

Thermo-mechanically induced finite element based nonlinear static response of elastically supported functionally graded plate with random system properties

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Abstract. The present work proposes the thermo mechanically induced statistics of nonlinear transverse central deflection of elastically supported functionally graded (FG) plate subjected to static loadings with random system properties. The FG plate is supported on two parameters Pasternak foundation with Winkler cubic nonlinearity. The random system properties such as material properties of FG material, external loading and foundation parameters are assumed as uncorrelated random variables. The material properties are assumed as non-uniform temperature distribution with temperature dependent (TD) material properties. The basic formulation for static is based on higher order shear deformation theory (HSDT) with von-Karman nonlinear strain kinematics through Newton-Raphson method. A second order perturbation technique (SOPT) and direct Monte Carlo simulation (MCS) are used to compute the nonlinear governing equation. The effects of load parameters, plate thickness ratios, aspect ratios, volume fraction, exponent, foundation parameters, and boundary conditions with random system properties are examined through parametric studies. The results of present approaches are compared with those results available in the literature and by employing direct Monte Carlo simulation (MCS).

Keywords: integrated design; evaluation; current practice; integrated platform; online survey; designers

1. Introduction

The functionally graded materials have attracted much attention in the many engineering applications from last decade due to high temperature resistance and maintain structural integrity by gradation of composition along the thickness direction through the appropriate volume fraction change (Birman and Byrd 2007, Koizumi 1997, Suresh and Mortensen 1998).

The FG materials are being increasingly used in many engineering sectors such as in aerospace for spacecraft antennas and thermal barrier coating, in nuclear for making a wall of fission

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bending response of shear deformable functionally graded plate subjected to thermo-mechanical loads, based on Reddy's higher order shear deformation plate theory using the semi analytical method. Ferreira *et al.* (2005) presented a static analysis of functionally graded material plate using third-order shear deformation theory based on mesh less methods. Ghannadpour and Alinia (2006) presented a large deflection analysis of rectangular functionally graded plates under pressure loads using the Von-Karman theory with potential energy. Na and Kim (2006) presented nonlinear bending of clamped rectangular FGM plate subjected to a transverse uniform pressure and thermal loads using a 3-D finite element method, in this study the thermal loads were assumed as uniform, linear and sinusoidal temperature rises across the thickness direction. Yang *et al.* (2005) presented the bending response of shear deformable FG materials plate with system randomness using first-order shear deformation plate theory (FSDT) combined with FOPT. Khabbaz *et al.* (2009) presented the energy concept with the first and third-order shear deformation theories (FSDT and TSDT) for nonlinear analysis of FGM plates under pressure loads. Alinia and Ghannadpour (2009) studied the nonlinear analysis of pressure loaded FGM plates based on classical plate theory. Shen and Wang (2010) presented the nonlinear bending of simply supported FGM plates subjected to combined thermo-mechanical loadings resting on elastic foundations using temperature dependent material properties. Singha *et al.* (2011) presented the finite element analysis of functionally graded plates to evaluate the transverse central deflection under transverse distributed load using FSDT considering the exact neutral surface position through Newton-Raphson iteration method. Praveen and Reddy (1998) evaluated the transverse central of functionally graded ceramic, metal plates in thermal environment using finite element method combined with first order shear deformation theory using von-Karman nonlinearity. Huang and Shen (2004) evaluated the nonlinear transverse central deflection and free vibration response of functionally graded plate subjected to thermo mechanical loadings using HSDT with von-Karman nonlinearity through semi analytical approach. Wang and Shen (2013) examined the nonlinear dynamic response of sandwich FGM plate resting elastic foundation in thermal environment using HSDT through semi-analytical method. Shen (2007) presented the nonlinear thermal bending response of simply supported, shear deformable FGM plates subjected to combined action of thermal and electrical loading due to heat conduction based on higher order shear deformation theory. Zhang (2014) evaluated the transverse central deflection response of FG materials plate resting on two-parameter elastic foundations using on physical neutral surfaces and high-order shear deformation theory.

All the above mentioned literatures are based on deterministic study which gives only mean structural response and unaccounted the effect of random system properties on the structural performance for higher reliability and safety.

The studies related to the stochastic response of FG and other material structures subjected static loading are very limited due to complexities involved in quantification of the random system properties. In this direction, Yang *et al.* (2005) presented the elastic buckling response of shear deformable FG materials plate with uncertain system randomness using FSDT combined with FOPT. Onkar *et al.* (200, 2006) presented the nonlinear bending and post buckling of composite laminates with random material properties under random loading based on FOPT. Singh *et al.* (2001, 2003, 2008) evaluated the vibration and bending of composite laminate plates supported with or without elastic foundation using C^0 finite element method based on HSDT in conjunction with FOPT. Lal *et al.* (2007, 2009) presented the effect of random material properties on nonlinear free vibration and buckling response of laminated composite plates supported with and without elastic foundation in the thermal environment using HSDT based C^0 nonlinear FEM based on the

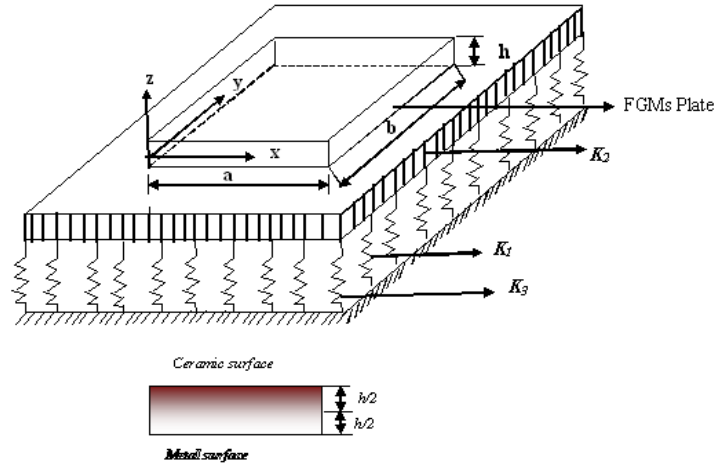


Fig. 1 Geometry of FGM plate resting on elastic foundation

induced nonlinear bending analysis of elastically supported FGM plates subjected to static transverse uniformly distributed mechanical loadings in thermal environment involving randomness in random system properties are rarely available. For the optimum performance and accurate prediction of response it is obligatory to understand the thermo mechanically induced transverse central deflection response through stochastically. The main objective of this paper is to evaluate statistics in terms of mean and coefficient of variance (COV) of thermo mechanically induced nonlinear transverse central deflection response of FG materials plate supported by an elastic foundation with random system parameters.

2. Mathematical formulations

Consider a rectangular FGM plate consists of metal and ceramic at the top and bottom layer having length a , width b , and total thickness h , defined in (x, y, z) system with x - and $-y$ axes located the middle plane and its origin placed at the corner of the plate. The plate is assumed to be attached to the elastic foundation excluding any separation takes place in the process of deformation as shown in Fig. 1. The interaction between the plate and the supporting foundation follows the two parameter model (Pasternak-type) with Winkler cubic nonlinearity as (Shen *et al.* 2010)

$$p = K_1 w + K_3 w^3 - K_2 \nabla^2 w \quad (1)$$

Where p is the foundation reaction per unit area, and $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ is second order Laplace differential operator. The parameters K_1 , K_2 and K_3 are linear normal, shear and nonlinear normal spring stiffness foundations, respectively. This model is simply known as Winkler type when $K_2 = 0$ (Lal *et al.* 2007). The symbol comma ($,_x$) denotes as partial differential with respect to x .

The properties of the FGM plate are assumed to vary through the thickness of the plate only, such that the top surface $z = h/2$ is ceramic-rich and the bottom surface $z = -h/2$ is metal reach as shown in Figs. 2(a)-(d). The effective mechanical and thermal properties of the FGMs plate of an arbitrary point within the plate domain are expressed as (Jagtap *et al.* 2012).

Where P denotes the effective material property, P_m and P_c represents the properties of the metal and ceramic, respectively, V_c is the volume fraction of the ceramic and n is the volume fraction exponent and is always positive. The effective material properties of the plate, including Young's modulus E , density ρ vary according to Eq. (2) and ν is assumed to be constant due to weakly dependent on temperature change.

2.1 Displacement field model

In the present analysis, the assumed displacement field based on Reddy's HSDT having C^1 continuity by the satisfaction of conditions that the transverse shear stresses vanish at the top and bottom of the plate and nonzero elsewhere is modified by C^0 continuity by considering derivatives of the out-of-plane displacements as separate degree of freedom (Shaker *et al.* 2008, Jagtap *et al.* 2012, William *et al.* 1992, Singh and Lal 2010, Lal *et al.* 2013, Lal *et al.* 2012a, Shankara and Iyengar 1996). The modified displacement based on C^0 continuity field, along the X , Y , and Z directions for an arbitrary plate is now written as (Singh *et al.* 2001).

$$\begin{aligned}\bar{u} &= u + f_1(z)\psi_x + f_2(z)\phi_x; \\ \bar{v} &= v + f_1(z)\psi_y + f_2(z)\phi_y; \\ \bar{w} &= w;\end{aligned}\tag{4}$$

Where \bar{u} , \bar{v} , and \bar{w} denote the displacements of a point along the (x, y, z) coordinates, u , v , and w are corresponding displacements of a point on the mid plane. ψ_x and ψ_y are the rotations of normal to the mid plane about the y -axis and x -axis respectively, with $\phi_x = \partial w / \partial x$ and $\phi_y = \partial w / \partial y$. The parameter $f_1(z)$ and $f_2(z)$ are defined as

$$f_1(z) = C_1 z - C_2 z^3; f_2(z) = -C_4 z^3 \quad \text{with} \quad C_1 = 1; C_2 = C_4 = \frac{4}{3h^2}$$

The displacement vector for the modified C^0 continuous model is denoted as

$$\{\Lambda\} = [u \quad v \quad w \quad \phi_x \quad \phi_y \quad \psi_x \quad \psi_y]^T\tag{5}$$

2.2. Strain displacement relations

For the structures considered here, the relevant total strain vector consisting of strains in terms of mid-plane deformation, rotation of normal and higher order terms associated with the displacement for FGM including thermal strain is expressed as (Shegokar and Lal 2013a, 2013b)

$$\{\varepsilon\} = \{\varepsilon_l\} + \{\varepsilon_{nl}\} - \{\bar{\varepsilon}_t\}\tag{6}$$

Where $\{\varepsilon_l\}$, $\{\varepsilon_{nl}\}$ and $\{\bar{\varepsilon}_t\}$ are the linear and nonlinear strain vectors, thermal strain vector, respectively.

Using Eq. (6) the linear strain vector can be obtained using linear strain displacement relations (Singh *et al.* 2008). Assuming that the strains are much smaller than the rotations (in the von-Karman sense), one can obtain nonlinear strain vector $\{\varepsilon_{nl}\}$ as (Lal *et al.* 2007).

$$\{\varepsilon_{nl}\} = \frac{1}{2} [A_{nl}] \{\phi\}\tag{7}$$

respectively with $k_{tb}=k_t-k_b$.

2.3 Constitutive relations

The constitutive relationship between stress and strain vectors in the plane stress state for an isotropic layer accounting thermal effect can be written as (Lal *et al.* 2009)

$$\{\sigma\} = [Q_{ij}] \{\varepsilon\} \quad (11)$$

Where $\{Q_{ij}\}$, $\{\sigma\}$ and $\{\varepsilon\}$ are transformed stiffness matrix, stress and strain vectors of the isotropic lamina, respectively. For FGM material the elastic constant (Q_{ij}) are defined as

$$Q_{11} = Q_{22} = \frac{E(z,T)}{1-\nu^2}, \quad Q_{12} = \frac{\nu E(z,T)}{1-\nu^2}, \quad Q_{44} = Q_{55} = Q_{66} = \frac{E(z,T)}{2(1+\nu)} \quad (12)$$

2.4 Strain energy of the plate

The strain energy (Π_1) of the FG material plates considering linear and nonlinear strain can be expressed as

$$\Pi_1 = \frac{1}{2} \int_A \{\varepsilon_l + \varepsilon_{nl}\}^T [\sigma] dA \quad (13)$$

Substituting Eq. (6)-(7a) in Eq. (13) can be written as

$$\begin{aligned} \Pi_1 = & \frac{1}{2} \int_A \{\bar{\varepsilon}_l\}^T [D] \{\bar{\varepsilon}_l\} dA + \frac{1}{2} \int_A [\bar{\varepsilon}_l]^T [D_3] \{A\} \{\theta\} dA \\ & + \frac{1}{2} \int_A \{A\}^T \{\theta\} [D_4] \{\bar{\varepsilon}_l\} dA + \frac{1}{2} \int_A \{A\}^T \{\theta\} [D_5] \{A\} \{\theta\} dA \end{aligned} \quad (14)$$

Where (D), (D_3), (D_4) and (D_5) are the FG material stiffness matrices as given in Appendix (A.1) and

$$\{\bar{\varepsilon}\} = (\varepsilon_1^0 \quad \varepsilon_2^0 \quad \varepsilon_6^0 \quad k_1^0 \quad k_2^0 \quad k_6^0 \quad k_1^2 \quad k_2^2 \quad k_6^2 \quad \varepsilon_4^0 \quad \varepsilon_5^0 \quad k_4^2 \quad k_5^2) \quad (14a)$$

2.5 Strain energy due to foundation

Strain energy due to elastic foundation having a shear deformable layer with Winkler cubic nonlinearity is expressed as (Lal *et al.* 2012b)

$$\Pi_2 = \frac{1}{2} \int_A p w dA \quad (15)$$

Substituting value of p from Eq. (1) into Eq. (15), and applying variation principle the strain energy due to foundation can be written as (Shen

$$\Pi_2 = \frac{1}{2} \int_A \left\{ K_1 w^2 + \frac{1}{2} K_3 w^4 + K_2 \left[(w_{,x})^2 + (w_{,x})^2 \right] \right\} dA \quad (16)$$

And

$$x = \sum_{i=1}^{NN} \varphi_i x_i; \quad y = \sum_{i=1}^{NN} \varphi_i y_i \quad (21)$$

Where φ_i the interpolation function for the i^{th} node, $\{q\}$ is the vector of unknown displacements for the i^{th} node, NN is the number of nodes per element and x_i and y_i are Cartesian Coordinate of the i^{th} node.

The linear mid plane strain vector as given in Eq. (7) can be expressed in terms of mid plane displacement field and then the energy is computed for each element and then summed over all the elements to get the total strain energy. Following this, and using Eq. (21), Eq. (14) can be written as

$$\Pi_1 = \sum_{e=1}^{NE} \Pi_1^{(e)} \quad (22)$$

Where, NE is the number of elements and $\Pi^{(e)}$ is the elemental total potential energy. Following the assembly procedure, Eq. (22) can be further written as

$$\Pi_1 = \frac{1}{2} \{q\}^T [K_l + K_{nl}(q)] \{q\} - \{q\}^T \{F^T\} \quad (23)$$

Where

$$[K_{nl}(q)] = \frac{1}{2} [K_{nl1}(q)] + [K_{nl2}(q)] + \frac{1}{2} [K_{nl3}(q)]$$

Where $[K_l]$, and $[K_{nl}(q)]$ are the global linear and nonlinear stiffness matrices defined in Appendix (A.2). The parameters $\{q\}$ and $\{F^T\}$ are the global displacement and thermal load vectors and defined in the Appendix (A.3).

3.2 Foundation analysis

Similarly, using finite element model Eqs. (20)-(21), Eq. (16) after the assembly procedure can be written as

$$\Pi_2 = \sum_{e=1}^{NE} (\Pi_3^{(e)}) = \{q^{(e)}\} [K_{fl} + K_{fnl}(q)] \{q^{(e)}\} = \{q\} [K_{fl} + K_{fnl}(q)] \{q\} \quad (24)$$

Where (K_{fl}) and $(K_{fnl}(q))$ is global linear and nonlinear foundation stiffness matrices, respectively and defined in Appendix (A.4).

3.3 Thermal buckling analysis

Using finite element model Eq. (20), Eq. (17) after the assembly procedure can be written as

$$\Pi_3 = \sum_{e=1}^{NE} \Pi_2^{(e)} = \frac{1}{2} \lambda \{q\}^T [K_g] \{q\} \quad (25)$$

Where λ and (K_g) are defined as the critical thermal buckling load parameter and the global geometric stiffness matrix defined in Appendix (A.5), respectively.

with Newton-Raphson approach using SOPT and MCS methods to evaluate the nonlinear transverse central deflection of FGM elastically supported plate which is described below.

5.1 A newton-raphson method for the solution of nonlinear governing equation

After assembling the element stiffness matrices and force vectors, a new system of nonlinear algebraic equations from Eq. (29) can be written as (William *et al.* 1992, Reddy and Chin 1998)

$$[K(q)]\{q\} = \{F\} \quad (30)$$

This nonlinear system should be linearized to be solved and to get the nodal displacements $\{q\}$. The Newton-Raphson iterative linearization method is used in this study for evaluation of nonlinear analysis.

In the Newton Raphson procedure, the linearized element Eq. (30) is written in the following form as

$$\{q\}^i = \{q\}^{(i-1)} - [T\{q\}^{(i-1)}]^{-1} \{R(\{q\}^{(i-1)})\} \quad (31)$$

Where the residual

$$\{R(\{q\}^{(i-1)})\} = (K\{q\}^{(i-1)}) - \{F\} \quad (32)$$

The tangent stiffness matrix $[T\{q\}^{(i-1)}]$ element is calculated using the definition given as

$$[T\{q\}^{(i-1)}] = \left(\frac{\partial \{R(\{q\})\}}{\partial \{q\}} \right)^{(i-1)} \quad (33)$$

The next step is to divide the load into small increments as discussed below.

The force vector in Eq. (32) can be written as

$$\{F\} = \sum_{i=1}^N \{\Delta F_i\} \quad (34)$$

Where $\{\Delta F_i\}$ are the incremental forces applied for i^{th} iteration.

The displacement vector $\{q\}$ for the first step and second step can be written as

$$[K(\{q_0\})]\{q_1\} = \{\Delta F_1\} \quad (35)$$

$$[K(\{q_1\})]\{q_2\} = \{\Delta F_1\} + \{\Delta F_2\} \quad (36)$$

This process is continuing until $\{F\}$ is converged.

In both methods, direct and Newton-Raphson, the first iteration can be calculated using linear stiffness matrix, i.e., assume $\{q\}^{(i-1)} = 0$, and calculate $\{q\}^i$ using Eq. (31) or Eq. (32). Then calculate the residual and repeat iteration process till reach a sufficient residual. At the exact solution, the residual equals zero.

5.2 Solution approach of stochastic finite element method

Where α_i , and α_j are the random system parameters.

Substituting Eq. (38) in Eq. (37) and collecting the similar order of terms, following equations are obtained

$$\{q_0\} = [K_0^{-1}] \{F_0\} \quad (40)$$

$$\{q_i^I\} = [K_0^{-1}] \{F_i^{*I} - [K_i^{*I}] \{q_0\}\} \quad (41)$$

$$\{q_{ij}^{II}\} = [K_0^{-1}] \{F_{ij}^{*II} - [K_i^{*II}] \{q_j^{*I}\} - [K_j^{*II}] \{q_i^{*I}\} - [K_{ij}^{*II}] \{q_0\}\} \quad (42)$$

Obviously, Zeroth order Eq. (40) is the deterministic and gives the mean response. The first order Eq. (41) and second order Eq. (42) on the other hand represents its random counterpart and solution of this equation provides the statistics of the nonlinear bending response, which can be solved using the probabilistic methods like perturbation technique, Monte Carlo simulation, Newman's expansion technique etc.

From these mean and covariance matrix of deflection $\{q\}$ can be obtained as as (Halder and Mahadevan 2000, Jagtap *et al.* 2012, Kumar *et al.* 2014).

$$\langle q \rangle \approx \{q_0\} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \{q_{ij}^{II}\} Cov[\alpha_i, \alpha_j] \quad (43)$$

$$Cov[q, q] \approx \sum_{i=1}^N \sum_{j=1}^N \{q_i^I\} \{q_j^I\}^T Cov[\alpha_i, \alpha_j] \quad (44)$$

After $Cov[\alpha_i, \alpha_j]$ is substituted in terms of correlation coefficients ρ_{ij} in Eq. (45), the final expression for $Cov[q, q]$ is obtained as (Jagtap *et al.* 2012, Kumar *et al.* 2014, Singh and Lal 2010).

$$Cov[\{q\}, \{q\}] = \sum_{i=1}^N \sum_{j=1}^N \left[\frac{\partial \{q\}}{\partial \alpha_j} \Big|_{\alpha=0} \left(\rho_{ij} \sigma_{\alpha_i} \sigma_{\alpha_j} \right) \frac{\partial \{q^T\}}{\partial \alpha_j} \Big|_{\alpha=0} \right] \quad (45)$$

$$\text{Where } [\sigma_\alpha] = \begin{bmatrix} \sigma_{b_1} & \dots & \dots & 0 \\ 0 & \sigma_{b_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \sigma_{b_m} \end{bmatrix} \text{ and } [\rho_{ij}] = \begin{bmatrix} 1 & \rho_{12} & \dots & \rho_{1m} \\ \rho_{21} & 1 & \dots & \rho_{2m} \\ \dots & \dots & \dots & \dots \\ \rho_{m1} & \rho_{m2} & \dots & 1 \end{bmatrix}$$

Where $[\sigma_\alpha]$, $[\rho_{ij}]$ and m are the standard deviation (SD) of input random variables, the correlation coefficient matrix and number of random variables, respectively. In the present analysis, the uncorrelated Gaussian random variables are taken into consideration. Therefore, covariance is equal to the variance.

The variance of the deflection of random variables b_i ($i=1, 2, \dots, R$) and correlation coefficients can be expressed as (Halder and Mahadevan 2000, Shegokar and Lal 2014).

$$\text{var} \{q\} = \sum_{i=1}^N \sum_{j=1}^N \left[\left(\frac{\partial \{q\}}{\partial b_i^R} \right) [\sigma_\alpha] [\rho_{ij}] [\sigma_\alpha] \left(\frac{\partial \{q\}}{\partial b_j^R} \right)^T \right] \quad (46)$$

It is to be noted that to estimate the second order variance, the information on the third and

Table 1 The material properties of ZrO₂/Ti-6Al-4V FGMs with TD material properties Reddy and Chin CD (1998)

Types of material	Properties	P ₀	P ₋₁	P ₁	P ₂	P ₃
ZrO ₂	E(Pa)	244.27e+9	0	-1.371e-3	1.214e-6	-3.681e-6
	α (1/K)	12.766e-6	0	-1.491e-3	1.006e-5	-6.778e-11
Ti-6Al-4V	E(Pa)	122.56e+9	0	-4.586e-4	0	0
	α (1/K)	7.5788e-6	0	6.638e-4	3.147e-6	0

Table 2 Comparison and convergences study of the transverse central deflection of simply supported FGMs (Al/ZrO₂) square plate for various volume fraction index with different mesh size having b/h=5

Mesh size	Volume fraction index (n)				
	Ceramic	0.5	1	2	Metal
Present (2×2)	0.0039	0.0047	0.0053	0.0061	0.0087
Present (3×3)	0.0238	0.0293	0.0329	0.0371	0.0512
Present (4×4)	0.0212	0.0262	0.0294	0.0331	0.0458
Present (5×5)	0.0218	0.0252	0.0278	0.0326	0.0433
Present (6×6)	0.0224	0.0261	0.0306	0.0346	0.0448
Ferreira <i>et al.</i> (2005)	0.0205	0.0262	0.0294	0.0323	0.0443
Percentage Difference†	3.3018	0.0	0.0	2.4767	3.2751

†Percentage Difference is evaluated in between Present (4×4) and Ferreira *et al.* (2005)

Where \bar{w}_0 is the dimensional nonlinear transverse central deflection of FGM plate. The material properties are position dependent and can be expressed as (Shen and Wang 2010)

$$P = P_t V_t(z) + P_b V_b(z)$$

Where P_t and P_b represent the temperature dependent properties (TD) of the top and bottom faces of the plate, respectively and can be expressed as

$$P = P_0(P_{-1}T^{-1} + 1 + P_1T + P_2T^2 + P_3T^3)$$

The material properties such as P_0 , P_{-1} , P_1 , P_2 , P_3 , and T considered for present analysis is shown in Table 1. The value of temperature T is taken as 300K for the whole of the analysis unless otherwise stated.

For the temperature independent material properties (TID) the value of P_{-1} , P_1 , P_2 , and P_3 , are equal to zero. It is noted that, the results computed in this paper is for ZrO₂/Ti-6Al-4V material plate unless otherwise stated. The temperature dependent (TD) material property of functionally graded materials is given in Table 1.

6.1 Comparative study for statistics of nonlinear transverse central deflection

The accuracy and convergence of present deterministic approach for transverse central deflection of simply supported FGMs (Al/ZrO₂) square plate subjected to a uniform transverse load is shown in Table 2 with numerical results of (Ferreira *et al.* 2005). Based on established approach and analyses of the foregoing sections, it is acknowledged that (4×4) mesh is founded

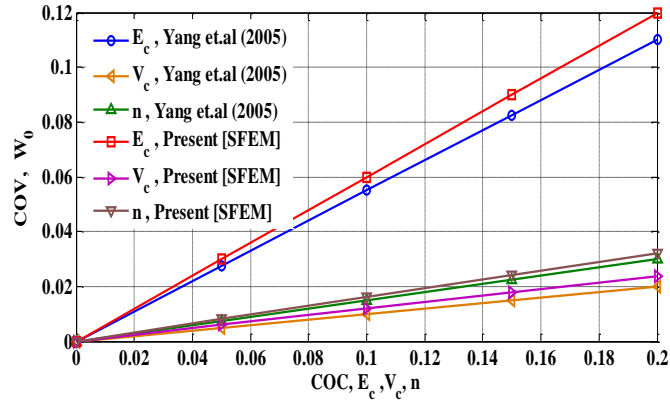


Fig. 4 Validation study of COV on nonlinear transverse central deflection of square FGM simply supported plate for random change in $b_1=E_c$, $b_3=E_m$, and $b_5=n$

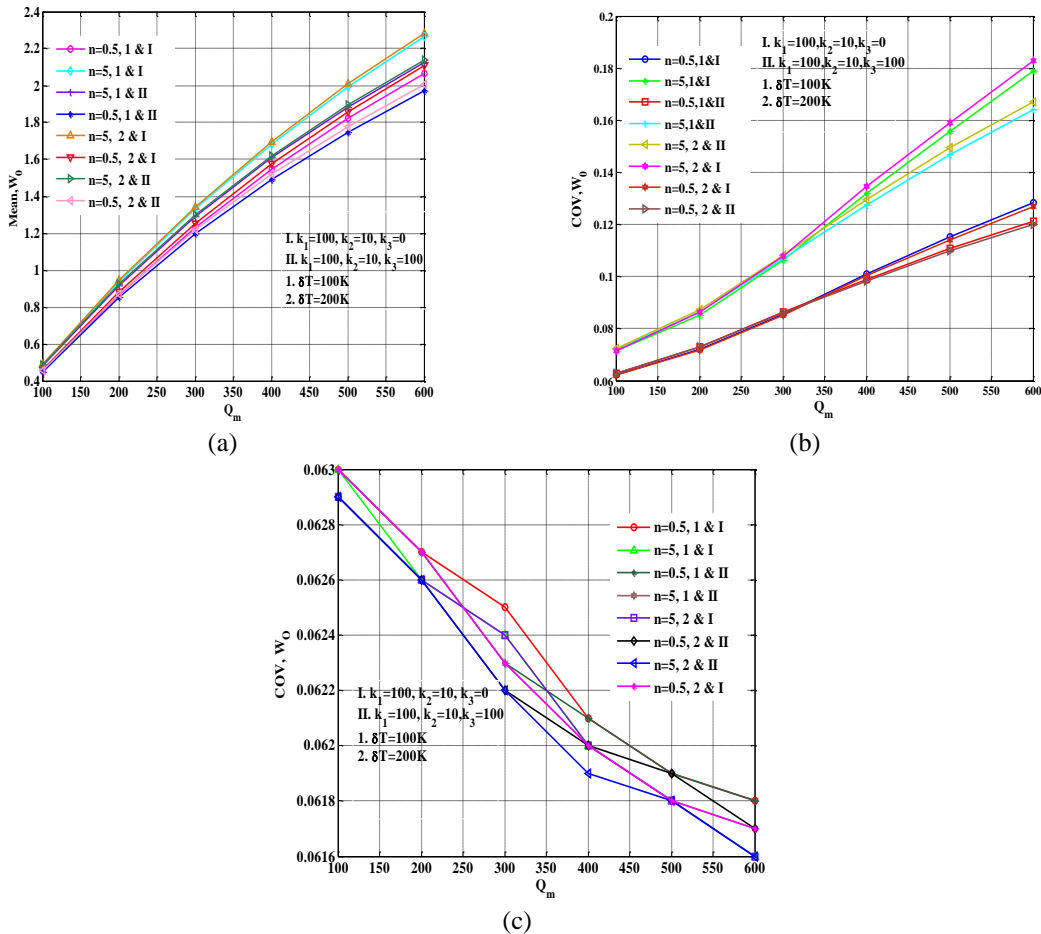


Fig. 5 The effect of temperature change, foundation parameters, volume fraction index and load parameters with random system properties on the (a) expected mean (b) $COV, \{b_i (i=1, \dots, 5)=0.1\}$ and (c) $COV \{b_i(i=6)=0.1\}$ of transverse central deflection of square FGM simply supported plate resting on elastic foundation in thermal environments

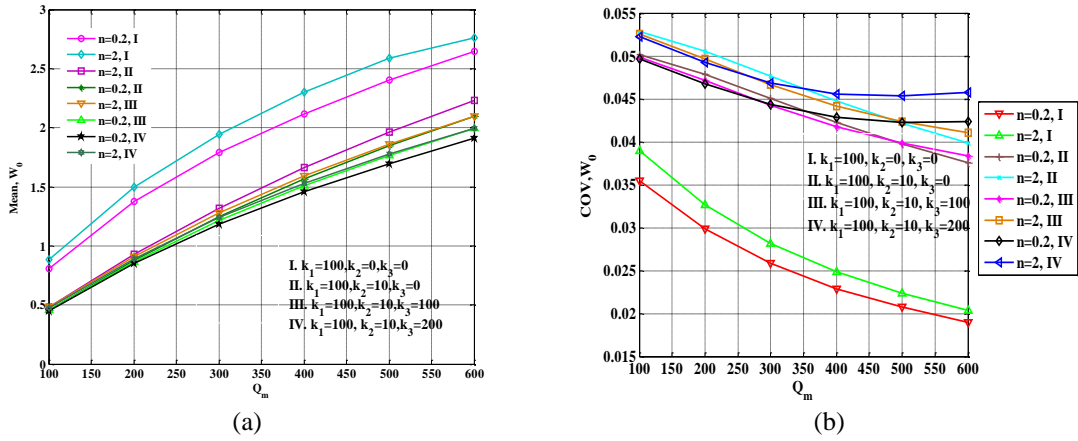


Fig. 6 The effect of foundation parameters, volume fraction index and load parameters with random system properties on the (a) expected mean, and (b) $COV\{b_i (i=11,12,13)=0.1\}$ of transverse central deflection of FGM square simply supported plate resting on elastic foundations in thermal environments

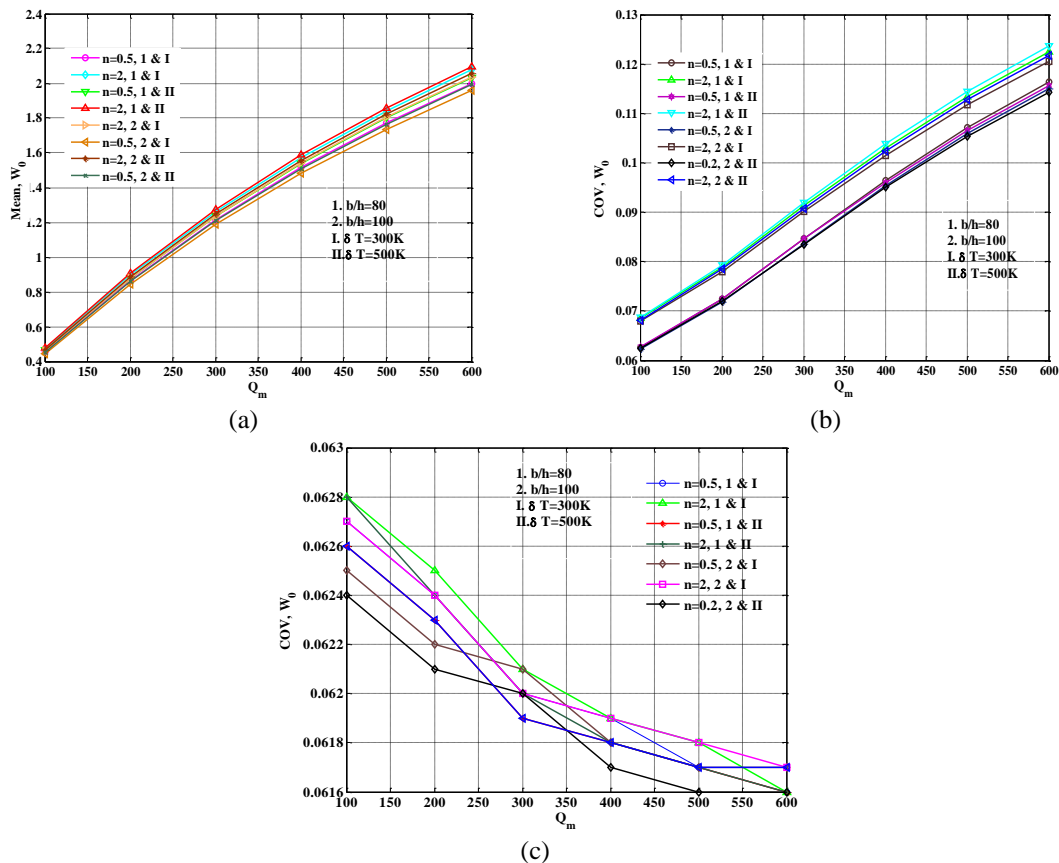


Fig. 7 The effect of plate thickness ratio, temperature change, volume fraction index, load parameters with random system properties on the (a) expected mean (b) $COV, \{b_i (i=1...5)=0.1\}$, and (c) $COV, \{b_i (i=6)=0.1\}$ of transverse central deflection of FGM square simply supported plate resting on a nonlinear elastic foundation ($k_1=100, k_2=10, k_3=100$) in thermal environments

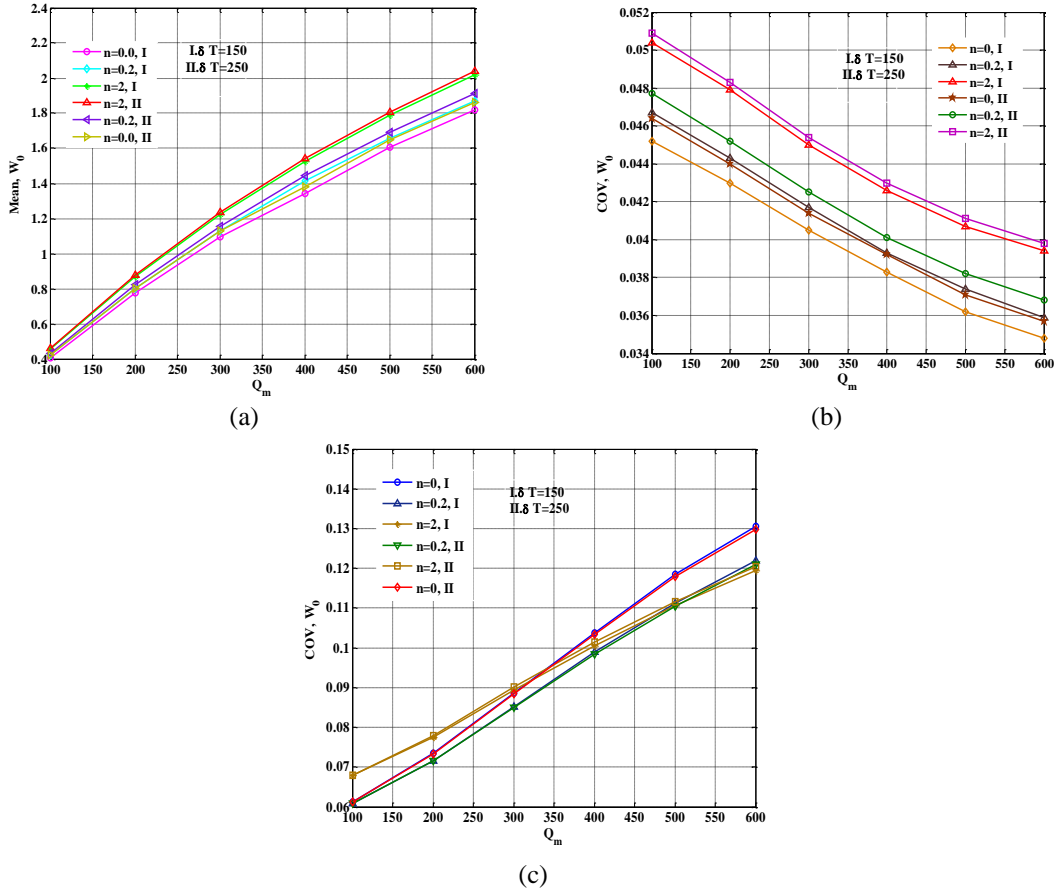


Fig. 10 The effect of volume fraction index, temperature change, and load parameters with random system properties on the (a) expected mean (b) COV, $\{b_i(i=1..5)=0.1\}$, and (c) COV, $\{b_i(i=11,12,13)=0.1\}$ of transverse central deflection of FGM square simply supported plates resting on elastic foundation in thermal environments

COV, $\{b_i(i=6)=0.1\}$ of transverse central deflection of square simply supported FGM plate resting on elastic foundation ($k_1=100, k_2=10, k_3=100$) in thermal environments is shown in Fig. 7. For the same temperature change, volume fraction, exponent and load parameters, with the increase of plate thickness ratio, the mean transverse central deflection decreases and corresponding COV increases. All other effects are already explained in the previous figure discussion.

Fig. 8 shows the effect of plate aspect ratios (b/a), temperature change, volume fraction index, and load parameters with random system properties on the (a) expected mean (b) COV, $\{b_i(i=1..5)=0.1\}$ of transverse central deflection of simply supported FGM plate resting on elastic foundation ($k_1=100, k_2=10, k_3=100$) in thermal environments for $b/h=90$. For the same temperature change, volume fraction index and load parameters, with the increase of the plate aspect ratio, mean and corresponding COV with random change in material properties increases. The effect of volume fraction on the mean and corresponding COV is more sensitive for rectangular plate as compared to square plate.

The effect of support conditions (namely SSSS, CCCC, and CSCS), volume fraction index and

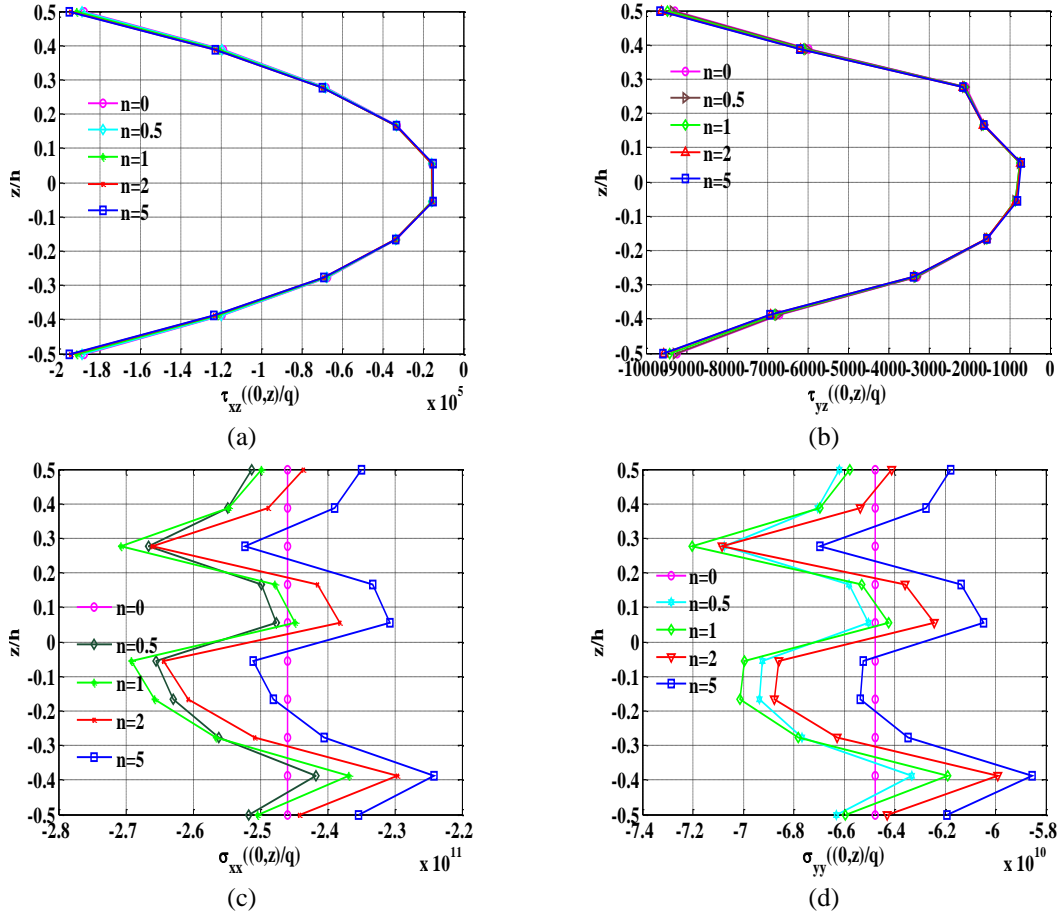


Fig. 12 Through the thickness variation of shear stresses (a) τ_{xy} , (b) τ_{yz} , and normal stresses (c) σ_{xx} and (d) σ_{yy} of simply supported FGM ($ZrO_2/TI-6Al-4V$) plate with various volume fraction index

Fig. 11(a)-(c) show variation of displacements in x, y, and z axis across the thickness of simply supported FGM ($ZrO_2/TI-6Al-4V$) plate with various volume fraction index and having TD material properties, UT , $a/h=20$, $\Delta T=100$ K and $Q=100$. It is clear that very little deformation has been occurred in the u-and v-directions, respectively. No deformation is observed in the w direction due to independent of thickness.

Variation of transverse shear stress and in plane stress across the thickness of simply supported FGM ($ZrO_2/TI-6Al-4V$) plate with various volume fraction index with TD material properties, uniform temperature distribution, $a/h=20$, $\Delta T=100$ K and $Q=100$ is shown in Fig. 12(a)-(d). With the increase of volume fraction index, the direct stresses (σ_{xx} and σ_{yy}) are highly sensitive while, shear stresses (τ_{xy} , and τ_{yz}) are least sensitive.

7. Conclusions

The stochastic finite element method using SOPT and independent MCS methods combined

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TK

Appendix

$$[D] = \begin{bmatrix} A1_{ij} & B_{ij} & E_{ij} & 0 & 0 \\ B_{ij} & D1_{ij} & F1_{ij} & 0 & 0 \\ E_{ij} & F1_{ij} & H_{ij} & 0 & 0 \\ 0 & 0 & 0 & A2_{ij} & D2_{ij} \\ 0 & 0 & 0 & D2_{ij} & F2_{ij} \end{bmatrix}, [D_3] = \begin{bmatrix} A1_{ij} & 0 \\ B_{ij} & 0 \\ E_{ij} & 0 \\ 0 & A2_{ij} \\ 0 & D2_{ij} \end{bmatrix}, [D_4] = [D_3]^T \text{ and } [D_5] = \begin{bmatrix} A1_{ij} & 0 \\ 0 & A2_{ij} \end{bmatrix} \quad (\text{A.1})$$

$$\text{With } (A1_{ij}, B_{ij}, D1_{ij}, E_{ij}, F1_{ij}, H_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2, z^3, z^4, z^6) dz; \quad (i, j=1, 2, 6)$$

$$(A2_{ij}, D2_{ij}, F2_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z^2, z^4) dz; \quad (i, j=4, 5) \quad (\text{A/1a})$$

$$[K_l] = \int_A [B_l]^T [D] [B_l] dA,$$

$$[K_{nl} \{q\}] = \int_A [B_{nl}]^T [D_1] [B_l] dA + \frac{1}{2} \int_A [B_l]^T [D_2] [B_{nl}] dA + \frac{1}{2} \int_A [B_{nl}]^T [D_3] [B_{nl}] dA \quad (\text{A.2b})$$

$$\{q\} = \sum_{e=1}^{NE} \{q\}^{(e)}, [F^T] = \sum_{i=1}^n \int_{A^{(e)}} \left[[B_{li}^{(e)}]^T [N^T] + [B_{bi}^{(e)}]^T [M^T] + [B_{b2i}^{(e)}]^T [P^T] \right] dA \quad (\text{A.3})$$

$$[K_{fl}] = \frac{1}{2} \int_A [B_f]^T [D_f] [B_f] dA, [K_{fml}] = \frac{1}{2} \int_A [B_f]^T [D_{fml}] [B_f] dA \quad (\text{A.4})$$

$$[K_G] = \int_A [B_{nl}]^T \{N\} dA = \int_A [G]^T [\bar{N}] [G] dA \quad (\text{A.5})$$