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Static deflection and dynamic behavior of higher-order hyperbolic shear deformable compositionally graded beams

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Abstract. In this work we introduce a higher-order hyperbolic shear deformation model for bending and frees vibration analysis of functionally graded beams. In this theory and by making a further supposition, the axial displacement accounts for a refined hyperbolic distribution, and the transverse shear stress satisfies the traction-free boundary conditions on the beam boundary surfaces, so no need of any shear correction factors (SCFs). The material properties are continuously varied through the beam thickness by the power-law distribution of the volume fraction of the constituents. Based on the present refined hyperbolic shear deformation beam model, the governing equations of motion are obtained from the Hamilton's principle. Analytical solutions for simply-supported beams are developed to solve the problem. To verify the precision and validity of the present theory some numerical results are compared with the existing ones in the literature and a good agreement is showed.

Keywords: deflection; dynamic analysis; functionally graded material; hyperbolic shear deformation theory; refined theory

1. Introduction

Functionally graded materials (FGMs) are the novel class of composite materials that have continuous mutation in material properties from one surface to a further along the thickness direction. This genius concept of FGMs was primary initiated in 1984 by a group of material scientists while preparing a space-plane plan, in Japan (Koizumi 1997). The primary constituents for these materials are metal with ceramic or from a combination of materials. The FGM is thus appropriate for various applications, such as thermal coatings of barrier for ceramic engines, gas turbines, nuclear fusions, optical thin layers, biomaterial electronics, and in many other fields.

Consequently, studies and computational (numerical) techniques are devoted to analyze the static and dynamic behaviors of FGM beams and plates are also in huge demand in research sectors day-by-day. However, the behavior of FG beams can be predicted using either, the classical beam theory (CBT), first-order shear deformation beam theory (FSBT), third-order shear

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deformation theory to study the bending and free vibration responses of isotropic, functionally graded, sandwich and laminated composite plates. More recently, a new three unknowns model as the case of the classical plate theory (CPT) was elaborated by Tounsi *et al*. (2016) and Houari *et al*. (2016) for static, buckling and vibration analysis of both functionally graded and sandwich plates. Some plate theories are used also to explore the behavior of nanostructures as is described in Refs (Bounouara *et al*. 2016, Belkorissat *et al*. 2015).

In this study, static bending and vibration behaviors of compositionally graded beams are analyzed based on a novel simple higher order refined beam model which captures the shear deformation effects using a shear strain function without the need of any shear correction factor. The governing equations of motion in the framework of the present refined hyperbolic beam model are derived through Hamilton's principle and resolved applying an analytical solution for simply-supported boundary conditions. The material properties are graded through the beam's depth by the power-law model. Numerical examples are supplied to demonstrate the impacts of power-law exponent, length of the FG beam, thickness to length ratios on the bending and free vibration of functionally graded beams.

2. Mathematical formulation

We consider a functionally graded beam having length L and rectangular cross section $b \times h$, with b represents the width and h the thickness which its coordinates is depicted in Fig. 1. The beam is made of elastic and isotropic material with material properties varying smoothly in the z thickness direction.

2.1 Material properties

The effective material properties of the non-homogeneous beam such as Young's modulus E_f, shear modulus G_f and mass density ρ_f are supposed to vary continuously in the thickness direction according to a power function of the volume fractions of the constituents.

According to the Voigt rule of mixture, the effective material properties, P_f, can be expressed as (Simsek and Yurtcu 2013, Bouremana *et al*. 2013, Ould Larbi *et al*. 2013, Ebrahimi and barati 2016) like

$$P_f = P_c V_c + P_m V_m \tag{1}$$

Where P_m, P_c, V_m and V_c are the material properties and the volume fractions of the metal and the ceramic constituents related by

$$V_c + V_m = 1 \tag{2}$$

The volume fraction of the ceramic constituent of the FG beam is supposed to be given by

$$V_c = \left(\frac{z}{h} + \frac{1}{2} \right)^k \tag{3}$$

k is a variable parameter that dictates material variation profile through the thickness and z is the distance from the mid-plane of the FG beam. The FG beam becomes a fully ceramic beam

$$\varepsilon_x = \varepsilon_x^0 + z\kappa^b + f(z)\kappa^s \text{ and } \gamma_{xz} = g(z)\gamma^0 \tag{7}$$

Where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \; \kappa^b = -\frac{\partial^2 w}{\partial x^2}, \; \kappa^s = -\frac{\partial^2 \varphi}{\partial x^2}$$
$$\gamma^0 = \frac{\partial \varphi}{\partial x}, \; g(z) = 1 - f'(z), \text{ and } f'(z) = \frac{df(z)}{dz} \tag{8}$$

Supposing that the material of FG beam obeys Hooke's law, the constitutive relations can be given as

$$\sigma_x = Q_{11}(z)\varepsilon_x \text{ and } \tau_{xz} = Q_{55}(z)\gamma_{xz} \tag{9}$$

Where

$$Q_{11}(z) = E(z) \text{ and } Q_{55}(z) = \frac{E(z)}{2(1+\nu)} \tag{10}$$

2.3 Equations of motion

The equations of motion are derived by using Hamilton's principle. The principle can be stated in an analytical form as (Reddy 2002)

$$\delta \int_0^T (U + V - K) dt = 0 \tag{11}$$

Where δU is the variation of the strain energy; δV represents the potential energy; and the variation of the kinetic energy is given by δK. The variation of the strain energy of the beam can expressed by the following form

$$\delta U = \int_0^L \int_A (\sigma_x \delta\varepsilon_x + \tau_{xz}\delta\gamma_{xz}) dA dx$$
$$= \int_0^L \left(N \frac{d\delta u_0}{dx} - M \frac{d^2 \delta w_0}{dx^2} - P \frac{d^2 \delta\varphi}{dx^2} + Q \frac{d\delta\varphi}{dx} \right) dx \tag{12}$$

Where N, M, P and Q represent the stress resultants and they are expressed as

$$(N, M, P) = \int_A (1, z, f)\sigma_x dA \text{ and } Q = \int_A g\,\tau_{xz} dA \tag{13}$$

The variation of work done by externally transverse loads q can be given as

$$\delta V = -\int_0^L q\delta w_0\, dx \tag{14}$$

3. Analytical solution

The equations of motion cited above are analytically resolved for bending and free vibration problems. The Navier solution procedure is employed to determine the analytical solutions for a simply supported FG beam. The variables u_0, w_0, φ can be written by assuming the following variations

$$\begin{Bmatrix} u_0 \\ w_0 \\ \varphi \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_n \cos(\lambda x) e^{i\omega t} \\ W_n \sin(\lambda x) e^{i\omega t} \\ \phi_n \sin(\lambda x) e^{i\omega t} \end{Bmatrix}, \tag{20}$$

Where U_n, W_n, and Φ_n are arbitrary parameters to be determined, ω is the eigenfrequency associated with m^{th} eigenmode, and $\lambda = m\pi/L$. The transverse load q is also expanded in Fourier series as

$$q(x) = \sum_{m=1}^{\infty} Q_m \sin(\lambda x) \tag{21}$$

Where Q_m is the load amplitude calculated from

$$Q_m = \frac{2}{L} \int_0^L q(x) \sin(\lambda x)\, dx, \tag{22}$$

The coefficients Q_m are given below for some typical loads. For the case of a sinusoidally distributed load, we have

$$m=1 \ \text{ and } Q_1 = q_0 \tag{23}$$

And for the case of uniform distributed load, we have

$$Q_m = \frac{4q_0}{m\pi}, \quad (m = 1, 3, 5....) \tag{24}$$

Substituting the expansions of u_0, w, φ, and q from Eqs. (20) and (21) into the equations of motion Eq. (18), the analytical solutions can be obtained from the following equations

$$\left(\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \right) \begin{Bmatrix} U_n \\ W \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ Q_m \\ 0 \end{Bmatrix}, \tag{25}$$

Where

$$\begin{gathered} s_{11} = A\lambda^2 \ , \ \ s_{12} = -B\lambda^3 \ , s_{13} = -B_s\lambda^3 \ , \ s_{22} = D\lambda^4 \ , \ s_{23} = D_s\lambda^4 \ , \ s_{33} = H_s\lambda^4 + A_s\lambda^2 \\ m_{11} = I_0 \ , \ \ m_{12} = -I_1\lambda \ , \ m_{13} = -J_1\lambda \ , \ m_{22} = I_0 + I_2\lambda^2 \ , \ m_{23} = I_0 + J_2\lambda^2 \ , \\ m_{33} = I_0 + K_2\lambda^2 \end{gathered} \tag{26}$$

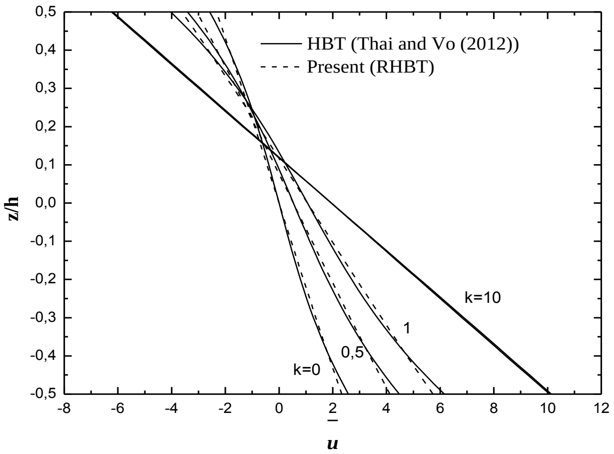

Fig. 2 Variation of the longitudinal displacement $\bar{u}$ through-the-thickness of a FG beam

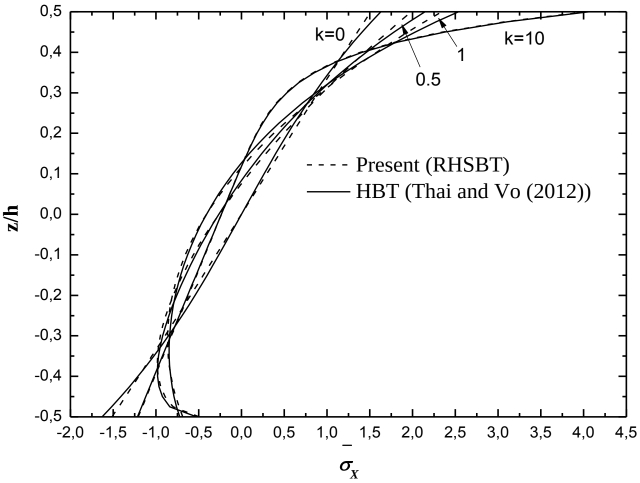

Fig. 3 The variation of the axial stress $\bar{\sigma}_x$ across-the-thickness of a FG beam (L=2 h)

subjected to a uniform load. For convenience, the following dimensionless forms are used

$$\bar{w}=100\frac{E_m h^3}{q_0 L^4}w\left(\frac{L}{2}\right),\ \bar{u}=100\frac{E_m h^3}{q_0 L^4}u\left(0,-\frac{h}{2}\right),\ \bar{\sigma}_x=\frac{h}{q_0 L}\sigma_x\left(\frac{L}{2},\frac{h}{2}\right),\ \bar{\tau}_{xz}=\frac{h}{q_0 L}\tau_{xz}\left(0,0\right),$$

$$\bar{\omega}=\frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}\ ,$$

The results obtained of our model are displayed in Table 1, and for various nondimensional displacements and stresses of FG beams under uniform load q_0 for different values gradient index k and slenderness ratio L/h. One can observe that our results are in good correlations with the provided results by the existing efficient shear deformation beam theories (Li *et al*. 2010, Thai and Vo 2012, Ould *et al*. 2013).

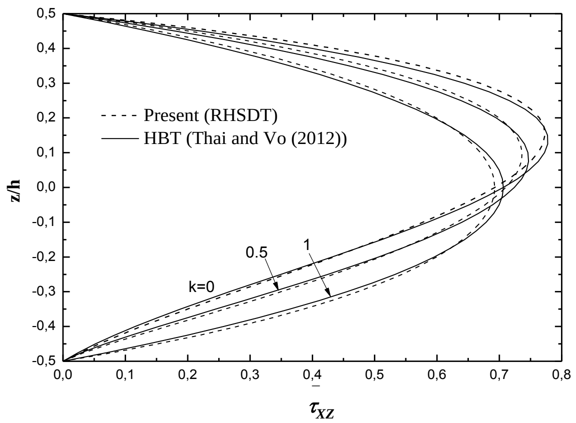

Fig. 4 The variation of the transverse shear stress $\bar{\tau}_{xz}$ across-the-thickness of a FG beam (*L*=2 h)

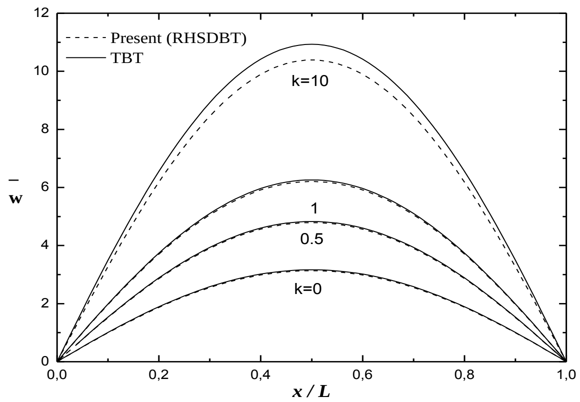

Fig. 5 Variation of the transverse displacement $\bar{w}$ versus non-dimensional length of a FG beam (*L*=5 h) bending stiffness of the FG beam

Figs. 2-4 demonstrate the variation of the longitudinal displacement $\bar{u}$, longitudinal stresses $\bar{\sigma}_x$ and transverse shear stress $\bar{\tau}_{xz}$ through the depth of the FG thick beam (*L*=2*h*), for the case of uniform load. A comparison with the analytical solutions given by Thai and Vo (2012) is also depicted in these figures using different values of the gradient index *k*. It is observed that there is a good concordance between the actual higher-order hyperbolic beam model and those of Thai and Vo (2012). It can also observed from these Figs. 2-4 that the rise of the power law exponent *k* leads to an increase of the axial displacement $\bar{u}$, longitudinal stresses $\bar{\sigma}_x$ and transverse shear stress $\bar{\tau}_{xz}$. This is due to the fact that an increase in the volume fraction of metal will decrease the Fig. 5 presents the evolution of the dimensionless transversal displacement $\bar{w}$ versus nondimensional length for various values of the volume fraction exponent *k*. It is shown also that the present refined beam model offers very closes results to Reddy (TBT). Furthermore, the results show that the rise of the power law exponent *k* leads to an increase of transversal displacement $\bar{w}$.

4.2 Free vibration analysis

Table 2 Variation of dimensionless frequency $\bar{\omega}$ with various gradient indices *k* for FG beam

L/*h*	Model	*k*					
		0	0.5	1	2	5	10
5	Simsek (2010)	5.1527	4.4111	3.9904	3.6264	3.4012	3.2816
	Thai and Vo (2012)	5.1527	4.4107	3.9904	3.6265	3.4014	3.2817
	Present model	5.1527	4.4109	3.9904	3.6264	3.4013	3.2856
10	Simsek (2010)	5.4603	4.6516	4.2050	3.8361	3.6485	3.5389
	Thai and Vo (2012)	5.4603	4.6516	4.2050	3.8361	3.6485	3.5390
	Present model	5.4603	4.6616	4.2050	3.8361	3.6484	3.5391

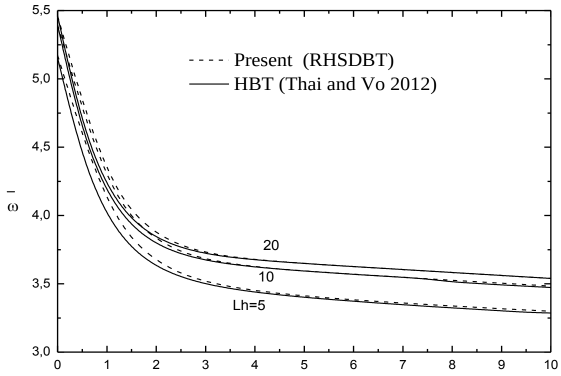

Fig. 6 Influence of the power law index *k* on the dimensionless frequency $\bar{\omega}$ of FG beam with various span-to-depth ratio *L*/*h*

For the evaluate of the present refined hyperbolic shear deformation beam model in the case of free vibration, dimensionless fundamental frequencies *w* obtained by the present theory are compared with those obtained by Thai and Vo (2012) and Simsek (2010) of FG beams for different values of power law index *k* and slenderness ratio *L*/*h* and the results are tabulated in Table 2. It can be observed that the results are in good correlations with the results obtained by Thai and Vo (2012), Simsek (2010).

The non-dimensional frequency of FG beam as a function of gradient index *k* and for different values of slendness ratio *L*/*h* using both the present theory and HBT (Thai and Vo 2012) is plotted in Fig. 6 and a close agreement between the present theory and HBT is shown. It is also seen that the frequency values decrease with increasing the power law index *k*. The full ceramic beams (k=0) lead to a highest frequency. However, the lowest frequency values are obtained for full metal beams ($k\rightarrow\infty$). The reason is an increasing in the value of the power indexes lead to grow the percentage of metal phase which make the FG beam more flexible, and thus a reduction in the fundamental frequency values.

5. Conclusions

In the present research work, static deflection and vibration analysis of functionally graded

(FG) beams are proposed according to a higher-order hyperbolic shear deformation beam model; in which the transverse shear stress vary hyperbolically through the thickness satisfying shear stress free surface conditions on the top and bottom surfaces of the beam without the need any shear correction factors. The governing differential equations of motion and the boundary conditions of FG beam are formulated through Hamilton's principle, and resolved analytically by Navier-type model for simply-simply boundary condition. According to the obtained results, it is found that the proposed model can provide very accurate results compared o the other solution results. The impacts of the power-law index and span-to-depth ratio on the deflection, stresses, and natural frequencies as well as load-frequency are explored.

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