lambda} = \frac{\tau}{\lambda} \rightarrow \frac{dP}{dx} = \frac{1}{\lambda^2}\left[\frac{d\tau}{dx}\lambda - \tau\frac{d\lambda}{dx}\right] = \frac{1}{\lambda^2}\left[\frac{d\tau}{dx}\lambda - \tau\underbrace{\frac{d^2\varphi}{dx^2}}_{0}\right] = \frac{1}{\lambda}\frac{d\tau}{dx} \tag{18}$$

the system of local balance equation can further be re-written in terms of the Kirchhoff stress

$$\begin{cases} \dfrac{d\tau}{dx} + \lambda b = 0 \\ \vartheta\dot{\bar{\eta}}^e = -\dfrac{dQ}{dx} + r + D_{mech} \end{cases} \tag{19}$$

By exploiting the state equation in Eq. (9), we can obtain the explicit expression for elastic part of entropy

$$\bar{\eta}^e = \alpha E(\bar{\varepsilon} - \bar{\varepsilon}^p) + \rho c ln\left(\frac{\vartheta}{\vartheta_0}\right) \rightarrow \dot{\bar{\eta}}^e = \alpha E(\dot{\bar{\varepsilon}} - \dot{\bar{\varepsilon}}^p) + \rho c\frac{\dot{\vartheta}}{\vartheta} \rightarrow \vartheta\dot{\bar{\eta}}^e = \alpha E\vartheta(\dot{\bar{\varepsilon}} - \dot{\bar{\varepsilon}}^p) + \rho c\dot{\vartheta} \tag{20}$$

With this result on hand, the equilibrium system becomes

$$\begin{aligned} \frac{d\tau}{dx} + \lambda b &= 0 \\ \rho c\dot{\vartheta} = -\frac{dQ}{dx} + r + \bar{D}_{mech} &- \alpha E\vartheta(\dot{\bar{\varepsilon}} - \dot{\bar{\varepsilon}}^{p}) \end{aligned} \tag{21}$$

Note that the Fourier law for heat flow in the finite deformation problem is

$$Q = -k\frac{d\vartheta}{dx^{\varphi}} = -\frac{k}{\lambda}\frac{d\vartheta}{dx} \tag{22}$$

We note that the set of equilibrium equations in Eq. (22) consists of the mechanical balance equation and the heat transfer equation. This set of differential equations cannot in general be solved analytically, thus an approximate solution should be carried out. Among number of possible solutions (see Romero 2009, Simo and Miehe 1992, Ibrahimbegovic 2009), we introduce the 'adiabatic' splitting procedure (e.g., Ngo *et al*. 2013), which was proved to provide a stable solution for the coupled thermomechanics problem (e.g., see Niekamp *et al*. 2014).

2.2 Adiabatic operator split procedure

In adiabatic operator split procedure, the general thermo-mechanical problem is divided in two the 'mechanical' process and the 'thermal' process; two processes connecting to each other by the 'adiabatic' constrain. In this procedure, the 'mechanical' process is assumed to develop while enforcing the adiabatic condition ($\dot{\bar{\eta}}^{e} = 0$) (see Table 1). In particular, the mechanical process is solved with the evolution of temperature due to structural heating only. The remaining evolution of temperature (due to external heating, internal heating and mechanical dissipation) is solved latter in the 'thermal' process.

Table 1 Adiabatic operator split procedure

Phase 1 - Mechanical part with "adiabatic"condition	Phase 2- Thermal part with 'adiabatic' condition
$\begin{cases} 0 = \dfrac{d\tau}{dx} + \lambda b \\ \dot{\eta}^{e} = 0 \rightarrow \rho c\dot{\vartheta} = -\vartheta E\alpha\left(\dot{\varepsilon} - \dot{\varepsilon}^{p}\right) \end{cases}$	$\rho c\dot{\vartheta} = -\dfrac{dQ}{dx} + r + D_{mech}$

2.2.1 Constitutive equation of Kirchhoff stress in the mechanical part

From the adiabatic condition, we have the possibility to compute the corresponding temperature evolution

$$\dot{\bar{\eta}}^{e} = 0 \rightarrow \dot{\vartheta} = -\frac{\vartheta\alpha E}{\rho c}(\dot{\bar{\varepsilon}} - \dot{\bar{\varepsilon}}^{p}) \tag{23}$$

Applying further this evolution of temperature to the state equation for Kirchhoff stress, we get

$$\tau = \mathrm{E}\left(\bar{\varepsilon}\text{-}\bar{\varepsilon}^{\mathrm{p}}\right)\text{-}\alpha\mathrm{E}\left(\vartheta\text{-}\vartheta_{0}\right) \rightarrow \dot{\tau} = \mathrm{E}\left(\dot{\bar{\varepsilon}}\text{-}\dot{\bar{\varepsilon}}^{\mathrm{p}}\right)\text{-}\alpha\mathrm{E}\dot{\vartheta} \tag{24}$$

$$\dot{\tau} = \underbrace{\left(\mathrm{E} + \frac{\alpha^{2}\mathrm{E}^{2}\vartheta}{\rho c}\right)}_{\mathrm{C}^{\mathrm{e,ad}}}\left(\dot{\bar{\varepsilon}}\text{-}\dot{\bar{\varepsilon}}^{\mathrm{p}}\right) \tag{25}$$

We can appeal to the plastic consistency condition

$$\dot{\bar{\phi}} = \frac{\partial \bar{\phi}}{\partial \tau}\dot{\tau} + \frac{\partial \bar{\phi}}{\partial \bar{q}}\dot{\bar{q}} + \frac{\partial \bar{\phi}}{\partial \vartheta}\dot{\vartheta} = 0 \tag{26}$$

Assuming that the plastic threshold it remains constant in each time step, we can write: $\frac{\partial \bar{\phi}}{\partial \vartheta} = 0$. We also have $\frac{\partial \bar{\phi}}{\partial \tau} = sign(\tau)$and $\frac{\partial \bar{\phi}}{\partial \bar{q}} = 1$. Thus, the condition in Eq. (26) can be re-written

$$\dot{\bar{\phi}} = \text{sign}(\tau)\dot{\tau} + \bar{K}\dot{\bar{\xi}} = 0 \rightarrow \dot{\tau} = \text{-sign}(\tau)\bar{K}\dot{\bar{\xi}} \tag{27}$$

where we assumed a linear hardening for the plasticity : $\bar{q} = -\bar{K}\bar{\xi}$. If we further write the evolution equations for plastic strain and hardening variable

$$\dot{\bar{\varepsilon}}^{\text{p}} = \dot{\bar{\gamma}}\frac{\partial \phi}{\partial \tau} = \dot{\bar{\gamma}}\text{sign}(\tau) \tag{28}$$

$$\dot{\bar{\varepsilon}}^{\text{p}} = \dot{\bar{\gamma}}\frac{\partial \phi}{\partial \tau} = \dot{\bar{\gamma}}\text{sign}(\tau) \tag{29}$$

the last two equations allow us to write $\dot{\bar{\xi}} = sign(\tau)\dot{\bar{\varepsilon}}^{p}$. By replacing this result into Eq. (49) we obtain

$$\dot{\tau} = \bar{\text{K}}\dot{\bar{\varepsilon}}^{\text{p}} \tag{30}$$

From (25) and (30), we can express the stress rate constitutive equation for the Kirchhoff stress in terms of the 'adiabatic' tangent modulus

$$\dot{\tau} = \frac{\left(\text{E}+\frac{\alpha^2\text{E}^2\vartheta}{\rho\text{c}}\right)}{\left(1+\left(\frac{\text{E}}{\bar{\text{K}}}+\frac{\alpha^2\text{E}^2\vartheta}{\bar{\text{K}}\rho\text{c}}\right)\right)}\dot{\bar{\varepsilon}} = \underbrace{\frac{\bar{\text{K}}\left(\text{E}+\frac{\alpha^2\text{E}^2\vartheta}{\rho\text{c}}\right)}{\bar{\text{K}}+\text{E}+\frac{\alpha^2\text{E}^2\vartheta}{\rho\text{c}}}}_{\text{C}^{\text{p,ad}}}\dot{\bar{\varepsilon}} \tag{31}$$

We note that the 'adiabatic' tangent modulus plays the same role as the normal tangent modulus in the mechanical problem only.

2.2.2 Local computation for mechanical variables in the mechanical process

In the mechanical phase, the internal variables are computed by the return mapping algorithm (e.g., see Ibrahimbegovic 2009). Here, it is assumed that we have already computed the mechanical variables of time step 'n': (τ_n, $\bar{\varepsilon}_n^p$, $\bar{\xi}_n$) and that we need to compute their new values at the next time step 'n+1':

$$\begin{aligned}\tau_{n+1} &= \tau_n + \Delta\tau_{n+1}\\ \bar{\varepsilon}_{n+1}^p &= \bar{\varepsilon}_n^p + \Delta\bar{\varepsilon}_{n+1}^p\\ \bar{\xi}_{n+1} &= \bar{\xi}_n + \Delta\bar{\xi}_{n+1}\end{aligned}$$

which jointly satisfy the plastic criterion at that time: $\bar{\phi}_{n+1} = |\tau_{n+1}| - \left(\tau_y + \bar{K}\bar{\xi}_{n+1}\right) = 0$. We start by computing the elastic trial value of the Kirchhoff stress by assuming the internal variables remain frozen; this leads to

$$\tau_{\text{n+1}}^{\text{trial}} = \tau_{\text{n}} + \text{C}^{\text{e,ad}}\Delta\varepsilon_{\text{n+1}} \tag{32}$$

where $C^{e,ad}$ was introduced in the Eq. (25).

By noting that $\tau_{n+1} = \tau_n + C^{e,ad}(\Delta\varepsilon - \Delta\varepsilon^p)$and $\Delta\varepsilon^p = \bar{\gamma}_{n+1}\frac{\partial\bar{\phi}}{\partial\tau}$, we can further write

$$\rightarrow \tau_{n+1} = \tau_{n+1}^{trial} - C^{e,ad}\bar{\gamma}_{n+1}\frac{\partial\bar{\phi}}{\partial\tau} \tag{33}$$

The trial value of the plastic criterion can then be computed as

$$\bar{\phi}_{n+1}^{trial} = \left|\tau_{n+1}^{trial}\right| - \left(\tau_y + \bar{K}\xi_n\right) \tag{34}$$

If this trial value remains negative or zero, the trial guess is indeed the correct solution, and the step is elastic. In the opposite, the correct positive value of the plastic multiplier must be computed by enforcing the plastic criterion

$$\bar{\phi}_{n+1} = \bar{\phi}_{n+1}^{trial} - \bar{\gamma}_{n+1}\left(C^{e,ad} + \bar{K}\right) = 0 \rightarrow \bar{\gamma}_{n+1} = \frac{\bar{\phi}_{n+1}^{trial}}{C^{e,ad} + \bar{K}} \tag{35}$$

With this plastic multiplier value, the internal variable and the Kirchhoff stress at the step 'n+1' can easily be computed

$$\bar{\xi}_{n+1} = \bar{\xi}_n + \bar{\gamma}_{n+1} \tag{36}$$

$$\bar{\varepsilon}_{n+1}^p = \bar{\varepsilon}_n^p + \bar{\gamma}_{n+1}\mathrm{sign}(\tau_{n+1}^{trial}) \tag{37}$$

The stress update at the same time step is then carried out as

$$\sigma_{n+1} = \left(1 - \frac{E\bar{\gamma}_{n+1}}{|\sigma_{n+1}^{trial}|}\right)\sigma_{n+1}^{trial} \tag{38}$$

and the mechanical dissipation at the consider time step is computed by the equation:

$$\bar{D}_{mech}^{n+1} = \bar{\gamma}_{n+1}\tau_y \tag{39}$$

3. Localized failure in one-dimensional finite deformation thermo-plasticity problem

3.1 Discontinuity in displacement

The localized failure happening in a steel truss-bar or cable required the corresponding kinematic enhancement in terms of 'discontinuity' or a 'jump' in the displacement field, so as to preserve the mesh objectivity (e.g., see Ngo *et al*. 2013 and references therein). In the framework of finite deformation, the displacement discontinuity has to be introduced in deformed configuration. In this case, we can cast the displacement field in terms of regular part and 'enhanced' part representing the 'jump'

$$\varphi = x + u(x) = x + \hat{u} + \bar{\bar{u}}\underbrace{[H_{\bar{x}}(x) + \varphi_0(x)]}_{M_1(x)} \tag{40}$$

where $\varphi_0(x)$ is at a sufficiently smooth C^1-function to limit the influence of the 'jump' in displacement within a particular zone, and $H_{\bar{x}}(x)$ is the Heaviside function

$$H_{\bar{x}} = \begin{cases} 0, x \le \bar{x} \\ 1, x > \bar{x} \end{cases} \tag{41}$$

The stretchmeasure of finite deformation thus becomes

$$\lambda = \frac{d\varphi}{dx} = 1 + \frac{d\hat{u}}{dx} + \left(\frac{d\varphi_0}{dx} + \delta_{\bar{x}}\right)\bar{\bar{u}} = 1 + \underbrace{\frac{d\hat{u}}{dx} + \frac{d\varphi_0}{dx}\bar{\bar{u}}}_{\bar{\varepsilon}} + \bar{\bar{u}}\delta_{\bar{x}} = (1+\bar{\varepsilon})\left(1 + \underbrace{\frac{\bar{\bar{u}}}{1+\bar{\varepsilon}}}_{\bar{\bar{\varepsilon}}^p}\delta_{\bar{x}}\right) \tag{42}$$

By taking further the logarithmic strain measure, we can re-write the last result as

$$\varepsilon := \ln(\lambda) = \ln(1+\bar{\varepsilon}) + \ln(1+\bar{\bar{\varepsilon}})\,\delta_{\bar{x}} \;\rightarrow\; \varepsilon = \bar{\varepsilon} + \bar{\bar{\varepsilon}}\delta_{\bar{x}} = \bar{\varepsilon}^e + \bar{\varepsilon}^p + \bar{\bar{\varepsilon}}^p\delta_{\bar{x}} \tag{43}$$

where the singular part of the strain field acts as the 'plastic' strain at the localized failure point; with

$$\bar{\bar{\varepsilon}}^{\mathrm{p}}\delta_{\bar{\mathrm{x}}} \;;\; \bar{\bar{\varepsilon}}^{\mathrm{p}} := \frac{\bar{\bar{\mathrm{u}}}}{1+\bar{\varepsilon}} \tag{44}$$

3.2 Mechanical phase computation at the localized failure point

The constitutive equation of the Kirchhoff stress in Eq. (31) at the localized failure point will take the following form

$$\dot{\tau}_{\bar{\mathrm{x}}} = \frac{\bar{\mathrm{K}}\left(\mathrm{E}+\frac{\alpha^2\mathrm{E}^2\vartheta_{\bar{\mathrm{x}}}}{\rho\mathrm{c}}\right)}{\bar{\mathrm{K}}+\mathrm{E}+\frac{\alpha^2\mathrm{E}^2\vartheta_{\bar{\mathrm{x}}}}{\rho\mathrm{c}}}\left(\dot{\bar{\varepsilon}} + \dot{\bar{\bar{\varepsilon}}}^{\mathrm{p}}\delta_{\bar{\mathrm{x}}}\right) \tag{45}$$

We note that $\dot{t}_{\bar{x}} = \dot{t}_{\bar{x}|\pm}$ and that the traction should be bounded, which implies that the singular part in Eq. (45) above should be equal to zero:

$$\frac{\bar{\mathrm{K}}\left(\mathrm{E}+\frac{\alpha^2\mathrm{E}^2\vartheta_{\bar{\mathrm{x}}}}{\rho\mathrm{c}}\right)}{\bar{\mathrm{K}}+\mathrm{E}+\frac{\alpha^2\mathrm{E}^2\vartheta_{\bar{\mathrm{x}}}}{\rho\mathrm{c}}}\dot{\bar{\bar{\varepsilon}}}^{\mathrm{p}} = 0 \rightarrow \bar{\mathrm{K}} = 0 \tag{46}$$

Applying the result in Eq. (25) at the localized failure point, we will have

$$\begin{aligned}\dot{t}_{\bar{x}} &= \left(E + \frac{\alpha^2E^2\vartheta}{\rho c}\right)\left(\dot{\bar{\varepsilon}} - \dot{\varepsilon}^p - \dot{\bar{\bar{\varepsilon}}}^p\delta_{\bar{x}}\right) \\ &= \left(E + \frac{\alpha^2E^2\vartheta}{\rho c}\right)\left(\dot{\bar{\varepsilon}} - \dot{\gamma}sign(\tau_{\bar{x}}) - \dot{\bar{\gamma}}sign(\tau_{\bar{x}})\delta_{\bar{x}}\right)\end{aligned} \tag{47}$$

where $\bar{\bar{\gamma}}$ denote the singular part of the plastic multiplier, which controls the evolution of 'plastic' deformation (displacement 'jump') at the localized failure point.

$$\dot{\bar{\bar{\mathrm{u}}}} = \dot{\bar{\bar{\gamma}}}\mathrm{sign}(\tau_{\bar{\mathrm{x}}}) \tag{48}$$

The activation of the localized failure is controlled by the following plasticity criterion

$$0 \geq \bar{\bar{\phi}}(\tau_{\bar{\mathrm{x}}}, \bar{\bar{\mathrm{q}}}, \vartheta_{\bar{\mathrm{x}}}) := \tau_{\bar{\mathrm{x}}}\text{-}(\tau_{\mathrm{u}}\text{-}\bar{\bar{\mathrm{q}}}) \tag{49}$$

where τ_u denotes the Kirchhoff type of ultimate traction, $\bar{\bar{q}}$ is the traction-like variable controlling the reduction of traction limit due to softening.

The evolution of the traction at the localized failure point should be in agreement with plastic consistency condition at that point:

$$\dot{\bar{\phi}} = \frac{\partial \bar{\phi}}{\partial \tau_{\bar{x}}} \dot{\tau}_{\bar{x}} + \frac{\partial \bar{\phi}}{\partial \bar{q}} \dot{\bar{q}} + \frac{\partial \bar{\phi}}{\partial \vartheta_{\bar{x}}} \dot{\vartheta}_{\bar{x}} = 0 \rightarrow \dot{\tau}_{\bar{x}} + \dot{\bar{q}} + \frac{\partial \bar{\phi}}{\partial \vartheta_{\bar{x}}} \dot{\vartheta}_{\bar{x}} = 0 \tag{50}$$

By assuming that the localized failure threshold is constant with respect to temperature at a particular time step ($\frac{\partial \bar{\bar{\phi}}}{\partial \vartheta_{\bar{x}}} = 0$) and by using a linear softening modulus for the localized failure with $\bar{\bar{q}} = -\bar{\bar{K}}\bar{\bar{\xi}} = -\bar{\bar{K}}\bar{\bar{\gamma}} = -\bar{\bar{K}}\bar{\bar{\varepsilon}}^{p}$, we can further obtain from Eq. (50)

$$\dot{\tau}_{\bar{x}} = \bar{\bar{K}}\dot{\bar{\varepsilon}}^{p} \tag{51}$$

The result in Eq. (51) corresponds to a rigid-plastic model for the traction–displacement 'jump' at the localized failure point (see Fig. 2).

With this decomposition of strain field, the following function of thermodynamic energy is established:

$$\psi(\varepsilon, \varepsilon^{p}, \xi, \vartheta) = \bar{\psi}(\bar{\varepsilon}, \bar{\varepsilon}^{p}, \bar{\xi}, \vartheta) + \bar{\bar{\psi}}\left(\bar{\bar{u}}, \bar{\bar{\xi}}, \vartheta_{\bar{x}}\right)\delta_{\bar{x}} \tag{52}$$

where $\bar{\psi}(\bar{\varepsilon}, \bar{\varepsilon}^{p}, \bar{\xi}, \vartheta)$ is the regular part controlling the energy at the bulk and $\bar{\bar{\psi}}\left(\bar{\bar{u}}, \bar{\bar{\xi}}, \vartheta_{\bar{x}}\right)$ is the singular part controlling the energy at the localized failure point.

The internal dissipation can also be decomposed into regular part for the bulk and the singular part for the localized failure zone

$$D_{int} = \underbrace{\overbrace{\tau\dot{\bar{\varepsilon}}^{p}}^{\bar{D}^{p}_{mech}} + \bar{q}\dot{\bar{\xi}} + \overbrace{\vartheta\dot{\eta}^{p}}^{\bar{D}_{ther}}}_{\bar{D}_{int}} + \underbrace{\overbrace{\tau_{\bar{x}}\dot{\bar{\bar{u}}} + \bar{\bar{q}}\dot{\bar{\bar{\xi}}}}^{\bar{\bar{D}}_{mech}} + \overbrace{\vartheta_{\bar{x}}\dot{\bar{\bar{\eta}}}^{p}}^{\bar{\bar{D}}_{ther}}}_{\bar{\bar{D}}_{int}}\delta_{\bar{x}} \tag{53}$$

We note from Eq. (53) that the internal dissipation in localized failure case consists of the regular part controlling the dissipation in the bulk and the singular part controlling the local dissipation in the localized failure point. The latter can also be recast in the following form

$$\bar{\bar{D}}_{mech} = \tau_{\bar{x}}\dot{\bar{\bar{u}}} + \bar{\bar{q}}\dot{\bar{\bar{\xi}}} = \dot{\bar{\bar{\gamma}}}\left(\tau_{\bar{x}} - (\tau_{u} - \bar{\bar{q}})\right) + \dot{\bar{\bar{\gamma}}}\tau_{u} = \dot{\bar{\bar{\gamma}}}\tau_{u} \tag{54}$$

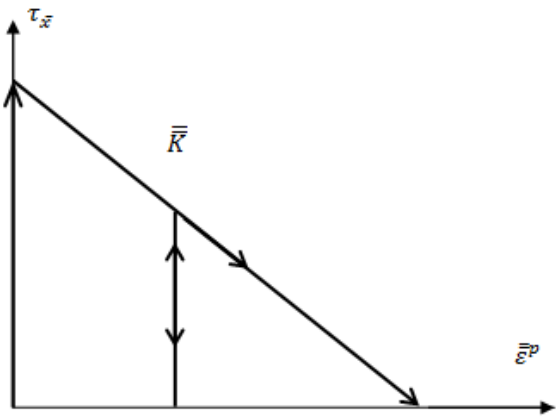


Fig. 2 Rigid-plastic model for the traction at localized failure point

3.3 Thermal phase at the localized failure point

By assuming that the heat capacity ρc is equal to zero at the localized failure point, and that the thermal expansion coefficient α also equals to zero, from the heat transfer equation in Eq. (23), we obtain

$$\left[\frac{dQ}{dx}\right]_{\bar{x}} = \bar{\bar{D}}_{mech} \rightarrow [\![Q]\!] = \bar{\bar{D}}_{mech}\delta_{\bar{x}} \tag{55}$$

Thus, we obtain the 'jump' in the heat flux at the localized failure point which is produced by the corresponding value of dissipation at that point.

3.4 Set of local balance equation

3.4.1 Mechanical balance equation

Given that the traction must remain continuous at the localized failure point, we can write the following equation

$$P(x = \bar{x}|_{\pm}) = t^{\varphi}_{\bar{x}^{\varphi}} \rightarrow \tau(x = \bar{x}|_{\pm}) = \tau_{\bar{x}} \tag{56}$$

where t^{φ} is the traction force at the localized failure point written at the deformed configuration, whereas P is the first Piola-Kirchhoff stress which takes the same value at the same point in the initial configuration. The complete set of mechanical balance equations can be written as:

$$\begin{cases} \frac{d\tau(\lambda,\vartheta)}{dx} + \lambda b = 0 \quad \text{for } x \neq \bar{x} \\ \tau(x = \bar{x}|_{\pm}) = \tau_{\bar{x}} \text{for} x = \bar{x} \end{cases} \tag{57}$$

3.4.2 Thermal balance equation

The thermal balance equation consists of the heat transfer equation written for the bulk and also the 'jump' in heat flow at the localized failure point

$$\begin{cases} \rho c\dot{\vartheta} = -\frac{dQ}{dx} + r \qquad \text{for} x \neq \bar{x} \\ [\![Q]\!] = \bar{\bar{D}}_{mech}\delta_{\bar{x}} \text{for} x = \bar{x} \end{cases} \tag{58}$$

3.4.3 Local balance equation system

By combining (42) and (43), we finally have the local balance system for the localized failure problem:

$$\begin{cases} \begin{cases} \frac{d\tau(\lambda,\vartheta)}{dx} + \lambda b = 0 \qquad \text{for } x \neq \bar{x} \\ \tau(x = \bar{x}|_{\pm}) = \tau_{\bar{x}} \text{for} x = \bar{x} \end{cases} \\ \begin{cases} \rho c\dot{\vartheta} = -\frac{dQ}{dx} + r \qquad \text{for} x \neq \bar{x} \\ [\![Q]\!] = \bar{\bar{D}}_{mech}\delta_{\bar{x}} \text{for} x = \bar{x} \end{cases} \end{cases} \tag{59}$$

4. Numerical solution procedure

4.1 Adiabatic operator split

The adiabatic operator split will have a new appearance when considering the localized failure:

Table 2Adiabatic operator split procedure considering the localized failure

Phase 1 - Mechanical part with "adiabatic"condition	Phase 2- Thermal part with 'adiabatic' condition
$\begin{cases} 0 = \dfrac{d\tau}{dx} + \lambda b \\ \dot{\vec{\eta}}^e = 0 \rightarrow \rho c \dot{\vartheta} = -\vartheta E \alpha \left(\dot{\vec{\varepsilon}} - \dot{\vec{\varepsilon}}^p\right) \end{cases}$	$\rho c \dot{\tilde{\vartheta}} = -\dfrac{dQ}{dx} + \bar{D}_{mech} + r$
(at localized failure point):	(at localized failure point):
$\tau_{-\vert\bar{x}} = \tau_{+\vert\bar{x}} = \tau_{\bar{x}}$	$[\![Q]\!] = \left(\bar{\bar{D}}_{mech}\right)\vert_{\bar{x}}$

4.2 Mechanical part

The following interpolation function is chosen for the displacement:

$$u^h(x,t) = \sum_{a=1}^{2} N_a(x) d_a(t) + M_1(x)\alpha_1^m(t) \tag{60}$$

where α_1^m is the coefficient representing the 'jump' in the displacement field at the localized failure point. We denote with $M_1(x)$ an additional shape funtion modelling the 'jump' in displacement field due to localized failure. The function $M_1(x)$ and it derivative are represented in Fig. 3.

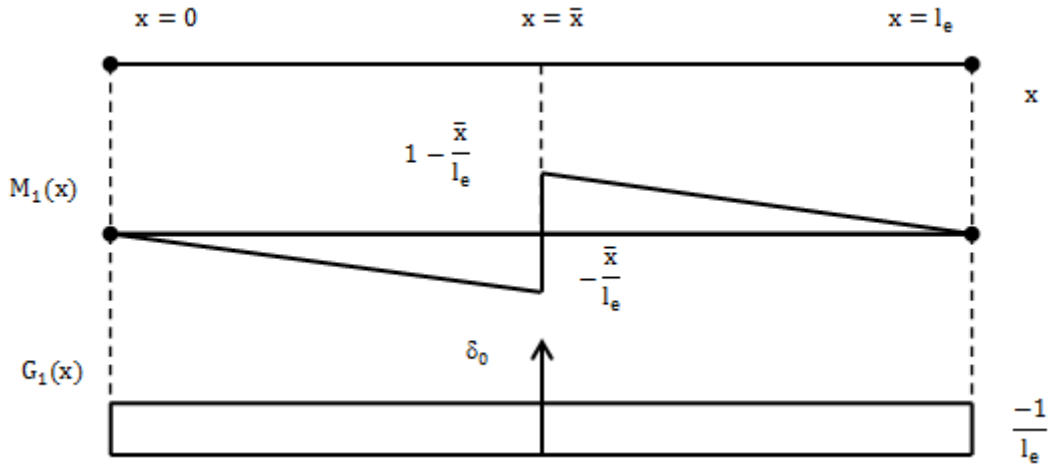


Fig. 3 Additional shape function $M_1(x)$ and its derivative

$$M_1(x) = H_{\bar{x}} - N_2(x) = \begin{cases} -\frac{x}{l_e} \text{ if } x \in [0, \frac{l_e}{2}[\\ 1 - \frac{x}{l_e} \text{ if } x \in]\frac{l_e}{2}, l_e] \end{cases} \tag{61}$$

$$G_1(x) = \frac{dM_1(x)}{dx} = \begin{cases} -\frac{1}{l^e} & \text{if } x \in [0, \frac{l_e}{2}[\cup]\frac{l_e}{2}, l_e] \\ -\frac{1}{l^e} + \delta_{\bar{x}} & \text{if } x = \bar{x} = \frac{l_e}{2} \end{cases} = \underbrace{-\frac{1}{l^e}}_{\bar{G}_1(x)} + \underbrace{\delta_{\bar{x}(\bar{x}=\frac{l_e}{2})}}_{\bar{\bar{G}}_1(x)\delta_{\bar{x}}} \tag{62}$$

In the finite element setting we choose the isoparametric shape function $N_2(x)$ as the C^1-function $\varphi_0(x)$.

The strain field therefore becomes

$$\varepsilon^h(x,t) = \sum_{a=1}^{2} B_a(x) w_a(t) + G_1(x)\beta_1^m(t) \tag{63}$$

where $B_a(x) = \frac{dN_a(x)}{dx}$. We assume the same kind of interpolation is used for the virtual strain

$$\frac{dw}{dx} = \sum_{a=1}^{2} B_a(x) d_a(t) + G_{1v}(x)\alpha_1^m(t) \tag{64}$$

where w_a denotes the virtual displacement nodal value and $\beta.$ is the variation of the displacement jump. In order to enforce the patch test satisfaction, we have to employ the following modified form of $G_{1v}(x)$

$$G_{1v}(x) = G_1(x) - \frac{1}{l^e}\int_{l^e} G_1(x)dx \tag{65}$$

For a two-node element, with the displacement discontinuity placed in its center, we obtain

$$G_v(x) = \bar{G}_v(x) + \bar{\bar{G}}_v(x)\delta_{\bar{x}} = -\frac{1}{l^e} + \delta_{\bar{x}} \tag{66}$$

The weak form of the mechanical balance equation can be written as:

$$\underbrace{\int_l wbdx + wb(l) - wb(0)}_{f_{ext}} - \underbrace{\int_l \frac{1}{\lambda}\frac{dw}{dx}\tau(\varepsilon,\vartheta)dx}_{f_{int}} = 0 \tag{67}$$

By applying the mentioned interpolation functions for the virtual strain, we have

$$\begin{cases} A_{e=1}^{N_{el}}\left[\int_{l^e}\frac{1}{\lambda}B^T\tau(\varepsilon,\vartheta)dx - \int_{l^e} N^T b dx\right] = 0 \\ \frac{1}{\lambda}\left[\int_{l^e}\bar{G}_v(x)\,\tau(\varepsilon,\vartheta)dx + \int_{l^e}\bar{\bar{G}}_v(x)\tau_{\bar{x}}\delta_{\bar{x}}dx\right] = 0 \end{cases} \tag{68}$$

$$\begin{gathered} A_{e=1}^{N_{el}}\left[\int_{l^e}\frac{1}{\lambda}B^T\tau(\varepsilon,\vartheta)dx - \int_{l^e} N^T b dx\right] = 0 \\ h^e = -\frac{1}{l^e}\int_{l^e}\tau(\varepsilon,\vartheta)dx + \tau_{\bar{x}} = 0 \end{gathered} \tag{69}$$

Eq. (70) can be brought to the linearized form of the balance equations can be written by making use of the following auxiliary result

$$\lambda = 1 + \frac{\mathrm{du}}{\mathrm{dx}} = 1 + \mathrm{B}^{\mathrm{T}}\mathrm{d} + \widetilde{\mathrm{G}_1}(\mathrm{x})\alpha_1^{\mathrm{m}} \rightarrow \Delta\lambda = \mathrm{B}^{\mathrm{T}}\Delta\mathrm{d} + \overline{\mathrm{G}}_1(\mathrm{x})\Delta\alpha_1^{\mathrm{m}} \tag{70}$$

We thus obtain the linearized form of the internal force vector for the bulk part

$$\mathrm{f}_{\mathrm{int,e}} = \int_{\mathrm{l^e}} \mathrm{B}^{\mathrm{T}} \frac{1}{\lambda}\tau(\mathrm{d_a}, \alpha_1^{\mathrm{m}})\mathrm{dx} \rightarrow \Delta\mathrm{f}_{\mathrm{int,e}} = \frac{\partial \mathrm{f}_{\mathrm{int,e}}}{\partial\lambda}\Delta\lambda = \int_{\mathrm{l^e}} \mathrm{B}^{\mathrm{T}} \left(-\frac{\tau}{\lambda^2} + \underbrace{\frac{\mathrm{d}\tau}{\mathrm{d}\varepsilon}}_{\mathrm{C}} \underbrace{\frac{\mathrm{d}\varepsilon}{\mathrm{d}\lambda}}_{\frac{1}{\lambda}} \right) \mathrm{dx}\,\Delta\lambda \tag{70a}$$

$$\rightarrow \Delta\mathrm{f}_{\mathrm{int,e}} = \int_{\mathrm{l^e}} \mathrm{B}^{\mathrm{T}} \left(\frac{\mathrm{C}}{\lambda} - \frac{\tau}{\lambda^2}\right) \mathrm{dx}\,\Delta\lambda \rightarrow \Delta\mathrm{f}_{\mathrm{int,e}} = \int_{\mathrm{l^e}} \left\{\mathrm{B}^{\mathrm{T}} \left(\frac{\mathrm{C}}{\lambda} - \frac{\tau}{\lambda^2}\right)\right\} \mathrm{dx} \left(\mathrm{B}^{\mathrm{T}}\Delta\mathrm{d} + \overline{\mathrm{G}}_1(\mathrm{x})\Delta\alpha_1^{\mathrm{m}}\right)$$

$$\Delta\mathrm{f}_{\mathrm{int,e}} = \underbrace{\int_{\mathrm{l^e}} \left\{\left(\frac{\mathrm{C}}{\lambda} - \frac{\tau}{\lambda^2}\right) \mathrm{B}^{\mathrm{T}}\mathrm{B}^{\mathrm{T}}\right\} \mathrm{dx}}_{\mathrm{K_e}} \Delta\mathrm{d} + \underbrace{\int_{\mathrm{l^e}} \left\{\left(\frac{\mathrm{C}}{\lambda} - \frac{\tau}{\lambda^2}\right) \mathrm{B}^{\mathrm{T}}\overline{\mathrm{G}}_1(\mathrm{x})\right\} \mathrm{dx}}_{\mathrm{F_e}} \Delta\alpha_1^{\mathrm{m}} \tag{71}$$

where *C* denotes the 'adiabatic' tangent modulus of the Kirchhoff stress and the logarithmic strain.

The linearized form of the mechanical internal force pertinent to the localized failure point can be written as

$$\Delta\left(\int_{l^e} \bar{G}_1(x)\tau(\varepsilon, \vartheta)dx\right) = \int_{l^e} \bar{G}_1(x) \left(\frac{d\tau}{d\varepsilon}\right) \left(\frac{d\varepsilon}{d\lambda}\right) dx\Delta\lambda \tag{70b}$$

$$\Delta\left(\int_{\mathrm{l^e}} \overline{\mathrm{G}}_1(\mathrm{x})\tau\mathrm{dx}\right) = \underbrace{\int_{\mathrm{l^e}} \overline{\mathrm{G}}_1(\mathrm{x}) \frac{\mathrm{C}}{\lambda} \mathrm{B}^{\mathrm{T}}\mathrm{dx}}_{\mathrm{L_e}} \Delta\mathrm{d} + \underbrace{\int_{\mathrm{l^e}} \overline{\mathrm{G}}_1(\mathrm{x}) \frac{\mathrm{C}}{\lambda} \mathrm{dx}\overline{\mathrm{G}}_1(\mathrm{x})\mathrm{dx}}_{\mathrm{H_e}} \Delta\alpha_1^{\mathrm{m}} \tag{72}$$

$$\Delta\mathrm{t}_{\bar{\mathrm{x}}} = \frac{\partial \mathrm{t}_{\bar{\mathrm{x}}}}{\partial \mathrm{d}}\Delta\mathrm{d} + \frac{\partial \mathrm{t}_{\bar{\mathrm{x}}}}{\partial \alpha_1^{\mathrm{m}}}\Delta\alpha_1^{\mathrm{m}} = \mathrm{P}_{\mathrm{e}}^{\mathrm{n+1}}\Delta\mathrm{d} + \overline{\overline{\mathrm{K}}}\Delta\alpha_1^{\mathrm{m}} \tag{73}$$

By further replacing the finite element interpolations in the linearized equations above, we obtain the corresponding set of linear algebraic equations expressing discrete linearized form of the mechanical balance equations

$$\begin{cases} \mathrm{A}_{\mathrm{e=1}}^{\mathrm{N_{elm}}}\left\{\mathrm{K}_{\mathrm{e}}^{\mathrm{n+1,i}}\Delta\mathrm{d}_{\mathrm{n+1}}^{\mathrm{i}} + \mathrm{F}_{\mathrm{e}}^{\mathrm{n+1,i}}\Delta\alpha_{1(\mathrm{n+1})}^{\mathrm{m,i}}\right\} = \mathrm{A}_{\mathrm{e=1}}^{\mathrm{N_{elm}}}\left\{\mathrm{f}_{\mathrm{n}}^{\mathrm{e,ext,i}} - \mathrm{f}_{\mathrm{n}}^{\mathrm{e,int}}\right. \\ \left. \mathrm{h}_{\mathrm{n}}^{\mathrm{e(i)}} + \left[\mathrm{L}_{\mathrm{e}}^{\mathrm{n+1,i}} + \mathrm{P}_{\mathrm{e}}^{\mathrm{n+1,i}}\right]\Delta\mathrm{d}_{\mathrm{n+1}}^{\mathrm{i}} + \left[\mathrm{H}_{\mathrm{e}}^{\mathrm{n+1,i}} + \overline{\overline{\mathrm{K}}}\right]\Delta\alpha_1^{\mathrm{m}} = 0 \right. \end{cases} \tag{74}$$

where

$$\mathrm{K_e} \coloneqq \int_{\mathrm{l^e}} \left(\frac{\mathrm{C}}{\lambda} - \frac{\tau}{\lambda^2}\right) \mathrm{B}^{\mathrm{T}}\mathrm{B}^{\mathrm{T}}\mathrm{dx} \rightarrow \mathrm{K_e} = \frac{\left(\frac{\mathrm{C}}{\lambda} - \frac{\tau}{\lambda^2}\right)\mathrm{A}}{\mathrm{l^e}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{75}$$

$$\mathrm{F_e} \coloneqq \int_{\mathrm{l^e}} \left(\frac{\mathrm{C}}{\lambda} - \frac{\tau}{\lambda^2}\right) \mathrm{B}^{\mathrm{T}}\overline{\mathrm{G}}_1(\mathrm{x})\mathrm{dx} \rightarrow \mathrm{F_e} = -\frac{\left(\frac{\mathrm{C}}{\lambda} - \frac{\tau}{\lambda^2}\right)\mathrm{A}}{\mathrm{l^e}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{76}$$

$$\mathrm{L_e} \coloneqq \int_{\mathrm{l^e}} \frac{\mathrm{C}}{\lambda} \mathrm{B}^{\mathrm{T}}\overline{\mathrm{G}}_1(\mathrm{x})\mathrm{dx} \rightarrow \mathrm{L_e} = -\frac{\mathrm{C}}{\lambda \mathrm{l^e}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{77}$$

$$\begin{gathered} P_e^{n+1,i} = \frac{\partial t_{\bar{x}}}{\partial d} = -\frac{C}{l^e}\int_{l^e} B^T dx \rightarrow P_e = -\frac{C}{l^e}\begin{bmatrix}1\\-1\end{bmatrix} \\ H_e := \int_{l^e} \frac{C}{\lambda}\bar{G}_1(x)\, dx \bar{G}_1(x) dx \rightarrow H_e = \frac{C}{\lambda l^e} \end{gathered} \tag{78}$$

In the set of equations in Eq. (75), we start by solving the second one pertaining to the localized failure point, leading to the element level computations

$$\Delta\alpha_1^{m,n+1} = -\left(H_e^{n+1,i} + \bar{\bar{K}}^i\right)^{-1}\left(L_e^{n+1,i} + P_e^{n+1,i}\right)\Delta d_{n+1}^i \tag{79}$$

By replacing this result into the first of the equations Eq. (75), we recover the 'static-condensation' form of the global mechanical balance equation

$$A_{e=1}^{N_{elm}}\left\{\hat{K}_e^{n+1,i}\Delta d_{n+1}^i\right\} = A_{e=1}^{N_{elm}}\left\{f_n^{e,ext,i} - f_n^{e,int,i}\right\} \tag{80}$$

where

$$\hat{K}_e^{n+1,i} = K_e^{n+1,i} - F_e^{n+1,i}\left(H_e^{n+1,i} + \bar{\bar{K}}^i\right)^{-1}\left(L_e^{n+1,i} + P_e^{n+1,i}\right) \tag{81}$$

$$\hat{K}_e^{n+1,i} = \left[1 - \left(\frac{C^i}{\lambda l^e} + \bar{\bar{K}}^i\right)^{-1}\left(\frac{1}{\lambda}+1\right)\frac{C^i}{\lambda l^e}\right]\left(\frac{C^i}{\lambda} - \frac{\tau}{\lambda^2}\right)\frac{A}{l^e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \tag{82}$$

4.2.1 Local computations of internal variables in bulk part

The internal variables are computed by the return mapping algorithm (e.g., see Ibrahimbegovic 2009). Assume that we also have: τ_n, $\bar{\varepsilon}_n^p$ and $\bar{\xi}_n$ and we need to compute

$$\begin{gathered} \tau_{n+1} = \tau_n + \Delta\tau_{n+1} \\ \bar{\varepsilon}_{n+1}^p = \bar{\varepsilon}_n^p + \Delta\bar{\varepsilon}_{n+1}^p \\ \bar{\xi}_{n+1} = \bar{\xi}_n + \Delta\bar{\xi}_{n+1} \end{gathered}$$

In order to satisfy the condition: $\bar{\phi}_{n+1} = |\tau_{n+1}| - \left(\tau_y + \bar{K}\bar{\xi}_{n+1}\right) = 0$, we start by computing the elastic trial value of the Kirchhoff stress

$$\tau_{n+1}^{trial} = \tau_n + C^{e,ad}\Delta\varepsilon_{n+1} \tag{83}$$

where $C^{e,ad}$ is the adiabatic tangent matrix as defined in Eq. (25). We note that $\tau_{n+1} = \tau_n + C^{e,ad}(\Delta\varepsilon - \Delta\varepsilon^p)$and $\Delta\varepsilon^p = \bar{\gamma}_{n+1}\frac{\partial\bar{\phi}}{\partial\tau}$, so that we will have

$$\rightarrow \tau_{n+1} = \tau_{n+1}^{trial} - C^{e,ad}\bar{\gamma}_{n+1}\frac{\partial\bar{\phi}}{\partial\tau} \tag{84}$$

The trial value of the plastic criterion can thus be computed

$$\bar{\phi}_{n+1}^{trial} = \left|\tau_{n+1}^{trial}\right| - \left(\tau_y + \bar{K}\xi_n\right) \tag{85}$$

If this value is negative or zero, the trial step is accepted as the correct guess. In the opposite, we have to compute the plastic multiplier value that will enforce the satisfaction of the plasticity criterion

$$\bar{\phi}_{n+1} = \bar{\phi}_{n+1}^{trial} - \bar{\gamma}_{n+1}\left(C^{e,ad} + \bar{K}\right) = 0 \rightarrow \bar{\gamma}_{n+1} = \frac{\bar{\phi}_{n+1}^{trial}}{C^{e,ad}+\bar{K}} \tag{86}$$

Given this value of the plastic multiplier, we can carry out the corresponding updates of the internal variables and the Kirchhoff stress at the step 'n+1'

$$\bar{\xi}_{n+1} = \bar{\xi}_n + \bar{\gamma}_{n+1} \tag{87}$$

$$\bar{\varepsilon}_{n+1}^{p} = \bar{\varepsilon}_n^{p} + \bar{\gamma}_{n+1}\mathrm{sign}(\sigma_{n+1}^{trial}) \tag{88}$$

$$\sigma_{n+1} = \left(1 - \frac{E\bar{\gamma}_{n+1}}{|\sigma_{n+1}^{trial}|}\right)\sigma_{n+1}^{trial} \tag{89}$$

The bulk part of mechanical dissipation at the time step 'n+1' can then be computed explicitly

$$\bar{\phi}_{n+1} = \bar{\phi}_{n+1}^{trial} - \bar{\gamma}_{n+1}\left(C^{e,ad} + \bar{K}\right) = 0 \rightarrow \bar{\gamma}_{n+1} = \frac{\bar{\phi}_{n+1}^{trial}}{C^{e,ad}+\bar{K}} \tag{90}$$

4.2.2 Local computations of internal variables at the localized failure point

Once the localized failure is initiated, it is accompanied by the softening behavior until the complete failure of the bar. This local computation seeks to provide the corresponding value of displacement 'jump' and the traction in the next time step (α_{n+1}^{m}, $\tau_{\bar{x}}^{n+1}$) starting with the displacement 'jump' in the current time step: (α_n^m, $\tau_{\bar{x}}^{n}$). The computed result should satisfy the localized failure criterion

$$\bar{\bar{\phi}}_{n+1} = \tau_{\bar{x},n+1} - \left(\tau_u + \bar{\bar{K}}\bar{\bar{\xi}}_{n+1}\right) = 0 \tag{91}$$

In order to test this, we first compute the elastic 'trial' traction

$$\tau_{\bar{x},n+1}^{trial} = \frac{1}{l^e}\int_{l^e} \tau_{n+1}^{trial}(\varepsilon,\vartheta)dx = \frac{1}{l^e}\int_{l^e} C^{e,ad}\left(B^T d_{n+1}^{i} + \bar{G}_1(x)\alpha_{1,n}^{m} - \varepsilon_{n+1}^{p}\right)dx \tag{92}$$

If the trial value of yield function is positive, we have to compute the true value of the traction, which can be written as

$$\tau_{\bar{x},n+1} = \frac{1}{l^e}\int_{l^e} \tau_{n+1}(\varepsilon,\vartheta)dx = \frac{1}{l^e}\int_{l^e} C^{e,ad}\left(B^T d_{n+1}^{i} + \bar{G}_1(x)\alpha_{1,n+1}^{m} - \varepsilon_{n+1}^{p}\right)dx \tag{93}$$

Combining the last two results, we can further obtain

$$\tau_{\bar{x},n+1} = \tau_{\bar{x},n+1}^{trial} + \frac{1}{l^e}\int_{l^e} C^{e,ad}\bar{G}_1(x)\Delta\alpha_{1,n+1}^{m}dx \tag{94}$$

With this result in hand, we can express the final value of yield function in terms of its trial value

$$0 = \bar{\bar{\phi}}_{n+1} = \bar{\bar{\phi}}_{n+1}^{trial} + \frac{1}{l^e}\int_{l^e} C^{e,ad}\bar{G}_1(x)\bar{\bar{\gamma}}_{n+1}dx + \bar{\bar{K}}\bar{\bar{\gamma}}_{n+1} = 0 \tag{95}$$

where

$$\bar{\bar{\phi}}_{n+1}^{trial} = \tau_{\bar{x},n+1}^{trial} - \left(\tau_u + \bar{\bar{K}}\bar{\bar{\xi}}_n\right) = 0 \tag{96}$$

The singular part of plastic multiplier can then be computed at the final time

$$\bar{\bar{\gamma}}_{n+1} = \frac{\bar{\bar{\Phi}}_{n+1}^{trial}}{\frac{C^{e,ad}}{l_e} + \bar{\bar{K}}} \tag{97}$$

This result allows us to provide the corresponding updates for the softening variable

$$\Delta\alpha_{1,n+1}^{m} = \Delta\bar{\bar{\xi}}_{n+1} = \bar{\bar{\gamma}}_{n+1} \tag{98}$$

We can also obtain the update of the traction

$$\tau_{\bar{x},n+1} = \tau_{\bar{x},n+1}^{trial} - \frac{C^{e,ad}}{l^e}\bar{\bar{\gamma}}_{n+1} \tag{99}$$

4.3Thermal transfer equation

The temperature transfer equation at the localized failure point is solved with two following hypotheses: 1) there is a 'jump' in temperature gradient at the localized failure point; 2) the structural heating is not taken into account in this process. However, we should also add the evolution of temperature due to the structural dissipation, which was compute in the mechanical phase to the final evolution of temperature.

$$\begin{cases} \rho c\dot{\bar{\vartheta}} = -\frac{dQ}{dx} + \bar{D}_{mech} + r & for x \neq \bar{x} \\ [\![Q]\!] = \bar{\bar{D}}_{mech}\delta_{\bar{x}} & for x = \bar{x} \end{cases} \tag{100}$$

The heat transfer obeys the Fourier heat conduction law for the bulk

$$Q = -k\frac{1}{\lambda}\frac{d\vartheta}{dx} \tag{101}$$

The local energy balance can be rewritten in the equivalent form to the heat equation

$$\rho c\dot{\vartheta} = k\frac{d^2\vartheta}{\lambda dx^2} + r + \bar{D}_{mech} + \bar{\bar{D}}_{mech}\delta_{\bar{x}} \tag{102}$$

The strong form Eq. (102) is further transferred into weak form by introducing an admissible variation in the temperature field, further denoted as ϑ, and by applying the virtual work laws

$$\int_0^l w_\vartheta \rho c\dot{\vartheta} dx + \int_0^l \frac{dw_\vartheta}{dx}\frac{k}{\lambda}\frac{d\vartheta}{dx}dx = \int_0^l w_\vartheta(r + \bar{D}_{mech} + \bar{\bar{D}}_{mech}\delta_{\bar{x}})dx \tag{103}$$

Where w_ϑ denotes the virtual temperature field. For constructing the finite element approximations, we consider a 2-node truss-bar element. The nodal values of temperature and the weighting temperature at node *i* are denoted as $d_{\vartheta i}$ and $w_{\vartheta i}$, respectively. More precisely, $\mathbf{d}_\vartheta$ and $\mathbf{w}_\vartheta$ denote the real and weighted nodal temperature vector, respectively; for a 2-node element, we have

$$d_\theta = \begin{Bmatrix} d_{\vartheta_1} \\ d_{\vartheta_2} \end{Bmatrix}; w_\vartheta = \begin{Bmatrix} w_{\vartheta_1} \\ w_{\vartheta_2} \end{Bmatrix}, \tag{104}$$

The real and weighting temperature fields along the element are constructed with isoparametric interpolation shape functions.

$$\vartheta(x) = \sum_{a=1}^{2} N_a(x) d_\vartheta^a(t) \tag{105}$$

$$w_\vartheta(x) = \sum_{a=1}^{2} N_a(x) w_\vartheta^a \tag{106}$$

By taking into account the interpolation of real and weight temperature fields, the weak form (103) is finally reduced to

$$A_{e=1}^{N_{el}} \left(\int_{l^e} \left[N(x) \rho c N(x) \dot{d}_\vartheta + B(x) \frac{k}{\lambda} B(x) d_\vartheta \right] dx \right) = A_{e=1}^{N_{el}} (Q_\vartheta^e) \tag{107}$$

Finally, the finite element equations to be solved for the 'thermal' phase are given by

$$A_{e=1}^{N_{el}} \left(M^e \dot{d}_\vartheta + K^e d_\vartheta \right) = A_{e=1}^{N_{el}} (Q_\vartheta^e) \tag{108}$$

where

$$M_{2x2}^e = \int_{l^e} N(x) \rho c N(x)\, dx \tag{109}$$

$$K_{2x2}^e = \int_{l^e} B(x) \frac{k}{\lambda} B(x)\, dx \tag{110}$$

$$\begin{gathered} Q_{\vartheta_{1x2}}^e = \int_{l^e} N(x) \left(r + \overline{D}_{mech} + \overline{\overline{D}}_{mech} \delta_{\bar{x}} \right) dx = \\ \int_{l^e} N(x) (r + \overline{D}_{mech})\, dx + 0.5 \overline{\overline{D}}_{mech} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \end{gathered} \tag{111}$$

There are many methods capable of solving the time-dependent Eq. (108). In this paper, the α-integration scheme is chosen (see Bathe 1996, Ibrahimbegovic 2009). Assuming that the time interval of interest for the heat transfer problem $[0,T]$ can be divided into n increments: $[t_0=0, t_1.., t_k, ..t_{n-1}, t_n = T]$, with a typical time step $h = t_{k+1} - t_k$.

By considering the equation of Newmark: $\Delta \dot{\boldsymbol{d}}_\vartheta = \Delta \boldsymbol{v} = \frac{1}{\alpha h} \Delta \boldsymbol{d}_\vartheta$ (where γ and β are the Newmark parameters) and by linearization, equation (111) becomes

$$A_{e=1}^{N_{el}} \left(\left(\frac{\gamma}{\beta \Delta t} M^e + K^e \right) \Delta d_\vartheta \right) = A_{e=1}^{N_{el}} \left(R^k \right) \tag{112}$$

where the residuals are computed by the following equation

$$R^k = Q_\vartheta^e - M^e \dot{d}_\vartheta^k - K^e d_\vartheta^k \tag{113}$$

Once $\boldsymbol{\Delta d}_\vartheta$ is known, the nodal temperature at the next time step can be updated by the formula

$$d_\vartheta^{k+1} = d_\vartheta^k + \Delta d_\vartheta \tag{114}$$

We note that the nodal temperature received in Eq. (113) should also be added the increment of temperature due to structural heating (adiabatic condition) which was explained in the mechanical process.

5. Numerical examples

5.1 Steel bar subjected to traction force

We consider a bar length l = 3 mm; fixed at the left end and subjected to an imposed displacement at the right end, increasing by 0.001mm/s to a final value, and also to an imposed temperature in the right end with, increasing by 1^0C/s (see Fig. 4). The material properties are given in the Table 3.

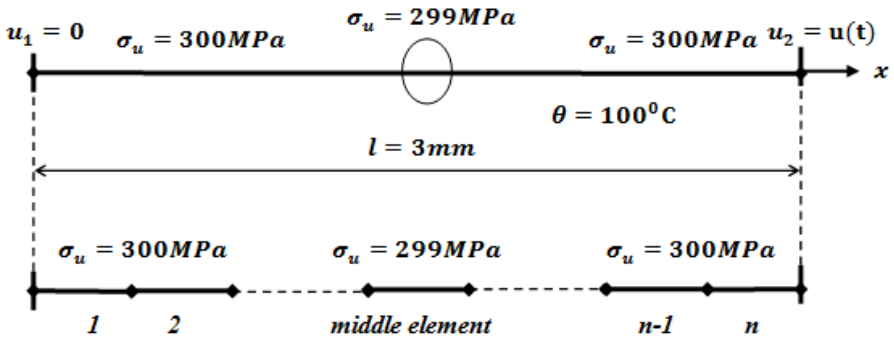


Fig. 4 Bar description and the chosen mesh

The computed force/displacement curve of the bar is given in Fig. 5 for different number of elements. It is clear that the model can provide the mesh invariant response.

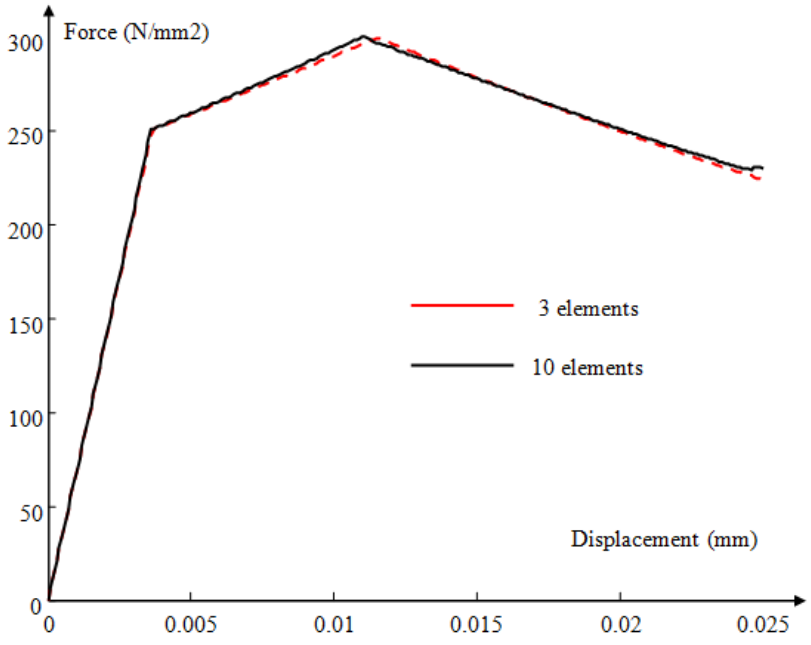


Fig. 5 Force-Displacement Curve

Table 3Material properties of the bar

Material Properties	Value	Dimension
Young modulus (E)	205000	MPa
Initial yield stress (σ_y)	250	MPa
Ultimate stress (σ_u)	300	MPa
Plastic hardening modulus (K_p)	20000	MPa
Localized softening modulus ($\bar{\bar{K}}$)	-50000	$MPam^{-1}$
Mass Density (ρ)	$7.865\ 10^{-9}$	Ns^2mm^{-4}
Thermal conductivity (k)	45	$Ns^{-1}K^{-1}$
Heat specific (c)	$0.46\ 10^{9}$	$mm^2s^{-2}K^{-1}$
Thermal elongation(α)	0.00001	

The temperature distribution along the bar is shown in Fig. 6, at different time values.

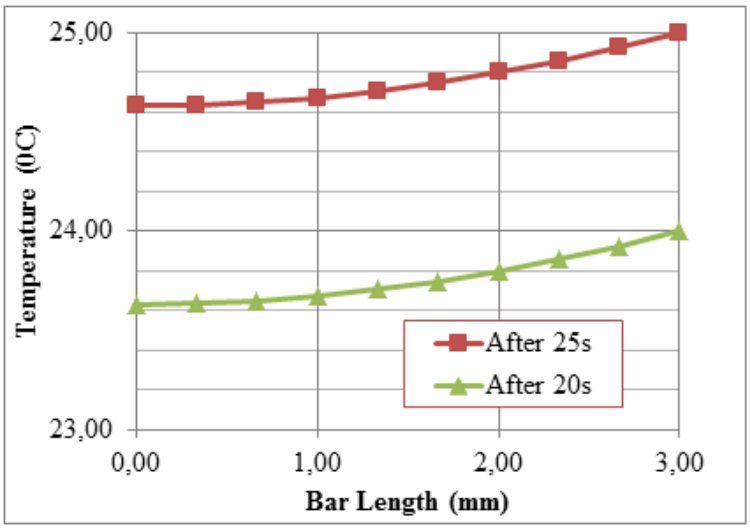


Fig. 6 Distribution of temperature along the bar

5.2 Truss instability example

We consider a truss as described in the Fig. 7, submitted to a thermo-mechanical loading.

The truss is 8 m high and 2.4 m in with, made of steel bar (area = $100mm^2$), made of steel bar (with the cross section area is equal to 100 mm^2). The material properties for steel are given in the Table 3. The truss is subjected to an increasingly horizontal loading acting on the top point (with the velocity = 1kN/s) and also a temperature rising in the middle of the second floor. The

temperature is constantly rising with the velocity of 2^0C per second.

Fig. 8 shows the deformed configuration of the truss after 17 seconds while shows the deformed configuration after 18 seconds of loading.

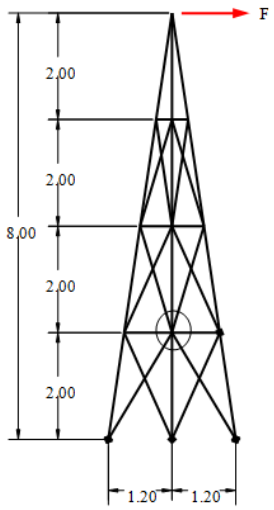
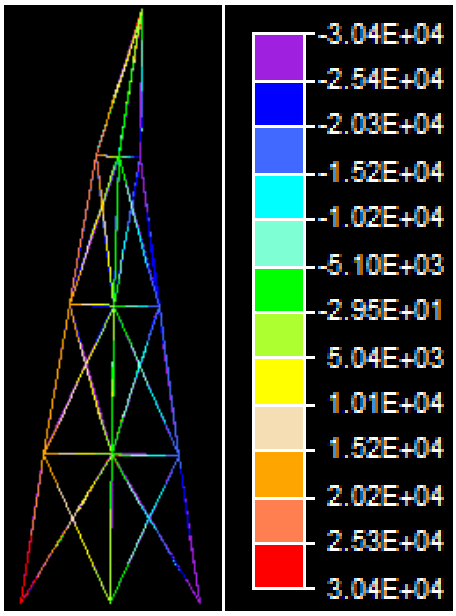
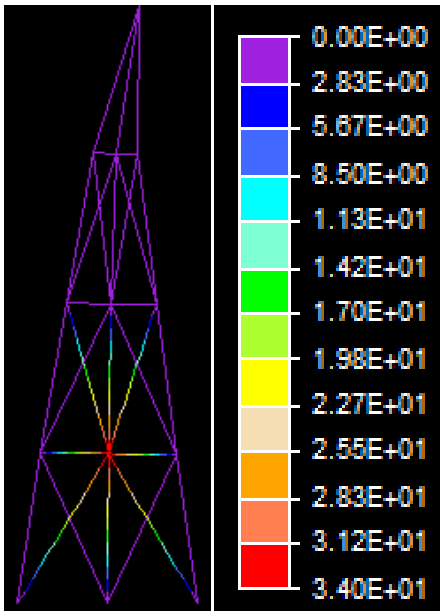


Fig. 7 Truss subjected to horizontal loading and temperature



(a) Internal force (N) of the truss after 17s



(b) Temperature (^{0}C) of the truss after 17 s

Fig. 8 Distribution of internal force and temperature in the truss after 17s

We note that in this state, the localised failure is detected in element firstly in the two top inclinedbars (element 4 and 12) and then in the two bottom side inclined bar (element 1 and 9).

The relation between horizontal loading and the horizontal displacement at the top of the truss is shown in Fig. 10.This figure also shows that for the considered loading type, the maximum (or the ultimate) horizontal loading which can be act in the truss tower is 18 kN.

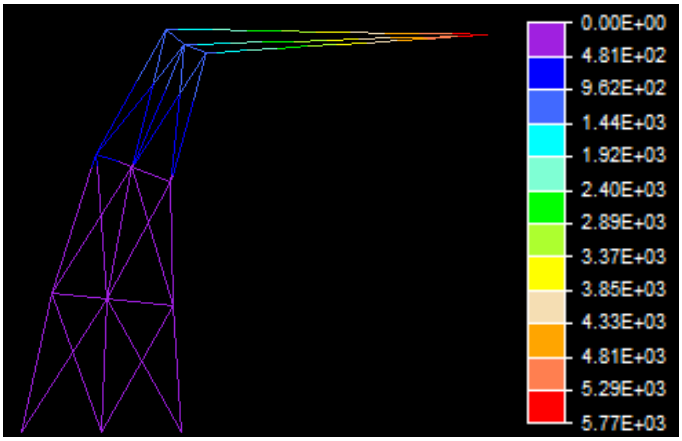


Fig. 9 Horizontal displacement of the truss after 18 seconds of loading

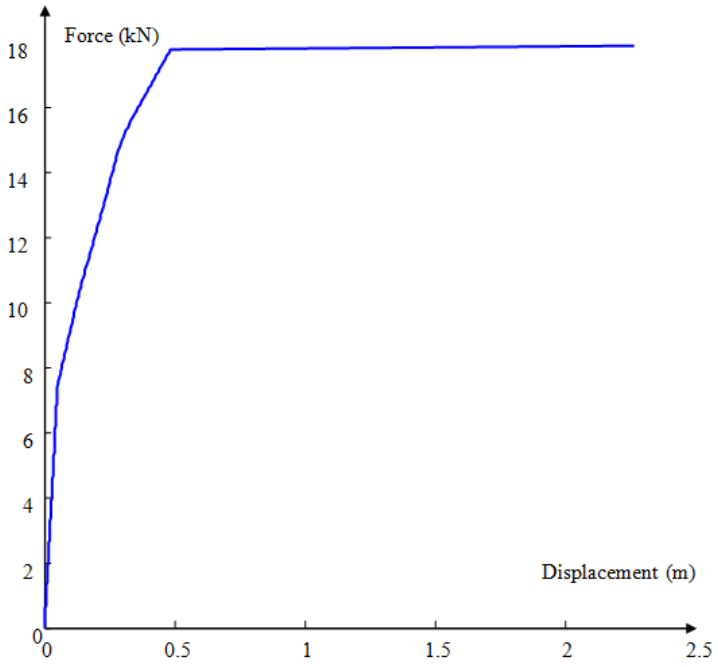


Fig. 10 Horizontal force/displacement curve at the top of the truss tower

6. Conclusions

In this paper, we have presented both the theoretical aspects and the numerical solution procedure of the thermo-plastic softening problem in large deformations. As the first novelty, the model is able to consider the localized failure of the material, which is necessary to simulate the ultimate limit state of the structures. The second contribution of the paper is that it provides a better understanding on the large deformation of the structure under coupled thermo-mechanical loading. The detailed developments presented herein, illustrating the model capability of representing the ultimate behavior at large strains of structures such as the truss-bar or cable, can readily be extended to other structural models (Ibrahimbegovic 1994, 1995 a,b).

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