A five-variable refined plate theory for thermal buckling analysis of composite plates

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(Received March 30, 2021, Revised June 16, 2021, Accepted June 18, 2021)

Abstract. This research is devoted to investigate the thermal buckling analysis behaviour of laminated composite plates, by applying an analytical model based on a refined plate theory (RPT) with five independent unknown variables. The theory accounts for parabolic distribution of the transvers shear strains through the plate thickness, and satisfied the zero traction boundary condition on the surface without using shear correction factors, hence a shear correction factor is not required. The governing differential equations and associated boundary conditions are derived by employing the principle of virtual work and solved via Navier-type analytical procedure to obtain critical buckling temperature for simply supported boundary condition of symmetric and antisymmetric cross-ply and angle-ply laminated plates. MATLAB 2018 program is used to investigate the effect of thickness ratio ($a/h$), aspect ratio ($a/b$), orthogonality ratio ($E_1/E_2$), coefficient of thermal expansion ratio ($\alpha_2/\alpha_1$) and numbers of layers on thermal buckling of laminated plate. It can be concluded that this theory gives good results when compared with other theory.

Keywords: thermal buckling; cross & angle-ply plate; critical buckling temperature; refined plate theory

1. Introduction

Designs of airframes for high speed flight and spacecraft structures have to consider carefully the effect of the thermal environment on structural and material behavior. The plate structures are often subjected to severe thermal environments during launching and reentry and may have significant and unavoidable initial geometric imperfections. When the plate is subjected to temperature change, thermally induced compressive stresses are developed in the constraint plate due to thermoeelastic properties and consequently buckling occurs. Therefore, the study of the buckling behavior of composite laminated plates under such environmental conditions is a matter of considerable importance in the design of aircraft. Thangaratnam and Ramachandran (1989) used finite element method using semiloof elements to analyse critical buckling temperature for composite laminates under thermal load. The equation of motion for critical temperature is obtained by equating the second variation of total potential energy to zero. Different boundary condition for cross-ply and angle-ply symmetric and antisymmetric plates. Chang and Leu (1991) studied thermal buckling of antisymmetric angle-ply laminated simply supported subjected to uniform thermal load using higher order deformation theory which account for transverse shear and transverse normal

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strain to obtain exact-closed form solution. Chen et al. (1991) studied the thermal buckling behavior of composite laminated plates subjected to uniform or non-uniform temperature fields are analyzed with the aid of the finite element method. Noor and Scott Burton (1992) presented three-dimensional analytical solution for thermal buckling multilayered angle-ply composite plates with temperature-dependent thermo elastic properties. The temperature is assumed to be independent of the surface coordinates, but has symmetric variation along plate thickness. Shu and Sun (1994) used a higher-order displacement field is developed for study the analysis of the thermomechanical buckling of composite plates subjected to thermal or mechanical load. Exact closed-form solutions of symmetric cross-ply laminates are obtained. Prabhu and Dhanaraj (1994) studied thermal buckling of laminated composite plates is analysed using the finite element method based on the Reissner-Mindlin first order shear deformation theory. The nine-node Lagrangian isoparametric element is employed for the thermal buckling analysis of symmetric cross-ply, anti-symmetric angle-ply and quasi-isotropic laminates subjected to uniform temperature distribution. Matsunaga (2006) investigated thermal buckling of angle-ply laminated composite and sandwich plates based on two-dimensional global higher order shear deformation theory. Abdul-Majeed et al. (2011) investigated thermal buckling of isotropic thermo elastic thin plates using governing differential equation and the Rayleigh-Ritz method. Three types of thermal distribution have been considered these are: uniform temperature, linear distribution and non-linear thermal distribution across thickness. Bourada et al. (2012) used a new four-variable refined plate theory for thermal buckling analysis of functionally graded material (FGM) sandwich plates. The thermal loads are assumed as uniform, linear, and nonlinear temperature rises across the thickness direction. Kumar et al. (2013) investigated the effect of temperature on the buckling response of a laminated composite plate subjected to thermo mechanical loadings. Mechanical loading consists of uniaxial, biaxial, and shear. The distribution of temperature on the surface is considered to be uniform. The mathematical formulation is based on higher order shear deformation theory. Jameel (2013) investigated critical buckling temperature of cross-ply and angle-ply composite laminated plate using classical laminated and higher order shear deformation plate theory. Equations of motion are solved using Navier and Levy methods for symmetric and anti-symmetric laminated plates. Kumar and Gupta (2014) investigated thermal buckling of symmetric cross-ply composite laminate using the classical laminated plate theory & first order shear deformation theory in conjunction with the Rayleigh-Ritz method is used for the evaluation of the thermal buckling parameters of structures made out of graphite fibres with an epoxy matrix. Symmetrically cross-ply laminated composite plates subjected to a combination of uniform temperature distribution through the thickness. Singh (2014) presented thermal buckling behavior of laminated composite curved panel embedded with shape memory alloy fiber based on higher order shear deformation plate theory. Variational principle with finite element modeling under uniform temperature loading is used to obtain the responses. Cetkovic (2016) Studied thermal buckling of laminated composite plates, based on Layer wise Theory of Reddy and new version of Layer wise Theory of Reddy. From the strong form, analytical solution is derived based on Navier’s type, while the weak form is analysed using the isoperimetric finite element approximation. Ounis and Belarbi (2017) studied the thermal buckling behavior of laminated plates with rectangular cut outs using classical plate theory as a base for finite element method. Xing and Wang (2017) concerned the critical buckling temperature of functionally graded rectangular thin plates. Closed form solutions for the critical thermal parameter are obtained for the plate with different boundary conditions under uniform, linear and nonlinear temperature fields using separation-of-variable method. Hussein and Alasadi (2018) investigation of the stress-strain for E-glass fiber/polyester composite plates subjected to the uniform temperature at various factors, such as fiber volume
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fraction and fiber orientation. Using finite element solution. Sadiq and Majeed (2019) studied critical buckling temperature of angle-ply laminated plate is developed using a new higher-order displacement field. Equations of motion based on higher-order theory angle-ply plates are derived through Hamilton’s principle, and solved using Navier-type solution to obtain critical buckling temperature for simply supported laminated plates. Tounsi et al. (2019) presented a novel higher-order shear deformation theory (HSDT) for buckling analysis of functionally graded plates. The present theory accounts for both shear deformation and thickness stretching effects by a parabolic variation of all displacements across the thickness, and satisfies the stress-free boundary conditions on the upper. The governing equations are obtained by the principle of virtual work. Analytical solutions for the buckling analyses are solved for simply supported sandwich plate. Belbachir et al. (2019). Investigated to describe the response of anti-symmetric cross-ply laminated plates subjected to a uniformly distributed nonlinear thermo-mechanical loading. By using refined plate theory. The undetermined integral terms are used and the variables number is reduced to four. The boundary conditions on the top and the bottom surfaces of the plate are satisfied. The principle of virtual work is used to obtain governing equations and boundary conditions. Navier solution for simply supported plates is used to derive analytical solutions. Abualnour et al. (2019). Studied The thermo-mechanical bending behavior of the antisymmetric cross-ply laminates is examined using a new simple four variable trigonometric plate theory. The proposed theory utilizes a novel displacement field which introduces undetermined integral terms and needs only four variables. Belbachir et al. (2020). Deals with the flexural analysis of anti-symmetric cross-ply laminated plates under nonlinear thermal loading using a refined plate theory with four variables. The undetermined integral terms are used and the number of variables is reduced to four. The principle of virtual work is used to obtain governing equations and boundary conditions. Navier solution for simply supported plates is used to derive analytical solutions. Abdul and Majeed (2020) a modified Fourier-Ritz approach for first time is used to study dynamic transverse response of laminated plates with different boundary conditions based on classical plate’s theory. The transverse displacement component of the plate is represented by Fourier series which is modified by adding auxiliary functions to cosine series so as to accelerate the convergence of the series and the solution. Ghadimi (2020) studied stability functions are calculated to obtain critical elastic buckling loads of asymmetric and axisymmetric one-span non-sway bending frames made up of laminated thin beams and columns with through-thickness mechanical properties variation subjected to axial compression. The shear and axial deformations are neglected. It is assumed that the members are perfect and axial compression is applied to neutral axis without eccentricity. The relative rotations of beams with respect to columns are occurred due to semi-rigid connections at joints of the bending frame. Menasria et al. (2020) Presented dynamic analysis of the FG-sandwich plate seated on elastic foundation with various kinds of support using refined shear deformation theory. The zero-shear stresses at the free surfaces of the FG-sandwich plate are ensured without introducing any correction factors. The four equations of motion are determined via Hamilton’s principle and solved by Galerkin’s approach for FG-sandwich plate with three kinds of the support. Chikr et al. (2020). Studied the buckling analysis of material sandwich plates based on a two-parameter elastic foundation under various boundary conditions is investigated on the basis of a new theory of refined trigonometric shear deformation. The governing equations and boundary conditions are obtained. Bensaid et al. (2021) investigate the static bending and buckling response of Functionally Graded (FG) nanobeams by employing a new refined first order shear deformation beam theory. The elegance of this novel theory is that, not only has one variable in terms of equations of motion as in classical beam theory (EBT) but also accounts for the effect of transverse shear deformation without any requirement of Shear Correction Factors.
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(2021). Presented Wave propagation analysis of porous functionally graded (FG) sandwich plate in a hygro-thermal environment. By using a simple four-unknown integral higher-order shear deformation theory (HSDT). The effect of moisture and temperature on wave propagation in porous FG sandwich plates is investigated by considering their role on the material’s expansion. Bakoura et al. (2021) Studied the mechanical buckling analysis of simply-supported functionally graded plates are carried out using a higher shear deformation theory (HSDT) in conjunction with the stress function method. Without using shear correction factor and gives rise to a variation of transverse shear stress such that the transverse shear stresses vary parabolically through the thickness satisfying the surface conditions without stress of shear.

In present work, critical temperature of simply supported composite cross-ply and angle-ply plate is obtained using refined five-parameter plate theory (RPT). The significant advantage of our proposed theory is that five unknown variable exists in its displacement formula and governing equation. The displacement $u$ in $x$ direction, the displacement $v$ in $y$ direction, the transverse displacement $W$ contains three components of extension $w_a$, bending $w_b$ and shear $w_s$ which these components are function of coordinates $x$ and $y$. the effect of thickness ratio $(a/h)$, aspect ratio $(a/b)$, orthogonality ratio $(E_1/E_2)$, coefficient of thermal expansion ratio $(\alpha_1/\alpha_2)$ and numbers of layers on thermal buckling of laminated plate for symmetric and antisymmetric thin and thick plate are investigated.

2. Theoretical analysis

2.1 Displacement field

Consider a rectangular plate of total thickness $(h)$ composed on $(n)$ orthotropic layers with the coordinate system (see Fig. 1) Kim (2009). Refined plate theory satisfies equilibrium conditions at the top and bottom forces of the plate without using shear correction factor. The transverse displacement $W$ includes three components of extension $w_a$, bending $w_b$ and shear $w_s$ the displacement field may be expressed as Kim (2009):

![Fig. 1 Developed samples from stir casting](image)
\[ U(x, y, z) = u(x, y) - z \left[ \frac{\partial w_b}{\partial x} \right] + z \left[ \frac{1}{4} - \frac{5}{3} \left( \frac{z}{h} \right)^2 \right] \frac{\partial w_s}{\partial x} \]
\[ V(x, y, z) = v(x, y) - z \left[ \frac{\partial w_b}{\partial y} \right] + z \left[ \frac{1}{4} - \frac{5}{3} \left( \frac{z}{h} \right)^2 \right] \frac{\partial w_s}{\partial y} \]
\[ W(x, y, z) = w_a(x, y) + w_b(x, y) + w_s(x, y) \]

For small strain Linear, the strain-displacement relations take the form Reddy 2004.

\[ \varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2} \gamma_{xy} \right), \varepsilon_{yz} = \frac{1}{2} \left( \frac{\partial w_a}{\partial y} \right) = \frac{1}{2} \gamma_{yz} \right), \varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w_s}{\partial x} \right) = \frac{1}{2} \gamma_{xz} \right) \]

The strain components will be derived, based on the displacement refined of plate, from Eq. (1) and (2) as:

\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z \frac{\partial^2 w_b}{\partial x \partial y} + 2z \left[ \frac{1}{4} - \frac{5}{3} \left( \frac{z}{h} \right)^2 \right] \frac{\partial^2 w_s}{\partial x \partial y} \right), \gamma_{yz} = \frac{\partial w_a}{\partial y} + \frac{5}{4} \left( \frac{z}{h} \right)^2 \frac{\partial^2 w_s}{\partial y} \right), \gamma_{xz} = \frac{\partial w_a}{\partial x} + \frac{5}{4} \left( \frac{z}{h} \right)^2 \frac{\partial^2 w_s}{\partial x} \right) \]

The strains associated with the displacements are:

\[ \varepsilon_x = \varepsilon_x^0 + zk_x^b + fk_x^s \right), \varepsilon_y = \varepsilon_y^0 + zk_y^b + fk_y^s \right), \varepsilon_z = 0 \right), \gamma_{xy} = \gamma_{xy}^0 + zk_{xy}^b + fk_{xy}^s \right), \gamma_{yz} = \gamma_{yz}^0 + gk_{yz}^s \right), \gamma_{xz} = \gamma_{xz}^0 + gk_{xz}^s \right) \]

where:

\[ \left\{ \begin{array}{c}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{array} \right\} = \left\{ \begin{array}{c}
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{array} \right\}, \left\{ \begin{array}{c}
k_x^b \\
k_y^b \\
k_{xy}^b
\end{array} \right\} = \left\{ \begin{array}{c}
- \frac{\partial^2 w_b}{\partial x^2} \\
- \frac{\partial^2 w_b}{\partial y^2} \\
- 2 \frac{\partial^2 w_b}{\partial x \partial y}
\end{array} \right\}, \left\{ \begin{array}{c}
k_x^s \\
k_y^s \\
k_{xy}^s
\end{array} \right\} = \left\{ \begin{array}{c}
- \frac{\partial^2 w_s}{\partial x^2} \\
- \frac{\partial^2 w_s}{\partial y^2} \\
- 2 \frac{\partial^2 w_s}{\partial x \partial y}
\end{array} \right\}, \left\{ \begin{array}{c}
f \\
g
\end{array} \right\} = \left\{ \begin{array}{c}
f = - \frac{1}{4} + \frac{5}{3} \left( \frac{z}{h} \right)^2 \\
g = \frac{5}{4} - \frac{5}{3} \left( \frac{z}{h} \right)^2
\end{array} \right\} \]

\[ \left\{ \begin{array}{c}
\gamma_{xz}^a \\
\gamma_{yz}^a
\end{array} \right\} = \left\{ \begin{array}{c}
\frac{\partial w_a}{\partial x} \\
\frac{\partial w_a}{\partial y}
\end{array} \right\} , \left\{ \begin{array}{c}
\gamma_{xz}^a \\
\gamma_{yz}^a
\end{array} \right\} = \left\{ \begin{array}{c}
\frac{\partial w_s}{\partial x} \\
\frac{\partial w_s}{\partial y}
\end{array} \right\} \]

\[ 2.2 \text{ Hamilton's principle} \]

Hamilton’s principle is used herein to derive the equations of motion appropriate to the displacement field. The principle can be stated in analytical form as Reddy (2004).

\[ 0 = \int_0^T (\delta U + \delta V) dt \]
The strain energy $\delta U$ can be written as:

$$\delta U = \int_V \left( \sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \sigma_{xy} \delta \gamma_{xy} + \sigma_{yz} \delta \gamma_{yz} + \sigma_{xz} \delta \gamma_{xz} \right) dV$$

(7)

Substituting Eq. (4) into Eq. (7) we get:

$$\delta U = \int_V \left[ \sigma_{xx} \left( \delta \varepsilon_x^0 + z \delta k^b_x + f \delta k^s_x \right) + \sigma_{xy} \left( \delta \varepsilon_y^0 + z \delta k^b_y + f \delta k^s_y \right) + \sigma_{yz} \left( \delta \gamma_{yz}^0 + g \delta k^s_x \right) + \sigma_{xz} \left( \delta \gamma_{xz}^0 + g \delta k^s_y \right) \right] dV$$

(8)

Substituting Eq. (5) into Eq. (7) we get:

$$\delta U = \int_A \left[ N_x \delta \varepsilon_x^0 + N_y \delta \varepsilon_y^0 + N_{xy} \delta \gamma_{xy}^0 + M_x^b \delta k^b_x + M_y^b \delta k^b_y + M_{xy} \delta k^b_{xy} + M_x^s \delta k^s_x + M_y^s \delta k^s_y + M_{xy}^s \delta k^s_{xy} + Q_x^a \delta \gamma_{xy}^a + Q_y^a \delta \gamma_{xy}^a + Q_{xz}^s \delta \gamma_{xz}^s + Q_{yz}^s \delta \gamma_{yz}^s \right] \partial x \partial y$$

where:

$$(N_x, N_y, N_{xy}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x, \sigma_y, \sigma_{xy}) \, dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} (\sigma_x, \sigma_y, \sigma_{xy}) \, dz$$

(9)

The Work done by applied thermal Forces can be written as:

$$\delta V = \int_A \left[ N_x \frac{\partial^2 (w_a + w_b + w_s)}{\partial x^2} + N_y \frac{\partial^2 (w_a + w_b + w_s)}{\partial y^2} + 2 N_{xy} \frac{\partial^2 (w_a + w_b + w_s)}{\partial x \partial y} \right] \, dA$$

(11)

### 2.3 Equation of motion

Substituting Eqs. (10)-(11) into Eq. (6) and then collecting the coefficient of ($\delta u$, $\delta v$, $\delta w_a$, $\delta w_b$ and $\delta w_s$) to zero separately, the equation of motion for the ply plate are obtained as follows:

$$\delta u: \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad , \quad \delta v: \frac{\partial N_x}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0$$

(12)
\[
\delta w_a : \frac{\partial Q^a_{xz}}{\partial x} + \frac{\partial Q^a_{yz}}{\partial y} + N^T(\omega) = 0 , \quad \delta w_b : \frac{\partial^2 M^b_{xz}}{\partial x^2} + \frac{\partial^2 M^b_{yz}}{\partial y^2} + 2 \frac{\partial^2 M^b_{xy}}{\partial x \partial y} + N^T(\omega) = 0
\]

where:

\[
N^T(\omega) = N^T_x \frac{\partial^2 (w_a + w_b + w_s)}{\partial x^2} + N^T_y \frac{\partial^2 (w_a + w_b + w_s)}{\partial y^2} + 2N^T_{xy} \frac{\partial^2 (w_a + w_b + w_s)}{\partial x \partial y}
\]

The transformed stress-strain relations of an orthotropic lamina in a plane state of stress are (Reddy 2004)

\[
\begin{bmatrix}
\sigma_x \\ \sigma_y \\ \sigma_{xy}
\end{bmatrix}
= \begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{12} & Q_{22} & Q_{26} \\
Q_{16} & Q_{26} & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}
\end{bmatrix}
- \begin{bmatrix}
\alpha_{xx} \\ \alpha_{yy} \\ 2\alpha_{xy}
\end{bmatrix} \Delta T,
\quad
\begin{bmatrix}
\sigma_{xz} \\ \sigma_{zy} \\
\sigma_{yz}
\end{bmatrix}
= \begin{bmatrix}
Q_{44} & Q_{45} & Q_{46} \\
Q_{45} & Q_{55} & Q_{56} \\
Q_{46} & Q_{56} & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\gamma_{xz} \\ \gamma_{zy} \\ \gamma_{yz}
\end{bmatrix}
\] (13)

The force results are:

\[
\begin{bmatrix}
N_x \\ N_y \\ N_{xy}
\end{bmatrix}
= \begin{bmatrix}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\ \sigma_y \\ \sigma_{xy}
\end{bmatrix}
+ \begin{bmatrix}
\frac{\partial Q^a_{xz}}{\partial x} \\ \frac{\partial Q^a_{yz}}{\partial y} \\ \frac{\partial Q^a_{xy}}{\partial x \partial y}
\end{bmatrix}
\Delta T
\]

(14)

where:

\[
\alpha_{xx} = \alpha_1 \cos^2 \theta + \alpha_2 \sin^2 \theta, \quad \alpha_{yy} = \alpha_1 \sin^2 \theta + \alpha_2 \cos^2 \theta
\]

\[
2\alpha_{xy} = 2(\alpha_1 - \alpha_2) \sin \theta \cos \theta (A_{ij}, B_{ij}, D_{ij}, B_{ij}^s, H_{ij}^s)
\]

\[
= \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij} (1, z, z^2, f, z f, f^2) \, dz \quad (i, j = 1, 2, 6)
\]

\[
(A_{ij}, A_{ij}^s, A_{ij}^s) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij} (1, g, g^2) \, dz \quad (i, j = 4, 5)
\]

Eq. (12) can be expressed in terms of displacements \((u, v, w_b, w_s, w_a)\) by substituting for the stress resultants from Eq. (14). the equations of motion (12) take the form:
\[
\begin{align*}
A_{11} \frac{\partial^2 u}{\partial x^2} + 2A_{16} \frac{\partial^2 u}{\partial x \partial y} + A_{66} \frac{\partial^2 u}{\partial y^2} + A_{16} \frac{\partial^2 v}{\partial x^2} + (A_{16} + A_{66}) \frac{\partial^2 v}{\partial x \partial y} + A_{26} \frac{\partial^2 v}{\partial y^2} & \\
- \left[ B_{11} \frac{\partial^3 w_b}{\partial x^3} + 3B_{16} \frac{\partial^3 w_b}{\partial x^2 \partial y} + (B_{12} + 2B_{66}) \frac{\partial^3 w_b}{\partial x \partial y^2} + B_{26} \frac{\partial^3 w_b}{\partial y^3} \right] & \\
- \left[ B_{11}^s \frac{\partial^3 w_s}{\partial x^3} + 3B_{16}^s \frac{\partial^3 w_s}{\partial x^2 \partial y} + (B_{12}^s + 2B_{66}^s) \frac{\partial^3 w_s}{\partial x \partial y^2} + B_{26}^s \frac{\partial^3 w_s}{\partial y^3} \right] &= 0
\end{align*}
\]

\[
A_{16} \frac{\partial^2 u}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u}{\partial x \partial y} + A_{66} \frac{\partial^2 u}{\partial y^2} + A_{16} \frac{\partial^2 v}{\partial x^2} + 2A_{26} \frac{\partial^2 v}{\partial x \partial y} + A_{22} \frac{\partial^2 v}{\partial y^2} & \\
- \left[ B_{16} \frac{\partial^3 w_b}{\partial x^3} + (B_{12} + 2B_{66}) \frac{\partial^3 w_b}{\partial x^2 \partial y} + 3B_{26} \frac{\partial^3 w_b}{\partial x \partial y^2} + B_{22} \frac{\partial^3 w_b}{\partial y^3} \right] & \\
- \left[ B_{16}^s \frac{\partial^3 w_s}{\partial x^3} + (B_{12}^s + 2B_{66}^s) \frac{\partial^3 w_s}{\partial x^2 \partial y} + 3B_{26}^s \frac{\partial^3 w_s}{\partial x \partial y^2} + B_{22}^s \frac{\partial^3 w_s}{\partial y^3} \right] &= 0
\]

\[
B_{11} \frac{\partial^2 u}{\partial x^3} + 3B_{16} \frac{\partial^2 u}{\partial x^2 \partial y} + (B_{12} + 2B_{66}) \frac{\partial^2 u}{\partial x \partial y^2} + B_{26} \frac{\partial^2 u}{\partial y^3} + B_{16} \frac{\partial^2 v}{\partial x^3} & \\
+ (B_{12} + 2B_{66}) \frac{\partial^2 u}{\partial x^2 \partial y} + 3B_{26} \frac{\partial^2 u}{\partial x \partial y^2} + B_{22} \frac{\partial^2 u}{\partial y^3} &
\end{align*}
\]

\[
D_{11}^s \frac{\partial^4 w_s}{\partial x^4} + 4D_{16}^s \frac{\partial^4 w_s}{\partial x^3 \partial y} + 2(D_{12}^s + 2D_{66}^s) \frac{\partial^4 w_s}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w_s}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w_s}{\partial y^4}
\]

\[
D_{11}^s \frac{\partial^4 w_s}{\partial x^4} + 4D_{16}^s \frac{\partial^4 w_s}{\partial x^3 \partial y} + 2(D_{12}^s + 2D_{66}^s) \frac{\partial^4 w_s}{\partial x^2 \partial y^2} + 4D_{26}^s \frac{\partial^4 w_s}{\partial x \partial y^3} + D_{22}^s \frac{\partial^4 w_s}{\partial y^4}
\]

\[
\begin{align*}
B_{11}^s \frac{\partial^3 u}{\partial x^3} + 3B_{16}^s \frac{\partial^3 u}{\partial x^2 \partial y} + (B_{12}^s + 2B_{66}^s) \frac{\partial^3 u}{\partial x \partial y^2} + B_{26} \frac{\partial^3 u}{\partial y^3} + B_{16} \frac{\partial^3 v}{\partial x^3} & \\
+ (B_{12}^s + 2B_{66}^s) \frac{\partial^3 u}{\partial x^2 \partial y} + 3B_{26} \frac{\partial^3 u}{\partial x \partial y^2} + B_{22} \frac{\partial^3 u}{\partial y^3} &
\end{align*}
\]

\[
- \left[ D_{11}^a \frac{\partial^4 w_a}{\partial x^4} + 4D_{16}^a \frac{\partial^4 w_a}{\partial x^3 \partial y} + 2(D_{12}^a + 2D_{66}^a) \frac{\partial^4 w_a}{\partial x^2 \partial y^2} + 4D_{26}^a \frac{\partial^4 w_a}{\partial x \partial y^3} + D_{22}^a \frac{\partial^4 w_a}{\partial y^4} \right] &
\]

\[
- \left[ H_{11}^a \frac{\partial^4 w_a}{\partial x^4} + 4H_{16}^a \frac{\partial^4 w_a}{\partial x^3 \partial y} + 2(H_{12}^a + 2H_{66}^a) \frac{\partial^4 w_a}{\partial x^2 \partial y^2} + 4H_{26}^a \frac{\partial^4 w_a}{\partial x \partial y^3} + H_{22}^a \frac{\partial^4 w_a}{\partial y^4} \right] &
\]

\[
+ A_{55}^a \frac{\partial^2 w_a}{\partial x^2} + A_{44}^a \frac{\partial^2 w_a}{\partial x \partial y} + 2A_{45}^a \frac{\partial^2 w_a}{\partial y^2} + A_{55}^a \frac{\partial^2 w_s}{\partial x^2} + A_{44}^a \frac{\partial^2 w_s}{\partial x \partial y} + 2A_{45}^a \frac{\partial^2 w_s}{\partial y^2} + A_{55}^a \frac{\partial^2 w_s}{\partial x \partial y} + N^T(\omega) = 0
\]

\[
A_{55}^a \frac{\partial^2 w_a}{\partial x^2} + A_{44}^a \frac{\partial^2 w_a}{\partial x \partial y} + 2A_{45}^a \frac{\partial^2 w_a}{\partial y^2} + A_{55}^a \frac{\partial^2 w_s}{\partial x^2} + A_{44}^a \frac{\partial^2 w_s}{\partial x \partial y} + 2A_{45}^a \frac{\partial^2 w_s}{\partial y^2} + A_{55}^a \frac{\partial^2 w_s}{\partial x \partial y} + N^T(\omega) = 0
\]

\[
2.4 \text{Navier Solution}
\]

In Navier’s method the generalized displacements are expanded a double trigonometric series in terms of unknown parameters. The choice of function in the series is restricted to those which satisfy the boundary condition of problem. The Navier method is employed to obtain the closed form solutions of the partial differential equations in Eq. (12) for simply supported rectangular plates. Two types of simply supported boundary conditions are Reddy (2004).
2.4.1 Navier solution of cross-ply laminated plates

Assuming the following displacements form to satisfy simply supported boundary conditions for cross-ply

\[
\begin{align*}
    u &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} \cos \alpha x \sin \beta y \\
    v &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} \sin \alpha x \cos \beta y \\
    W_b &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{bmn} \sin \alpha x \sin \beta y \\
    W_s &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{smn} \sin \alpha x \sin \beta y \\
    W_a &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{amn} \sin \alpha x \sin \beta y \\
\end{align*}
\]

2.4.2 Navier solution of angle-ply laminated plates

Assuming the following displacements form to satisfy simply supported boundary conditions for Angle-ply

\[
\begin{align*}
    u &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} \sin \alpha x \cos \beta y \\
    v &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} \cos \alpha x \sin \beta y \\
    W_b &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{bmn} \sin \alpha x \sin \beta y \\
    W_s &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{smn} \sin \alpha x \sin \beta y \\
    W_a &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{amn} \sin \alpha x \sin \beta y \\
\end{align*}
\]

Substituting Eq. (16) and Eq. (17) into Eq. (15), the Navier solution of cross and Angle-ply laminates can be determined from Matrix stiffness Eq. (18):

\[
\begin{bmatrix}
    s_{11} & s_{12} & s_{13} & s_{14} & 0 \\
    s_{12} & s_{22} & s_{23} & s_{24} & 0 \\
    s_{13} & s_{23} & s_{33} - k & s_{34} - k & -k \\
    s_{14} & s_{24} & s_{34} - k & s_{44} - k & s_{45} - k \\
    0 & 0 & -k & s_{45} - k & s_{55} - k \\
\end{bmatrix}
\begin{bmatrix}
    U_{mn} \\
    V_{mn} \\
    W_{bmn} \\
    W_{smn} \\
    W_{amn} \\
\end{bmatrix} = 0
\]

where \( K = (N_1^2 \alpha_2^2 + N_2^2 \beta_2^2) \) and \( s_{ij} \) is the element of stiffness.

3. Numerical results and discussion

Using above analytical solutions of the refined plate theory based on displacement field, a computer program is built using MATLAB18 programming for thermal buckling of laminated cross-ply and angle-ply composite plates. The parametric effect of side to thickness ration \( (a/h) \), aspect ratio \( (a/b) \), modulus ratio \( (E_1/E_2) \) and thermal expansion coefficient ratio \( (\alpha_2/\alpha_1) \) on critical buckling temperature of laminated composite plates are analysed. The results obtained by RPT are compared with other different theories those of the refine four parameters plate theory (RPT), FSDT, HSDT, LWT an Noor, 1992.
Table 1 A critical temperature of cross-ply (0/90/0/90) simply supported square plate (a/b = 1)

<table>
<thead>
<tr>
<th>a/h</th>
<th>Present</th>
<th>LWT¹</th>
<th>FSDT²</th>
<th>HSDT³</th>
<th>HSDT⁴</th>
<th>HSDT⁵</th>
<th>GRT⁵</th>
<th>RPT</th>
<th>P = 3 (TOT)</th>
<th>P = 5</th>
<th>P = 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.06652</td>
<td>0.0514</td>
<td>0.0613</td>
<td>0.0570</td>
<td>0.0554</td>
<td>0.0711</td>
<td>0.05580</td>
<td>0.05888</td>
<td>0.06109</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.1658</td>
<td>0.1400</td>
<td>0.1598</td>
<td>0.1479</td>
<td>0.1436</td>
<td>0.1749</td>
<td>0.14784</td>
<td>0.15344</td>
<td>0.15704</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.2121</td>
<td>0.1976</td>
<td>/</td>
<td>0.2088</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.23029</td>
<td>0.2245</td>
<td>/</td>
<td>0.2383</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.2331</td>
<td>0.2291</td>
<td>0.2438</td>
<td>0.2432</td>
<td>0.2431</td>
<td>0.2440</td>
<td>0.24331</td>
<td>0.24378</td>
<td>0.24359</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

¹Cetkovic (2016); ²Shukla (2001); ³Singh (2013); ⁴Shu and Sun (1994); ⁵Mansouri and Shariyat (2014)

Table 2 Dimensionless buckling temperature (T_cr a²h/π²D22) of cross-ply (0/90)_N simply supported square plate

<table>
<thead>
<tr>
<th>Lay-up</th>
<th>a/h</th>
<th>present</th>
<th>TOT¹</th>
<th>RPT</th>
<th>Present DQ results (11*11)</th>
<th>GRT</th>
<th>P = 3 (TOT)</th>
<th>P = 5</th>
<th>P = 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0/90)_s</td>
<td>4</td>
<td>0.06652</td>
<td>0.0575</td>
<td>0.07115</td>
<td>0.05580</td>
<td>0.05888</td>
<td>0.06109</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.165814</td>
<td>0.1522</td>
<td>0.17492</td>
<td>0.14784</td>
<td>0.15344</td>
<td>0.15704</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.233139</td>
<td>0.2435</td>
<td>0.24405</td>
<td>0.24331</td>
<td>0.24348</td>
<td>0.24359</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0/90)_{2s}</td>
<td>4</td>
<td>0.031579</td>
<td>0.0315</td>
<td>0.03348</td>
<td>0.03179</td>
<td>0.03205</td>
<td>0.03261</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.078715</td>
<td>0.0797</td>
<td>0.08231</td>
<td>0.07963</td>
<td>0.08021</td>
<td>0.08081</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.1106759</td>
<td>0.1148</td>
<td>0.11485</td>
<td>0.11478</td>
<td>0.11479</td>
<td>0.11481</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0/90)_{5s}</td>
<td>4</td>
<td>0.02401</td>
<td>0.0247</td>
<td>0.02541</td>
<td>0.02521</td>
<td>0.02526</td>
<td>0.02562</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.05985</td>
<td>0.0621</td>
<td>0.06247</td>
<td>0.06216</td>
<td>0.06250</td>
<td>0.06293</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.0841534</td>
<td>0.0872</td>
<td>0.08716</td>
<td>0.08715</td>
<td>0.08716</td>
<td>0.08717</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

¹Shu and Sun (1994)

3.1 Cross-ply composite plates

3.1.1 Verification of results

To verify the suggested above solution, obtained results are compared with the refined four parameters plate theory (RPT) and other higher order theories.

A critical temperature of cross-ply (0/90/0/90) simply supported square plate (a/b = 1) subjected to uniform temperature rise is analysed as listed in Table 1. Material constants are given as:

Material 1: \( \frac{E_1}{E_2} = 25, \frac{G_{12}}{E_2} = 0.5, \frac{G_{13}}{E_2} = 0.5, \frac{G_{23}}{E_2} = 0.2, v_{12} = v_{13} = v_{23} = 0.25, \frac{a_2}{a_1} = 3, a_4 = 1 \)

The critical temperature is normalized in the following form \( T_{cr} = \left( \frac{a^2 h}{\pi^2 D_2} \right) \). Result show that while present model of refined plate theory utilized more displacement parameters (five parameters), it is generally more accurate result than (RPT) and less accurate than higher other order theories. Table 2. Show the effect of number of layers on critical temperature for different thickness ratio. Obtained result compared with other theory and give good agreement, increasing number of layers...
A five-variable refined plate theory for thermal buckling analysis of composite plates

Table 3 A critical temperature of cross-ply (0/90) simply supported square plate (a/b = 1)

<table>
<thead>
<tr>
<th>theory</th>
<th>a/h</th>
<th>2</th>
<th>10/3</th>
<th>4</th>
<th>5</th>
<th>20/3</th>
<th>10</th>
<th>20</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>0.2865</td>
<td>0.1922</td>
<td>0.1572</td>
<td>0.1179</td>
<td>0.07656</td>
<td>0.03828</td>
<td>0.01035</td>
<td>0.4249e-3</td>
<td></td>
</tr>
<tr>
<td>LWT¹</td>
<td>0.3695</td>
<td>0.2391</td>
<td>0.1926</td>
<td>0.1419</td>
<td>0.09052</td>
<td>0.04449</td>
<td>0.01188</td>
<td>0.4858e-3</td>
<td></td>
</tr>
<tr>
<td>HSDT²</td>
<td>0.3198</td>
<td>0.2114</td>
<td>0.1729</td>
<td>0.1302</td>
<td>0.08524</td>
<td>0.04310</td>
<td>0.01177</td>
<td>0.4856e-3</td>
<td></td>
</tr>
</tbody>
</table>

¹Cetkovic (2016), ²Matsunaga (2005)

Table 4 A critical temperature of cross-ply (0/90/0) simply supported square plate (a/b = 1)

<table>
<thead>
<tr>
<th>theory</th>
<th>a/h</th>
<th>2</th>
<th>10/3</th>
<th>4</th>
<th>5</th>
<th>20/3</th>
<th>10</th>
<th>20</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>0.2945</td>
<td>0.2311</td>
<td>0.2033</td>
<td>0.1671</td>
<td>0.1206</td>
<td>0.0659</td>
<td>0.0191</td>
<td>8.0678e-4</td>
<td></td>
</tr>
<tr>
<td>LWT¹</td>
<td>0.3595</td>
<td>0.2625</td>
<td>0.2272</td>
<td>0.1848</td>
<td>0.1340</td>
<td>0.07628</td>
<td>0.02316</td>
<td>0.9964e-3</td>
<td></td>
</tr>
<tr>
<td>NoorAK,3D²</td>
<td>/</td>
<td>/</td>
<td>0.2140</td>
<td>0.1763</td>
<td>/</td>
<td>0.07467</td>
<td>0.02308</td>
<td>0.9961e-3</td>
<td></td>
</tr>
</tbody>
</table>

¹Cetkovic (2016), ²Noor, 3D, ³Matsunaga (2005), ⁴Singh (2013)

Table 5 Effect (α₂/α₄) on critical temperature of cross-ply (0/90/0/90) simply supported square plate (a/b = 1)

<table>
<thead>
<tr>
<th>a/h</th>
<th>T_cr (α₂/α₄)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.09357</td>
</tr>
<tr>
<td>4</td>
<td>0.0851</td>
</tr>
<tr>
<td>5</td>
<td>0.07806</td>
</tr>
<tr>
<td>10</td>
<td>0.1206</td>
</tr>
<tr>
<td>15</td>
<td>0.1671</td>
</tr>
<tr>
<td>20</td>
<td>0.14519</td>
</tr>
<tr>
<td>50</td>
<td>0.2033</td>
</tr>
<tr>
<td>100</td>
<td>0.2226</td>
</tr>
</tbody>
</table>

Table 6 Critical temperature of symmetric and antisymmetric cross-ply [0/90]ˡ⁻ˡ laminated thick and thin plates for different aspect ratio simply supported

<table>
<thead>
<tr>
<th>Lay-up</th>
<th>a/h</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0/90</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[0/90]ₓ</td>
<td>4</td>
<td>0.0665</td>
<td>0.07156</td>
<td>0.08053</td>
<td>0.08737</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.1658</td>
<td>0.2019</td>
<td>0.2872</td>
<td>0.36024</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.2122</td>
<td>0.2777</td>
<td>0.47497</td>
<td>0.70769</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.2331</td>
<td>0.31608</td>
<td>0.6037</td>
<td>1.0376</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0/90</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[0/90]ₓ</td>
<td>4</td>
<td>0.0192</td>
<td>0.0252</td>
<td>0.0283</td>
<td>0.0301</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.0455</td>
<td>0.09215</td>
<td>0.1268</td>
<td>0.1468</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.0569</td>
<td>0.1542</td>
<td>0.2773</td>
<td>0.3799</td>
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<td>100</td>
<td>0.06189</td>
<td>0.19736</td>
<td>0.4558</td>
<td>0.8133</td>
</tr>
</tbody>
</table>
Hussein A. Hashim and Ibtehal Abbas Sadiq

Effect of aspect ratio \((a/b)\)

Thermal expansion coefficient \((\alpha_2/\alpha_1)\) ratio

Fig. 1 Effect of aspect ratio \((a/b)\) and thermal expansion coefficient \((\alpha_2/\alpha_1)\) ratio of cross-ply (0/90/90/0) Plate on critical buckling temperature \(T_{cr}\)

caused decreasing critical temperature for all thickness ratio. Material used in this Table 2 is the same that use for Table 1.

Tables 3-4 show the effect of side to the thickness ratio \((a/h)\) on the critical temperature of cross-ply (0/90) and (0/90/0) simply supported square plate \((a/b = 1)\) subjected to uniform temperature respectively. Material properties for these tables as given as:

Material 3:

\[
\begin{align*}
E_1 & = 20, E_2 = 1, \\
G_{12} & = 0.5, G_{13} = 0.5, G_{23} = 0.5, \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0.25, \\
\alpha_2 & = 2, \alpha_1 = 0.1 \times 10^{-5}
\end{align*}
\]

Table 5 show the effect of changing \((\alpha_2/\alpha_1)\) on critical temperature for four symmetric cross-ply \((0/90/90/0)\) plates for different thickness ratio \((a/h)\), since stiffness increase when increasing orthotropic ratio therefore normalized critical temperature increase. The mechanical properties are the same in Table 1.

Changing of aspect ratio \((a/h)\) effect on critical buckling temperature of four symmetric and antisymmetric cross-ply \((0/90/90/0)\) laminated thick and thin plates, are listed in Table 6. Which
Table 7: Effect \((E_1/E_2)\) on critical temperature of symmetric and antisymmetric cross-ply \([0/90]_N\) laminated thick and thin plates simply supported.

<table>
<thead>
<tr>
<th>Lay-up</th>
<th>(E_1/E_2)</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0/90]_s</td>
<td>10</td>
<td>0.2341</td>
<td>0.3493</td>
<td>0.3988</td>
<td>0.4178</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.1169</td>
<td>0.2048</td>
<td>0.2531</td>
<td>0.2737</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.0469</td>
<td>0.0997</td>
<td>0.1400</td>
<td>0.1610</td>
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<tr>
<td></td>
<td>50</td>
<td>0.0336</td>
<td>0.0761</td>
<td>0.1127</td>
<td>0.1333</td>
</tr>
<tr>
<td>[0/90]_{2s}</td>
<td>10</td>
<td>0.1322</td>
<td>0.1973</td>
<td>0.2253</td>
<td>0.2360</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.0574</td>
<td>0.1006</td>
<td>0.1243</td>
<td>0.1345</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.0210</td>
<td>0.0447</td>
<td>0.0627</td>
<td>0.0721</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.0147</td>
<td>0.0334</td>
<td>0.0494</td>
<td>0.0585</td>
</tr>
<tr>
<td>[0/90]_{5s}</td>
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<td>0.1048</td>
<td>0.1565</td>
<td>0.1787</td>
<td>0.1872</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.0440</td>
<td>0.0771</td>
<td>0.0952</td>
<td>0.1031</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.0158</td>
<td>0.0335</td>
<td>0.0471</td>
<td>0.0542</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.0110</td>
<td>0.0250</td>
<td>0.0370</td>
<td>0.0438</td>
</tr>
<tr>
<td>[0/90]_2</td>
<td>10</td>
<td>0.0854</td>
<td>0.1246</td>
<td>0.1410</td>
<td>0.1471</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.0349</td>
<td>0.0590</td>
<td>0.0715</td>
<td>0.0767</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.0124</td>
<td>0.0253</td>
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<td>0.0390</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.0087</td>
<td>0.0188</td>
<td>0.0269</td>
<td>0.0313</td>
</tr>
<tr>
<td>[0/90]_5</td>
<td>10</td>
<td>0.0910</td>
<td>0.1354</td>
<td>0.1544</td>
<td>0.1617</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.0375</td>
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<td>0.0806</td>
<td>0.0871</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.0133</td>
<td>0.0282</td>
<td>0.0395</td>
<td>0.0453</td>
</tr>
<tr>
<td></td>
<td>50</td>
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<td>0.02131</td>
<td>0.0315</td>
<td>0.0372</td>
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</table>

show that critical temperature increases as aspect ratio \((a/b)\) increases, also it increases with increasing \((a/h)\) ratio which effected critical temperature larger than \((a/b)\) ratio. The mechanical properties are the same in Table 1.

In Table 7, show the effect of changing \((E_1/E_2)\) on critical temperature for four, eight and twenty layers symmetric and antisymmetric cross-ply plates for different thickness ratio \((a/h)\), notice that normalized critical temperature increase when aspect ratio increase, also normalized critical temperature decrease when orthotropic ratio increasing for both cross-ply symmetric and antisymmetric laminated plates. Using the mechanical properties are the same in Table 1.

Figs. 2(a)-2(d) shows first four buckling modes of moderately thick plate \((a/h = 10)\) rectangular \((a/b = 2)\) simply supported laminated plate.
Fig. 2 Thermal buckling mode for symmetric cross-ply (0/90/90/0) square plate, No. of layers = 4, \( a/h = 10 \), \( a/b = 1 \)

Table 8 A critical temperature \( T_{cr} = a_0 T \) of angle-ply (0/15/30/45) with simply supported square plate

<table>
<thead>
<tr>
<th>( a/h )</th>
<th>References</th>
<th>( \Theta = 0 )</th>
<th>( \Theta = 15 )</th>
<th>( \Theta = 30 )</th>
<th>( \Theta = 45 )</th>
</tr>
</thead>
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<tr>
<td></td>
<td></td>
<td>( (m, n) ) (1,2)</td>
<td>( (m, n) ) (1,2)</td>
<td>( (m, n) ) (1,1)</td>
<td>( (m, n) ) (1,1)</td>
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<td>present</td>
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<td>HODT</td>
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<td>0.2221</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Noor, 1992</td>
<td>0.1777</td>
<td>0.2087</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Discrepancy %</td>
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<td>12.9</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
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<td>0.1974</td>
<td>0.2596</td>
<td>0.2719</td>
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<td></td>
<td>HODT</td>
<td>0.1504</td>
<td>0.1849</td>
<td>0.2554</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Noor, 1992</td>
<td>0.1436</td>
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<td></td>
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<td>0.08124</td>
<td>0.1125</td>
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</tr>
<tr>
<td></td>
<td>Noor, 1992</td>
<td>0.05782</td>
<td>0.07904</td>
<td>0.1100</td>
<td>0.1194</td>
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<td>7.3</td>
<td>2.7</td>
<td>2.5</td>
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</table>
A five-variable refined plate theory for thermal buckling analysis of composite plates

Table 8  Continued

<table>
<thead>
<tr>
<th>a/h</th>
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<th>(\Theta = 0) (m, n) (1,2)</th>
<th>(\Theta = 15) (m, n) (1,2)</th>
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<td>0.017728</td>
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<td>0.02552</td>
<td>0.03472</td>
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<td>Noor, 1992</td>
<td>0.01739</td>
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<td>0.001115</td>
<td>0.001502</td>
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Table 9 A critical temperature \((T_{cr} = T \cdot \alpha_T \cdot 10^3)\) of antisymmetric six layers angle-ply \((45/-45)\) with simply supported square plate \((a/b = 1)\) subjected to uniform temperature rise is analysed

<table>
<thead>
<tr>
<th>a/h</th>
<th>Present</th>
<th>Chen and Liu, 1993</th>
<th>Discrepancy %</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>21.3428</td>
<td>21.3622</td>
<td>0.09</td>
</tr>
<tr>
<td>8</td>
<td>12.7245</td>
<td>12.7542</td>
<td>0.23</td>
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<tr>
<td>10</td>
<td>9.2833</td>
<td>9.2963</td>
<td>0.14</td>
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<td>15</td>
<td>4.7908</td>
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<td>1.3264</td>
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<tr>
<td>40</td>
<td>0.7580</td>
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<td>50</td>
<td>0.4887</td>
<td>0.4874</td>
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<td>80</td>
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<tr>
<td>100</td>
<td>0.1234</td>
<td>0.1230</td>
<td>0.32</td>
</tr>
</tbody>
</table>

3.2 Angle-ply composite plate

3.2.1 Verifications of results and design parameters

A critical temperature of antisymmetric angle-ply \((0/15/30/45)\) with simply supported square plate \((a/b = 1)\) subjected to uniform temperature rise is analyzed. Material constants are given as:

Material 4: \(\frac{E_1}{E_0} = 15, \frac{E_2}{E_0} = 1, \frac{G_{12}}{E_0} = \frac{G_{13}}{E_0} = \frac{G_{23}}{E_0} = 0.5, \frac{v_{12}}{E_0} = 0.3356, v_{13} = 0.3, \frac{\alpha_1}{\alpha_0} = 0.015, \frac{\alpha_2}{\alpha_0} = 1, E_0 = 1 \text{ Gpa}, \alpha_1 = 10^{-6}, \) No. of layer 10.

The critical temperature is normalized in the following form \((T_{cr} = \alpha_T T)\). Table 8 presents the convergence analysis of non-dimensional critical temperature \((T_{cr})\) of simply supported square plate for different side to thickness ratio \((a/h)\), which obtained results are compared with three dimensions elasticity theory proposed by Noor, (1992) which give good agreement with maximum discrepancy (12.9 %) for ten layers of antisymmetric angle-ply composite material.

A critical temperature of antisymmetric six layers angle-ply \((45/-45)\) with simply supported square plate \((a/b = 1)\) subjected to uniform temperature rise is analyzed. Material constants are given as:
Material 5: \( E_1 = 21, \ E_2 = E_3 = 1.7, \ G_{12} = G_{13} = 0.65, \ G_{23} = 0.639, \ \nu_{12} = \nu_{13} = 0.21, \ \alpha_2 = 16, \ \alpha_1 = -0.21, [45/-45]_3 \)

The critical temperature is normalized in the following form \( (T_{cr} = T_0 \alpha_0 10^3) \). Table 9 presents the convergence analysis of non-dimensional critical temperature \( (T_{cr}) \) of simply supported square plate, which obtained results are compared with Chen and Liu, (1993), which give good agreement with maximum discrepancy (0.32 %) for six layers of antisymmetric angle-ply composite material.

Table 10 Normalized critical temperature \( ([T_{cr} = T \alpha_1 \times 10^3 \times (b/h)^2]) \) for antisymmetric angle-ply plate \( a/h = 10 \)

<table>
<thead>
<tr>
<th>b/a</th>
<th>Angle</th>
<th>No. Of layers</th>
<th>Present</th>
<th>Chen and Lui 1993</th>
<th>Discrepancy%</th>
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<tbody>
<tr>
<td>2</td>
<td>30</td>
<td>4</td>
<td>4.1527</td>
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<td>2.28</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8</td>
<td>7.1719</td>
<td>7.7267</td>
<td>0.11</td>
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<tr>
<td></td>
<td></td>
<td>2</td>
<td>4.3127</td>
<td>4.2070</td>
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</tr>
<tr>
<td></td>
<td>45</td>
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<td></td>
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<td>2</td>
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<tr>
<td></td>
<td>60</td>
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<td>8</td>
<td>7.7179</td>
<td>7.7267</td>
<td>0.11</td>
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</table>

Table 11 Effects of Elastic moduli ratio on the dimensionless buckling temperature \( (T_{cr} \alpha_0 10^3) \) of the square simply supported antisymmetric angle-ply \( (45/-45)_3 \) plates \( a/h = 10 \)

<table>
<thead>
<tr>
<th>( \frac{E_1}{E_2} )</th>
<th>Present</th>
<th>LWT(^1)</th>
<th>TOT(^2)</th>
<th>RFT</th>
<th>Present DQ results</th>
<th>GRT (9x9)</th>
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<tr>
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<td>8.5386</td>
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<td>8.5390</td>
<td>8.6029</td>
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</table>
Table 11 Continued

<table>
<thead>
<tr>
<th>$E_1/E_2$</th>
<th>Present</th>
<th>LWT(^1)</th>
<th>TOT(^2)</th>
<th>Present DQ results</th>
<th>Present DQ results</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>RFT P = 3(TOT) P = 7 P = 5</td>
<td>RFT P = 3(TOT) P = 7 P = 5</td>
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<tr>
<td>40</td>
<td>35.90239</td>
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<td>50.6737</td>
<td>50.6840</td>
</tr>
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</table>

\(^1\)Cetkovic (2016), \(^2\)Matsunaga (2005)

Table 12 Effects of the thermal expansion coefficients ratio on the dimensionless buckling temperature ($T_{cr} = T_{c0}10^3$) of the square simply supported antisymmetric angle-ply $(45/−45)_3$ plates ($a/h = 10$)

<table>
<thead>
<tr>
<th>$\alpha_2/\alpha_1$</th>
<th>Present</th>
<th>LWT(^1)</th>
<th>TOT(^2)</th>
<th>Present DQ results</th>
<th>Present DQ results</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>RFT P = 3(TOT) P = 7 P = 5</td>
<td>RFT P = 3(TOT) P = 7 P = 5</td>
</tr>
<tr>
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<td>9.3134</td>
<td>10.3868</td>
<td>10.3854</td>
<td>10.3867</td>
</tr>
<tr>
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</table>

\(^1\)Cetkovic (2016), \(^2\)Matsunaga (2005)

Table 10 show another comparison with, Chen and Liu, 1993, for antisymmetric laminated square thick plate ($a/h = 10$) for different aspect ratio ($b/a$), angle orientation ($30, 45$ and $60$) and number of layer ($2, 4$ and $8$) which give closed results with maximum discrepancy ($2.28\%$). Material constant given as:

Material 6: $E_1/E_2 = 25$, $G_{12} = G_{13} = G_{23} = 0.5 E_2$, $v_{12} = 0.25$, $\alpha_2/\alpha_1 = 3$

Table 11 show the effect of changing ($E_1/E_2$) on critical temperature for six layers of antisymmetric angle-ply ($45/−45$) plates for thickness ratio ($a/h = 10$) since stiffness increase when increasing orthotropic ratio, therefore normalized critical temperature increase. Material constant are given as:

Material 7: $E_1/E_2 = \text{open}$, $E_2/E_0 = 1$, $G_{12}/E_0 = 0.65$, $G_{23}/E_0 = 0.639$, $v_{12} = 0.21$, $\frac{\alpha_2}{\alpha_0} = -0.21$, $\frac{\alpha_2}{\alpha_0} = 16$, $E_0 = 10 GPa$, $\alpha_0 = 10^{-6}$

Table 12. show the effect of changing $\alpha_2/\alpha_1$ on critical temperature for six layers of antisymmetric angle-ply ($45/−45$) plates for thickness ratio ($a/h = 10$) since stiffness decrease, when thermal expansion coefficient ratio increase therefore normalized critical temperature decrease. Material constant are given as:

Material 8: $E_1/E_0 = 30$, $E_2/E_0 = 1$, $G_{12}/E_0 = 0.65$, $G_{23}/E_0 = 0.639$, $v_{12} = 0.21$, $\frac{\alpha_2}{\alpha_0} = 1$, $\frac{\alpha_2}{\alpha_1} = \text{open}$, $E_0 = 10 GPa$
Figs. 3(a)-3(d) shows first four buckling mode of angle-ply moderately thick plate ($a/h = 10$) rectangular ($a/b = 2$) simply supported laminated plate.

5. Conclusions

The following conclusions may be derived:

• The most important characteristic of this work is that it contains five unknown displacement of refined plate theory, which compared with other theories those of the refined four parameters plate theory (RPT), FSDT, HSDT, LWT, and Noor, 1992 and give good agreement.

• The critical temperature buckling is affected by design parameters (aspect ratio ($a/b$), thickness ratio ($a/h$), lamination angle, modulus elastically ratio ($E_1/E_2$) and thermal expansion coefficient ratio ($\alpha_1/\alpha_2$)).

• The critical buckling temperature depend on the lamination scheme, especially for thick laminates and is greater for [0/90/0], compared to [0/90] laminates, when the same material properties of each layer are used.

• The critical buckling temperature decreases with the increase of modulus ratio $E_1/E_2$, this decreasing for only cross-ply plate.

• The critical buckling temperature increases with the increase of modulus ratio $E_1/E_2$ for angle-ply plate.
The critical buckling temperature decreases with the increase of thermal expansion coefficient ratio \((\alpha_2/\alpha_1)\) and is faster for thick, compared to thin laminates.

The critical buckling temperature increases with the increase of aspect ratio \(a/b\). This increase is again more pronounced for thick, compared to thin laminates. For \(a/b > 2\) the increase of critical temperature is almost linear, and thus the same for all buckling mode shapes.

For the same materials, angle-ply has a greater critical temperature buckling than cross-ply.

Acknowledgments

The research described in this paper was financially supported by the Natural Science Foundation.

References


Appendix

For stiffness cross-ply:

\[ s_{11} = A_{11}\alpha^2 + A_{66}\beta^2, \quad s_{12} = (A_{12} + A_{66})\alpha\beta, \quad s_{13} = -B_{11}\alpha^2 - (B_{12} + 2B_{66})\alpha\beta, \]
\[ s_{14} = -B_{11}\alpha^2 - (B_{11} + 2B_{66})\alpha, \quad s_{22} = A_{66}\alpha^2 + A_{22}\beta^2, \quad s_{23} = -(B_{11} + 2B_{66})\alpha\beta - B_{22}\beta^2, \]
\[ s_{24} = -(B_{12} + 2B_{66})\alpha\beta - B_{22}\beta^3, \quad s_{33} = D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4, \]
\[ s_{34} = D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4 + A_{16}\alpha^2 + A_{44}\beta^2, \]
\[ s_{45} = A_{55}\alpha^2 + A_{44}\beta^2, \quad s_{55} = A_{55}\alpha^2 + A_{44}\beta^2, \]
\[ A_{16} = A_{26} = D_{16} = D_{26} = D_{36} = H_{16} = H_{26} = B_{16} = B_{26} = B_{12} = B_{22} = B_{45} = A_{45} = A_{45}^a = A_{45}^b = 0 \]

For stiffness angle-ply:

\[ s_{11} = A_{11}\alpha^2 + A_{66}\beta^2, \quad s_{12} = (A_{12} + A_{66})\alpha\beta, \quad s_{13} = -(3B_{14}\alpha^2\beta + B_{26}\beta^3), \]
\[ s_{14} = -(3B_{14}\alpha^2\beta + B_{26}\beta^3), \quad s_{22} = A_{66}\alpha^2 + A_{22}\beta^2, \quad s_{23} = -(B_{14}\alpha^3 + 3B_{26}\alpha\beta^2), \]
\[ s_{24} = -(B_{14}\alpha^3 + 3B_{26}\alpha\beta^2), \quad s_{33} = D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4, \]
\[ s_{34} = D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4 + A_{15}\alpha^2 + A_{44}\beta^2, \]
\[ s_{45} = A_{55}\alpha^2 + A_{44}\beta^2, \quad s_{55} = A_{55}\alpha^2 + A_{44}\beta^2, \]
\[ A_{16} = A_{26} = D_{16} = D_{26} = D_{36} = H_{16} = H_{26} = B_{16} = B_{26} = B_{12} = B_{22} = B_{46} = B_{11} = B_{22} = B_{66} = B_{11} = B_{22} = B_{66} = 0 \]
\[ A_{45} = A_{45}^a = A_{45}^b = 0 \]

The plane stress reduced stiffness \( Q_{ij} \) are Reddy (2004)

\[ Q_{11} = \frac{\varepsilon_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}\varepsilon_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{11} = \frac{\varepsilon_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{13} \]

\[ \alpha = \frac{m\pi}{a}, \quad \beta = \frac{m\pi}{b}, \quad \text{and} \quad (U_{mn}, V_{mn}, W_{bmn}, W_{bmn}) \text{ are coefficients} \]