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Survey of nonlinear state estimation in aerospace systems with Gaussian priors

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Abstract. Nonlinear state estimation is a desirable and required technique for many situations in engineering (e.g., aircraft/spacecraft tracking, space situational awareness, collision warning, radar tracking, etc.). Due to high standards on performance in these applications, in the last few decades, there was an increasing demand for methods that are able to provide more accurate results. However, because of the mathematical complexity introduced by the nonlinearities of the models, the nonlinear state estimation uses techniques that, in practice, are not so well-established which, leads to sub-optimal results. It is important to take into account that each method will have advantages and limitations when facing specific environments. The main objective of this paper is to provide a comprehensive overview and interpretation of the most well-known methods for nonlinear state estimation with Gaussian priors. In particular, the Kalman filtering methods: EKF (Extended Kalman Filter), UKF (Unscented Kalman Filter), CKF (Cubature Kalman Filter) and EnKF (Ensemble Kalman Filter) with an aerospace perspective.

Keywords: Kalman filter; nonlinear state estimation; Gaussian priors; aircraft/spacecraft tracking

1. Introduction

The state estimation of stochastic systems is one of the basic problems of modern control theory and control system engineering, whose solution (an approximate solution) can be found in several filtering methods. The framework of these fields includes topics such as inertial navigation, aircraft/spacecraft tracking, multi-sensor data fusion, collision warning, space situational awareness and radar tracking, where the main problem is to estimate the state vector of an aerospace vehicle (e.g., aircraft, satellite, missile, space vehicles, etc.), given redundant and imperfect measurements of its position and velocity. In most applications, these measurements are based on radar signals that are contaminated with noise and system measurement errors (Bar-Shalom *et al.* 2001, Chandra and Gu 2019, Coelho and Bousson 2016, Roa *et al.* 2017, Tanizaki 1996). So, a good filtering algorithm should be able to remove the noise from data while retaining useful information (Welch and Bishop 2001). However, it is not an easy task, especially when facing online estimation with a statistical nature in a highly nonlinear environment. The process of

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state estimation requires robust filtering methods that can deal not only with the uncertainties of the system dynamics but also with the instrumental inaccuracies related to the data acquisition system and the environmental disturbances. In addition, it needs to consider that a real-physical system may have parameters whose values can be known only with specific accuracy and these uncertainties must also be considered since they might be crucial to evaluate the behaviours exhibit by the system (Akhlaghi *et al.* 2017, Ahmed 1998). All state estimations are based on the highly accurate information and therefore the development of methods that are able to provide such results are extremely important.

The filtering methods have their root on the Bayesian analysis of time-dependent behaviour, which belongs to the field of optimal filtering. (Gordon *et al.* 1993, Ho and Lee 1964, Jazwinski 1970, Maybeck 1979, Lee 2005, Bar-Shalom *et al.* 2001, Yuen *et al.* 2013, Särkkä and Nummenmaa 2009) are some examples of approaching the state estimation problem through a Bayesian perspective. The idea of constructing a mathematically optimal recursive estimator was initially presented for linear systems due to their mathematical simplicity. The optimal filtering history begins with (Wiener 1949 and Kolmogorov *et al.* 1962), where they solved the least-square estimation problem for stochastic systems. Wiener developed the solution for continuous-time and Kolmogorov developed the solution for discrete-time. This filter is known as the Wiener filter and is still important nowadays however, it is restricted to stationary signals only. Kalman (1960) extended Wiener's research for a more generic nonstationary process resulting in the Kalman filter. The main difference between these two filters is that the Wiener filter was developed in the frequency domain and is mainly used for signal estimation, whereas, the Kalman filter was developed in the time domain and is mainly used for state estimation.

The success of the Kalman filter (Kalman 1960, Grewal and Andrews 2001) in engineering applications is mostly due to the extended Kalman filter, which is based on the assumption that a local linearization of the system may be a sufficient description of the nonlinearities, so the linearized model is used instead of the original nonlinear function, documented by, (Coelho *et al.* 2017, Lefebvre *et al.* 2004, Mehra 1971, Simon 2006).

A large number of strategies and variations based on Kalman filters are available in the literature and many authors documented the efficiency of the Kalman filters to solve the nonlinear state estimation problems in the aerospace field. And some authors proposed improvements to the existing methods, for instance, Zhao *et al.* (2018) proposed a robust iterated extended Kalman filter based on the generalized maximum likelihood approach, Coelho *et al.* (2020) proposed an improved extended Kalman Filter with a new Jacobian matrix calculation point and a more accurate covariance calculation, Wu and Wang (2014) proposed a constrained unscented Kalman filter algorithm to improve the accuracy of numerical substructure modelling in hybrid testing and Ashrafifar and Jegarkandi (2020) applied an adaptive robust unscented Kalman filter to the problem of model-based fault detection and diagnosis for nonlinear supersonic air vehicles.

Before proceeding to the review of the nonlinear state estimation techniques, it is convenient to discuss some of the real-world applications where the Kalman filter played an important role:

• In the 1960s, in Apollo Mission (NASA 2019a), where a new approach was needed for computing the estimated state from onboard measurements, a way that would not overburden the simulation facilities or the onboard computational capability. In this situation, the Kalman filter proved to be a practical method for real-time onboard navigation. It was proved that the Kalman filter works as predicted and it was capable of solving the Apollo guidance and navigation problems. This mission also provided key steps on the development of the so-called Extended Kalman Filter.

• In the 1970s, the rotorcraft flight research began between NASA and the US Army, where a fully automatic digital flight and guidance system that had conventional autopilot capabilities (including autoland) was developed. The system used Kalman filtering for extracting aircraft position and inertial velocities from multiple ground-based and onboard sensors (NASA 2019b).

• On the Common Real-Time Debris Footprint (CRTF) Program (NASA 2019c), which contains a set of models that estimate the range of free-fall and impact locations of fragments that result from a vehicle breakup. In this program, the Kalman filter is applied to the estimation of the real-time vehicle state vector and uses tracking data from one or more sources. Each individual data source contains measurement error, which is implicit in the composite filter solution and it is represented by the filter's covariance matrix.

• In 2018, in the GPS Receiver and Orbit Determination Program (by ESA), where the main purpose was to develop a sophisticated navigation algorithm that could determine the positioning accuracy of a commercial receiver (ESA.int 2019a). The GPS payload encompasses a custom navigation computer that hosts an extended Kalman Filter, which uses raw GPS observables and an extremely precise orbital mathematical model in order to obtain position fixes with the required accuracy.

• The Advanced Concepts Team of ESA is assessing a novel approach to infer the status of a spacecraft and its instruments from reading a few strategically placed thermal sensors, where a thermal network model of a spacecraft is used (ESA.int 2019b). To cope with the strong nonlinearities of the resulting network the unscented Kalman filter is being used.

Presenting a survey of a field as diverse as nonlinear filtering is an ambitious task. Perhaps the most difficult issue is to discuss all nonlinear filtering methods within a limited amount of space. In order to achieve this goal, a conscious decision has been made to focus on the widely used classical approaches, such as:

1. Extended Kalman Filter (Welch and Bishop 2001, Doumiati *et al.* 2013)
2. Unscented Kalman Filter (Julier *et al.* 1995, Xiong *et al.* 2006, Gyorgy *et al.* 2014)
3. Cubature Kalman Filter (Arasaratnam and Haykin 2009, Zhang and Guo 2013)
4. Ensemble Kalman Filter (Kalnay *et al.* 2007)

The main objective of this paper is to provide a comprehensive overview and interpretation of these methods.

The paper is structured as follows section 2 addresses the problem statement, section 3, 4, 5, 6 and 7 discuss the Kalman filters by the following order: Linear, Extended, Unscented, Cubature and Ensemble Kalman filter, then section 8 provides an overview of the advantages and limitations of each method; and finally, section 9 provides the discussion and conclusion of this survey.

2. Problem statement

The main objective of the nonlinear state estimation problem is to accurately estimate the state of a moving target based on a sequence of noisy measurements, where the undesired noise shall be eliminated. On this survey, it is used a state-space approach with a discrete-time formulation, simply because it is more convenient for real-time application with online-onboard systems (Doumiati *et al.* 2013), which is the case of most aerospace vehicles.

To address the nonlinear state estimation problem, it is necessary at least two models, based on Markov models:

1. The **Nonlinear Dynamic Model**, which is responsible for describing the evolution of the system states and the time. It is given by:

$$\mathrm{x}_k = f\left(\mathrm{x}_{k-1}, u_{k-1}\right) + w_{k-1} \tag{1}$$

where $f(.)$ is the general nonlinear function of the dynamic model; $\mathrm{x}_k \in \mathbb{R}^n$ is the state vector at the time-step k, which can be defined as a set of variables that provide the complete status of the system at that time; w_k is white zero-mean, uncorrelated process noise, whose covariance matrix (Q_k) is known and defined by:

$$w_k \sim N(0, Q_k) \tag{2}$$

$$E\left[w_k w_j^T\right] = Q_k \delta_k \tag{3}$$

where δ_k is the Kronecker delta function and if $k=j$ then $\delta_k=1$; if $k\neq j$ then $\delta_k=0$ (Simon 2006).

2. The **Measurement Model**, which is responsible for relating the state of the system with the measurements. These measurements are imperfect because of the inherent noise and it must be eliminated in order to provide the best estimate possible:

$$\mathrm{y}_k = h\left(\mathrm{x}_k\right) + v_k \tag{4}$$

where $h(.)$ is the general function of the measurement model; $\mathrm{y}_k \in \mathbb{R}^m$ is the measurement vector at the time-step k; v_k is white zero-mean, uncorrelated noise, whose covariance matrix (R_k) is known and defined by:

$$v_k \sim N(0, R_k) \tag{5}$$

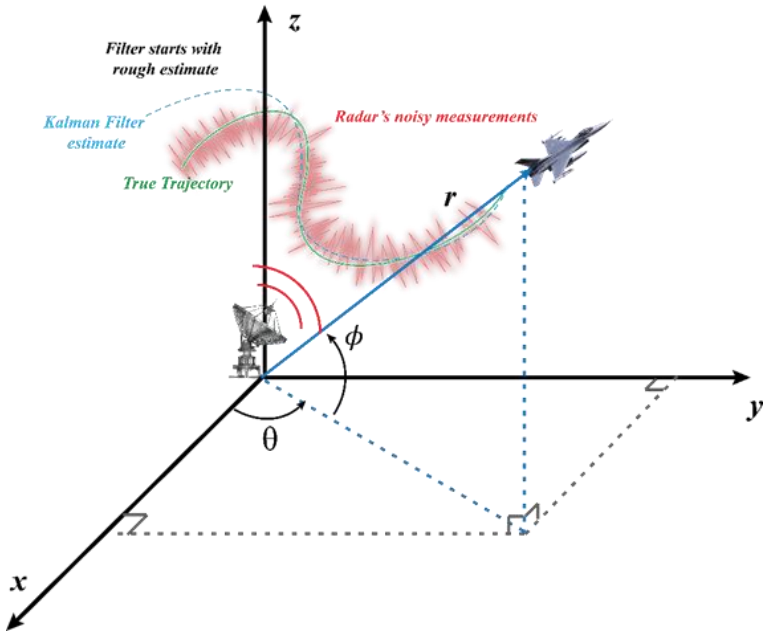

Fig. 1 Illustration of a radar tracking problem

$$E\left[v_k v_j^T\right] = R_k \delta_k \tag{6}$$

$$E\left[v_k w_j^T\right] = 0 \tag{7}$$

E is the k^{th} statistical moment of a continuous variable. The functions $f(.)$ and $h(.)$ depend on the time step number k, but for notational convenience, this dependence has not been explicitly denoted.

In the case of radar tracking, the state vector is defined as the sequence of the state dynamics of the moving target. The measurements are noise-corrupted observations provided by the radar (distances and directions/angles), as shown in Fig. 1. In this case, the Eq. (4) can be represented in the following form:

$$y_k = \begin{bmatrix} r_k \\ \theta_k \\ \phi_k \end{bmatrix} + \begin{bmatrix} v_k^r \\ v_k^\theta \\ v_k^\phi \end{bmatrix} \tag{8}$$

where r is the radial distance between the radar and the aerospace vehicle (target); θ is the angle measured from X-axis in XY plane of an inertial rectangular coordinate system to the projection of r onto XY plane; ϕ is the angle measured from the projection of r onto XY plane to the vector r.

A track is a sequence of state estimates up to the current time. False detections and multiple target scenarios form an additional level of complexity to the full process. The Kalman filters have been one of the most used tools to deal with radar tracking problems.

3. Linear Kalman Filter

The Kalman Filter (KF) is one of the most often-used tools to solve stochastic estimation problems. It was named after Rudolph E. Kalman, who in 1960 (Kalman 1960) presented an online recursive solution to the discrete-data linear filtering problem. The objective was to be able to estimate the state of a system based on noise-contaminated observations. The Kalman filter has a form of feedback control, which means that the filter estimates the process state at a specific time and then obtains feedback in the form of noisy measurements. So, it is represented by two groups of equations (also represented in Fig. 2):

a) Time update equations (prediction step): These set of equations are known as the predictor equations and they are responsible for projecting forward in time the current state and error covariance estimates, in order to obtain the *a priori* state estimate for the next time step.

$$\hat{\mathrm{x}}_k^- = A\hat{\mathrm{x}}_{k-1} + Bu_k \tag{9}$$

$$P_k^- = A_k P_{k-1} A_k^T + Q \tag{10}$$

b) Measurement update equations (correction step): These set of equations are known as the corrector equations and they are responsible for the feedback, which means, they are responsible for incorporating the new measurements (obtain by sensors, radars or other) into the *a priori*

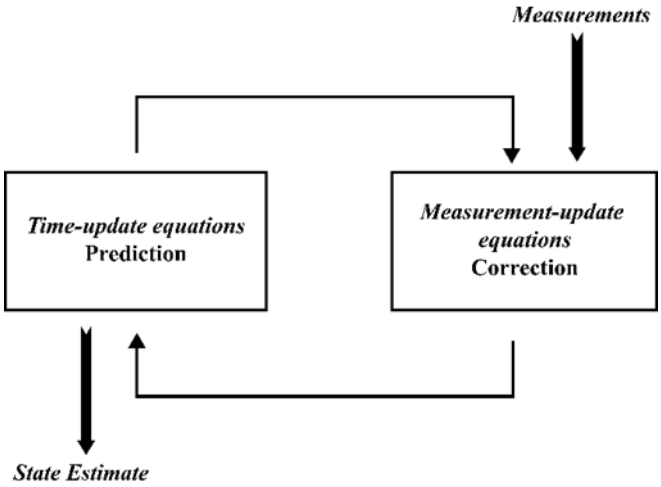

Fig. 2 A generic representation of Kalman filter equations groups. The Kalman filter estimates the process by using a form of feedback control, which means that at some point on time the filter will estimate the process state and then it will obtain feedback in form of noisy measurements

estimate (calculated on the step before) to obtain an improved *a posteriori* estimate.

$$K_k = P_k^- H^T \left(H P_k^- H^T + R \right)^{-1} \tag{11}$$

$$\hat{\mathrm{x}}_k = \hat{\mathrm{x}}_k^- + K_k \left(y_k - H\hat{\mathrm{x}}_k^- \right) \tag{12}$$

$$P_k = \left(I - K_k H \right) P_k^- \tag{13}$$

4. Extended Kalman Filter

Most of the real-world dynamics and measurements models are described as nonlinear systems so, the linear Kalman filter is not appropriate. To solve the nonlinear filtering problems, it was developed an extension of the filter: the extended Kalman filter (EKF). This is the most widely used method in engineering and a large number of applications can be found in literature, for instance, Yang and Zhou (2017) considered the EKF to solve a spacecraft attitude estimation problem, Jiang *et al.* (2014) applied the EKF to estimate the position of Mars rovers and suppress the measurement noise, Bishop and Antoulas (1994) study the performance of EKF to solve the problem of aircraft tracking and Rigatos and Tzafestas (2007) implemented the EKF algorithm to estimate the state vector of the autonomous vehicle by fusing data coming from odometry sensors and sonars.

The EKF model applies the classic Kalman filter to nonlinear systems through a local linearization, this means, the EKF has the ability to linearize the current mean and covariance, using first-order Taylor series expansion evaluated at the best current estimate of the state. The

linearization equations are given by:

$$\mathrm{x}_k = \tilde{\mathrm{x}}_k + A_k\left(\mathrm{x}_{k-1} - \hat{\mathrm{x}}_{k-1}\right) + w_{k-1} \tag{14}$$

$$\mathrm{y}_k = \tilde{\mathrm{y}}_k + H_k\left(\mathrm{x}_k - \tilde{\mathrm{x}}_k\right) + v_k \tag{15}$$

where $\mathrm{x}_k, \mathrm{y}_k$ are the actual state and measurement vectors; $\tilde{\mathrm{x}}_k, \tilde{\mathrm{y}}_k$ are the approximate state and measurement vectors; $\hat{\mathrm{x}}_k$ is the *posteriori* estimate state at step *k*; A_k is the Jacobian matrix of partial derivatives of *f* with respect to x and it is defined as:

$$A_k = \left.\frac{\partial f\left(\mathrm{x}, u\right)}{\partial \mathrm{x}}\right|_{\substack{\mathrm{x}=\hat{\mathrm{x}}_k \\ u=u_k}} \tag{16}$$

H_k is the Jacobian matrix of partial derivatives of *h* with respect to x and it is defined:

$$H_k = \left.\frac{\partial h\left(\mathrm{x}\right)}{\partial \mathrm{x}}\right|_{\mathrm{x}=\tilde{\mathrm{x}}_k} \tag{17}$$

The linearization is a very sensitive and important step. First, because it is too susceptible to error and second because allows the filter to get the best benefit from all the available *a priori* information.

The EKF algorithm can be presented in the following form:

a) Initialization

It is assumed that: $\hat{\mathrm{x}}_0 = \mathrm{x}_0$ and $\mathrm{P}_0 = \mathrm{P}_{initial}$.

b) Time update equations (prediction step):

The prediction of the state is given by:

$$\hat{\mathrm{x}}_k^- = f\left(\hat{\mathrm{x}}_{k-1}, u_k\right) \tag{18}$$

The predicted covariance is computed as:

$$P_k^- = A_k P_{k-1} A_k^T + w_k Q_{k-1} w_k^T \tag{19}$$

c) Measurement update equations (correction step):

The filter gain is computed as:

$$K_k = P_k^- H_k^T \left(H_k P_k^- H_k^T + V_k R_k V_k^T\right)^{-1} \tag{20}$$

The state estimation is calculated by:

$$\hat{\mathrm{x}}_k = \hat{\mathrm{x}}_k^- + K_k\left(\mathrm{y}_k - h\left(\hat{\mathrm{x}}_k^-\right)\right) \tag{21}$$

The estimated covariance is given by:

$$P_k = \left(I - K_k H_k\right) P_k^- \tag{22}$$

Exploiting the assumption that all transformations are quasi-linear, the EKF simply linearizes all nonlinear transformations and substitutes the Jacobian matrices for the linear transformations.

5. Unscented Kalman Filter

5.1 Unscented transform

The unscented Kalman filter was initially proposed by Julier and Uhlmann (Julier *et al.* 1995) and it is a numerical method based on the unscented transform (UT), well documented in the literature by (Akin *et al.* 2003, Julier and Uhlmann 1996, Julier *et al.* 2000, Li and Leung 2004, Ning and Fang 2007).

The UT was developed to address the deficiencies of the linearization by providing a more direct and explicit mechanism to transform the mean and covariance, and consequently, to ensure higher accuracy. The core idea is to deterministically choose a fixed number of sigma points that capture the mean and covariance of the original distribution of x. These sigma points are propagated through the nonlinearity and from them, the mean and covariance of the transformed variable will be estimated. Although the unscented transform resembles the Monte Carlo estimation, these approaches are significantly different, the sigma points are deterministically selected in the UT method, while they are randomly selected in the Monte Carlo methods.

The UT can be applied without the Gaussian assumption however, in this paper, the assumption is used only to provide a better comparison between different methods when implemented on the same conditions.

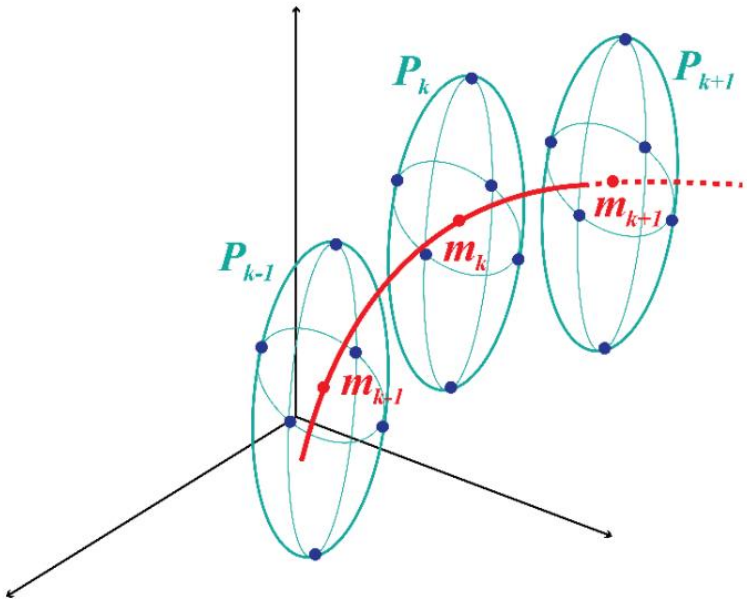

Fig. 3 Geometric representation of sigma points propagation. The mean (represented in red, m) of the state vector is the ellipsoid centre and the sigma points (represented in blue) are the extremes of the ellipsoid axes

The following procedure forms the Gaussian approximation based on the unscented transform. Considering the model as: x~N(m,P), y=g(x), where the variable x has dimension n, mean m and covariance P.

- ***Step 1*:** A set of (2*n*+1) sigma points are form by the following mathematical expressions:

$$\chi^{(0)} = \mathrm{m} \tag{23}$$

$$\chi^{(i)} = \mathrm{m}+\sqrt{n+\lambda}\left[\sqrt{\mathrm{P}}\right]_i \tag{24}$$

$$\chi^{(i+n)} = \mathrm{m}-\sqrt{n+\lambda}\left[\sqrt{\mathrm{P}}\right]_i, i=1,\ldots,n \tag{25}$$

where $[.]_i$ denotes the ith column of the matrix, λ is a scaling parameter defined in terms of algorithm parameters α and κ: $\lambda=\alpha^2(n+\kappa)-n$. The n is the total number of states to be estimated.

The parameters α determine the spread of the sigma points around the mean, in other words, it controls the size of the distribution. Ideally should be a smaller number: $0\leq\alpha\leq1$. The parameter κ is a secondary scaling parameter which is usually set to 0. It should be $\kappa\geq0$ to guarantee the semi-positive definiteness of the covariance matrix and that is why zero is a good default choice. The matrix square root denotes a matrix such that $\sqrt{\mathrm{P}}\sqrt{\mathrm{P}}^T=\mathrm{P}$. Each sigma point represents a vector, where one of the sigma vectors is the expected value of the augmented state vector and the remaining 2*n* sigma points are obtained from the columns of the matrix square root: $\pm\lambda P$.

- ***Step 2:*** Propagate the sigma points based on the nonlinear function *g*(.):

$$\gamma^{(i)} = \mathrm{g}\left(\chi^{(i)}\right), \quad i=0,\ldots,2n \tag{26}$$

which results in the transformed sigma points $\gamma^{(i)}$.

- ***Step 3*:** Compute the estimate of the mean and covariance of the transformed variable:

$$E\left[\mathrm{g}(\mathrm{x})\right] \simeq \mu_{\mathrm{U}} = \sum_{i=0}^{2n} W_i^{(\mathrm{m})}\gamma^{(i)} \tag{27}$$

$$E\left[\mathrm{g}(\mathrm{x})\right] \simeq \mu_{\mathrm{U}} = \sum_{i=0}^{2n} W_i^{(\mathrm{m})}\gamma^{(i)} \tag{28}$$

where the constant weights $W_i^{(\mathrm{m})}$ and $W_i^{(\mathrm{c})}$ are calculated by:

$$W_0^{(\mathrm{m})} = \frac{\lambda}{n+\lambda} \quad \text{and} \quad W_0^{(\mathrm{c})} = \frac{\lambda}{n+\lambda}+\left(1-\alpha^2+\beta\right) \tag{29}$$

$$W_i^{(\mathrm{m})} = \frac{1}{2(n+\lambda)}, \; i=1,\ldots,2n \quad \text{and} \quad W_i^{(\mathrm{c})} = \frac{1}{2(n+\lambda)}, \; i=1,\ldots,2n \tag{30}$$

β is an additional parameter used to incorporate the a *priori* information. For Gaussian distribution, the optimum value is β=2 (Chowdhary and Jategaonkar 2006).

If the unscented transform is applied to the augmented function $\tilde{\mathrm{g}}(\mathrm{x})=(\mathrm{x},\mathrm{g}(\mathrm{x}))$, the result is a set of sigma points, where $\chi^{(i)}$ and $\gamma^{(i)}$ are concatenated on the same vector (Wan and Merwe 2000).

5.2 Unscented Kalman Filter

The unscented Kalman filter (UKF) is an approximate filtering method based on the UT algorithm. Although resembles the Monte Carlo method, there is a fundamental difference: the samples are not drawn at random but rather according to a specific, determinist algorithm. In this case, the problem of a statistical convergence is not an issue and high order information about the distribution can be calculated using a very small number of sigma points (Romanenko *et al.* 2004, Wan and Van der Merwe 2000). The UKF was developed with the assumption that approximating a (Gaussian) distribution is easier than approximating a nonlinear transformation:

$$p(\mathrm{x}_k \,|\, y_{1:k}) \simeq \mathrm{N}(\mathrm{x}_k \,|\, \mathrm{m}_k, \mathrm{P}_k) \tag{31}$$

where, m_k,P_k are the mean and covariance computed by the algorithm.

The prediction and update steps for the additive form of the UKF are given by the following operations, performed at each measurement step k=1,2,3,… (Wan and Merwe 2000):

- Initialization:

Initialize with:

$$\hat{\mathrm{x}}_0 = E[\mathrm{x}_0] \quad \text{and} \quad \mathrm{P}_0 = E\left[(\mathrm{x}_0 - \hat{\mathrm{x}}_0)(\mathrm{x}_0 - \hat{\mathrm{x}}_0)^T\right] \tag{32}$$

- Prediction step:

Calculate the sigma points:

$$\chi_{k-1}^{(0)} = \mathrm{m}_{k-1} \tag{33}$$

$$\chi_{k-1}^{(i)} = \mathrm{m}_{k-1} + \sqrt{n+\lambda}\left[\sqrt{\mathrm{P}_{k-1}}\right]_i \tag{34}$$

$$\chi_{k-1}^{(i+n)} = \mathrm{m}_{k-1} - \sqrt{n+\lambda}\left[\sqrt{\mathrm{P}_{k-1}}\right]_i, i = 1,\ldots,n \tag{35}$$

where the parameter λ is defined by:

$$\lambda = \alpha^2(n+\kappa) - n \tag{36}$$

Propagate the sigma points through the dynamic model:

$$\hat{\chi}_k^{(i)} = \mathrm{f}\left(\chi_{k-1}^{(i)}\right), \; i = 0,\ldots,2n \tag{37}$$

Compute the predicted mean m_k^- and the predicted covariance P_k^-:

$$\mathrm{m}_k^- = \sum_{i=0}^{2n} W_i^{(\mathrm{m})} \hat{\chi}_k^{(i)} \quad \text{and} \quad \mathrm{P}_k^- = \sum_{i=0}^{2n} W_i^{(\mathrm{c})} \left(\hat{\chi}_k^{(i)} - \mathrm{m}_k^-\right)\left(\hat{\chi}_k^{(i)} - \mathrm{m}_k^-\right)^T + \mathrm{Q}_{k-1} \tag{38}$$

- Correction step:

Calculate the sigma points:

$$\chi_k^{-(0)} = \mathrm{m}_k^- \quad , \quad \chi_k^{-(i)} = \mathrm{m}_k^- + \sqrt{n+\lambda}\left[\sqrt{\mathrm{P}_k^-}\right]_i \tag{39}$$

$$\chi_k^{-(i+n)} = \mathrm{m}_k^- - \sqrt{n+\lambda}\left[\sqrt{\mathrm{P}_k^-}\right]_i, i = 1,\ldots,n \tag{40}$$

Propagate the sigma points through the measurement model:

$$\hat{\gamma}_k^{(i)} = \mathrm{h}\left(\chi_k^{-(i)}\right) \quad , i = 0,\ldots,2n \tag{41}$$

Compute the predicted mean μ_k, the predicted covariance S_k of the measurement, the cross-variance of the state and the measurement C_k:

$$\mu_k = \sum_{i=0}^{2n} W_i^{(\mathrm{m})} \hat{\gamma}_k^{(i)} \tag{42}$$

$$\mathrm{S}_k = \sum_{i=0}^{2n} W_i^{(\mathrm{c})} \left(\hat{\gamma}_k^{(i)} - \mu_k\right)\left(\hat{\gamma}_k^{(i)} - \mu_k\right)^T + \mathrm{R}_k \tag{43}$$

$$\mathrm{C}_k = \sum_{i=0}^{2n} W_i^{(\mathrm{c})} \left(\hat{\chi}_k^{-(i)} - \mathrm{m}_k^-\right)\left(\hat{\gamma}_k^{(i)} - \mu_k\right)^T \tag{44}$$

Given the measurement C_k, compute the filter gain K_k, the filtered state (the mean m_k and the covariance P_k):

$$\mathrm{K}_k = \mathrm{C}_k \mathrm{S}_k^{-1} \tag{45}$$

$$\mathrm{m}_k = \mathrm{m}_k^- + \mathrm{K}_k\left[\mathrm{y}_k - \mu_k\right] \tag{46}$$

$$\mathrm{P}_k = \mathrm{P}_k^- - \mathrm{K}_k \mathrm{S}_k \mathrm{K}_k^T \tag{47}$$

6. Cubature Kalman Filter

In 2009, Arasaratnam and Haykin proposed the Cubature Kalman filter (CKF), which is based on the cubature transform. It applies deterministic samplings to evaluate the demanding integrals encountered in filtering problems (Arasaratnam and Haykin 2009, 2010). The CKF is numerically more stable and more accurate than the UKF however, some drawbacks in computing the spherical-radial cubature rule cannot be ignored (Zhang and Guo 2013). Still, it can have great success in solving some nonlinear state estimation problems, for instance Pesonen and Piche (2010) applied the CKF in a hybrid navigation application for positioning, Zhao *et al.* (2018)

applied an adaptive CKF in an ultra-tightly coupled navigation system of a hypersonic vehicle and Mu (2012) implemented the CKF for a reentry ballistic target tracking problem.

The Cubature Kalman filter was developed assuming that the predictive density of the joint state-measurement random variable is Gaussian. The idea is to simplify the optimal Bayesian filter problem to a problem where the main focus is on how to compute various multi-dimensional Gaussian-weighted moment integrals. The solution was found in a third-degree spherical-radial cubature rule which entails a set of cubature points scaling linearly with the state-vector dimension. The CKF preserve completely the second-order information due to the maximum entropy principle.

It can be described as a derivative-free sequential-state estimator that relies on the integration for its operation. It is the closest known approximation to the Bayesian filter that could be designed in a nonlinear setting under the Gaussian assumption. The CKF is presented as an appealing option for nonlinear state estimation, first because unlike the EKF, it does not require the evaluation of Jacobian matrices during the estimation process and second, it does not depend on additional tuning parameters as the UKF, where the performance is completely dominated by the tuning parameters: α, β, κ.

The additive form of the cubature Kalman filter (CKF) algorithm is:

- Prediction step:

Calculate the sigma points:

$$\chi_{k-1}^{(i)} = \mathrm{m}_{k-1} + \sqrt{\mathrm{P}_{k-1}}\,\xi^{(i)} \quad , i = 1,\ldots,2n \tag{48}$$

being $\mathrm{P}=\sqrt{\mathrm{P}}\sqrt{\mathrm{P}}^{T}$ where $\sqrt{\mathrm{P}}$ is the Cholesky factor of the covariance matrix P or some other similar square root of the covariance matrix and $\xi^{(i_1,\ldots,i_n)}$ is an n-dimensional vector with one unit sigma point $\xi^{(i_k)}$ at the element *k* and in this case, is defined as:

$$\xi^{(i)} = \begin{cases} \sqrt{n}\mathrm{e}_i & , i = 1,\ldots,n \\ -\sqrt{n}\mathrm{e}_{i-n} & , i = n+1,\ldots,2n \end{cases} \tag{49}$$

where e_i denotes a unit vector in the direction of the coordinate axis *i*.

Propagate the sigma points:

$$\hat{\chi}_k^{(i)} = \mathrm{f}\left(\chi_{k-1}^{(i)}\right) \quad , i = 1,\ldots,2n \tag{50}$$

Compute the predicted mean m_k^- and the predicted covariance P_k^-:

$$\mathrm{m}_k^- = \sum_{i=0}^{2n} W_i^{(\mathrm{m})}\,\hat{\chi}_k^{(i)} \tag{51}$$

$$\mathrm{P}_k^- = \frac{1}{2n}\sum_{i=1}^{2n}\left(\hat{\chi}_k^{(i)} - \mathrm{m}_k^-\right)\left(\hat{\chi}_k^{(i)} - \mathrm{m}_k^-\right)^{T} + \mathrm{Q}_{k-1} \tag{52}$$

- Correction step:

Calculate the sigma points:

$$\chi_k^{-(i)} = \mathrm{m}_k^- + \sqrt{\mathrm{P}_k^-}\,\xi^{(i)} \quad , i = 1,\ldots,2n \tag{53}$$

where $\xi^{(i)}$ is defined as:

$$\xi^{(i)} = \begin{cases} \sqrt{n}\mathrm{e}_i & , i = 1,\ldots,n \\ -\sqrt{n}\mathrm{e}_{i-n} & , i = n+1,\ldots,2n \end{cases} \tag{54}$$

Propagate the sigma points through the measurement model:

$$\hat{\gamma}_k^{(i)} = \mathrm{h}\left(\chi_k^{-(i)}\right) \quad , i = 1,\ldots,2n \tag{55}$$

Compute the predicted mean μ_k, the predicted covariance S_k of the measurements, the cross-variance of the state and the measurement C_k:

$$\mu_k = \frac{1}{2n}\sum_{i=1}^{2n}\hat{\gamma}_k^{(i)} \tag{56}$$

$$\mathrm{S}_k = \frac{1}{2n}\sum_{i=1}^{2n}\left(\hat{\gamma}_k^{(i)} - \mu_k\right)\left(\hat{\gamma}_k^{(i)} - \mu_k\right)^T + \mathrm{R}_k \tag{57}$$

$$\mathrm{C}_k = \frac{1}{2n}\sum_{i=1}^{2n}\left(\chi_k^{-(i)} - \mathrm{m}_k^-\right)\left(\hat{\gamma}_k^{(i)} - \mu_k\right)^T \tag{58}$$

Given the measurement C_k, compute the filter gain K_k, the filtered state, the mean m_k, and the covariance P_k:

$$\mathrm{K}_k = \mathrm{C}_k\mathrm{S}_k^{-1} \tag{59}$$

$$\mathrm{m}_k = \mathrm{m}_k^- + \mathrm{K}_k\left[\mathrm{C}_k - \mu_k\right] \tag{60}$$

$$\mathrm{P}_k = \mathrm{P}_k^- - \mathrm{K}_k\mathrm{S}_k\mathrm{K}_k^T \tag{61}$$

The presented form is the CKF of Arasatnam and Haykin (2009) obtained when the third-order spherical cubature integration rule is applied to the classical Gaussian filter. It is important to note that in the CKF literature, the "third-order" characteristic of the cubature integration rule is often emphasized, however in the covariance computation, the rule is only exact for the first order polynomials. Thus, in that sense, CKF is a first-order method.

7. Ensemble Kalman filter

The Ensemble Kalman Filter (EnKF) was initially proposed by Evensen (1994) and is based on forecasting the error statistics using the Monte Carlo method, which belongs to a broader category of the particle filters. Although the EnKF is being extensively used in the context of weather forecasting and it is being recognized for producing accurate and computationally effective

estimation on systems with a very high dimension, it is almost unknown by the tracking community, except the work of Cui and Zhang (2008). There is some other research on the aerospace context using EnKF on topics such as simulated Doppler radar data by Snyder and Zhang (2003), simulated data of sounding, satellite and plane by Mitchell *et al.* (2002), but still a very small research number.

The EnKF was initially proposed as an attempt to improve the error covariance calculation, which in the Extended Kalman Filter is difficult to implement, first, because of the computational cost of error-covariance calculation in realistic systems with many degrees of freedom, second because of the nonlinearities of the systems itself and the poorly characterized error sources. In the EnKF, the prediction and analysis error covariances have ensemble representations. These ensembles have sizes which limit the number of degrees of freedom, in this way the filter error covariance calculations are a lot more practical for modest ensemble sizes.

The EnKF implementation can be summarized by the following steps:

1. Generation of the ensemble points of the augmented system. The states of the system are augmented when incorporating the measurements.

2. Propagation of ensemble points through process and measurement equations. Evensen (1994) called the step as forecast which is commonly known as predictor step by the engineering community.

3. Updates of ensemble points using Kalman filter scheme with the help of measured data and measurement noise statistics. This step is commonly known as analysis by weather scientists, whereas system engineers are used to calling it a measurement update.

The EnKF can be presented as follows:

- Filter Initialization:

Assuming a system described by equations (1) and (2), the state vector is augmented with the measurements obtained, in order to form the ensemble (Lorentzen and Naevdal 2001). Initially, to represent the error statistics in the forecast step, it is assumed that at time *k*, we have an ensemble of N forecasted state estimates with random noise. The ensemble is denoted as $X_k^f \in \mathbb{R}^{n\times N}$ where,

$$\mathbf{X}_k^f = \left(\mathrm{x}_k^{f_1},...,\mathrm{x}_k^{f_N}\right) \tag{62}$$

The subscript f_i refers to the *i*-th forecast ensemble member.

- Prediction Step/Forecast Step:

Define the ensemble error matrix $E_k^f \in \mathbb{R}^{n\times N}$ around the ensemble mean:

$$E_k^f = \left[\mathrm{x}_{k+1}^{f_1} - \overline{\mathrm{x}}_{k+1}^f \quad ... \quad \mathrm{x}_{k+1}^{f_N} - \overline{\mathrm{x}}_{k+1}^f\right] \tag{63}$$

where, $\overline{\mathrm{x}}_{k+1}^f = \frac{1}{N}\sum_{i=1}^{N}\mathrm{x}_{k+1}^{f_i}$. It is assumed that the forecast ensemble mean is the best forecast estimate of the state and the spread of the ensemble members around the mean are the error between the best estimate and the actual state.

Define the ensemble output error, where:

$$E_{\mathrm{y}_k}^a = \left[\mathrm{y}_k^{f_1} - \overline{\mathrm{y}}_k^f \quad ... \quad \mathrm{y}_k^{f_N} - \overline{\mathrm{y}}_k^f\right] \tag{64}$$

Approximate P_k^f by $\hat{P}_k^f$, $P_{\mathrm{xy}_k}^f$ by $\hat{P}_{\mathrm{xy}_k}^f$ and $P_{\mathrm{yy}_k}^f$ by $\hat{P}_{\mathrm{yy}_k}^f$:

$$\hat{P}_k^f = \frac{1}{N-1} E_k^f \left(E_k^f\right)^T \tag{65}$$

$$\hat{P}_{\mathrm{xy}_k}^f = \frac{1}{N-1} E_k^f \left(E_{\mathrm{y}_k}^f\right)^T \tag{66}$$

$$\hat{P}_{\mathrm{yy}_k}^f = \frac{1}{N-1} E_{\mathrm{y}_k}^f \left(E_{\mathrm{y}_k}^f\right)^T \tag{67}$$

The last step of the forecast is to calculate:

$$x_{k+1}^{f_i} = f\left(x_k^{a_i}, \mathrm{u}_k\right) + w_k^i \tag{68}$$

where, $\overline{\mathrm{x}}_k^a = \frac{1}{N}\sum_{i=1}^{N} \mathrm{x}_k^{a_i}$.

- ***Correction Step*** (corrector step or measurement update):

Calculate the Kalman gain matrix:

$$\hat{K}_k = \hat{P}_{\mathrm{xy}_k}^f \left(\hat{P}_{\mathrm{yy}_k}^f\right)^{-1} \tag{69}$$

Calculate the estimate of the state:

$$\mathrm{x}_k^{a_i} = \mathrm{x}_k^{f_i} + \hat{K}_k \left(\mathrm{y}_k - h\left(\mathrm{x}_k^{f_i}\right)\right) \tag{70}$$

On this step, the EnKF performs an ensemble of parallel data assimilation cycles, where i=1,…,N.

In a nonlinear process with a linear measurement system, the ensemble can be formed only with the states. In this case, the update equation presented above will need to be modified. Another important point is that the ensemble variance is calculated by dividing (N-1) instead of the expected N. This happens as a correction to the unknown population mean, known as Bessel's correction, in order to reduce the bias in the estimation of the population variance (Hunt *et al.* 2007).

8. Advantages and limitations - Overview

Most of the variables of interest (that compose the system's state) cannot be measured directly. So, the idea is to generate precise methods (Kalman filters) that are able to extract this valuable information from the noisy signals. In order to obtain an optimal estimate of the desired quantities, minimizing the error as possible. Therefore, the filter choice is very important and shall be adequate to the characteristics of the system, the desirable (and final) efficiency that we want as well as the computational time that will be needed to obtain such results. Because all algorithms will have advantages and limitations when applied in specific environments, this section will

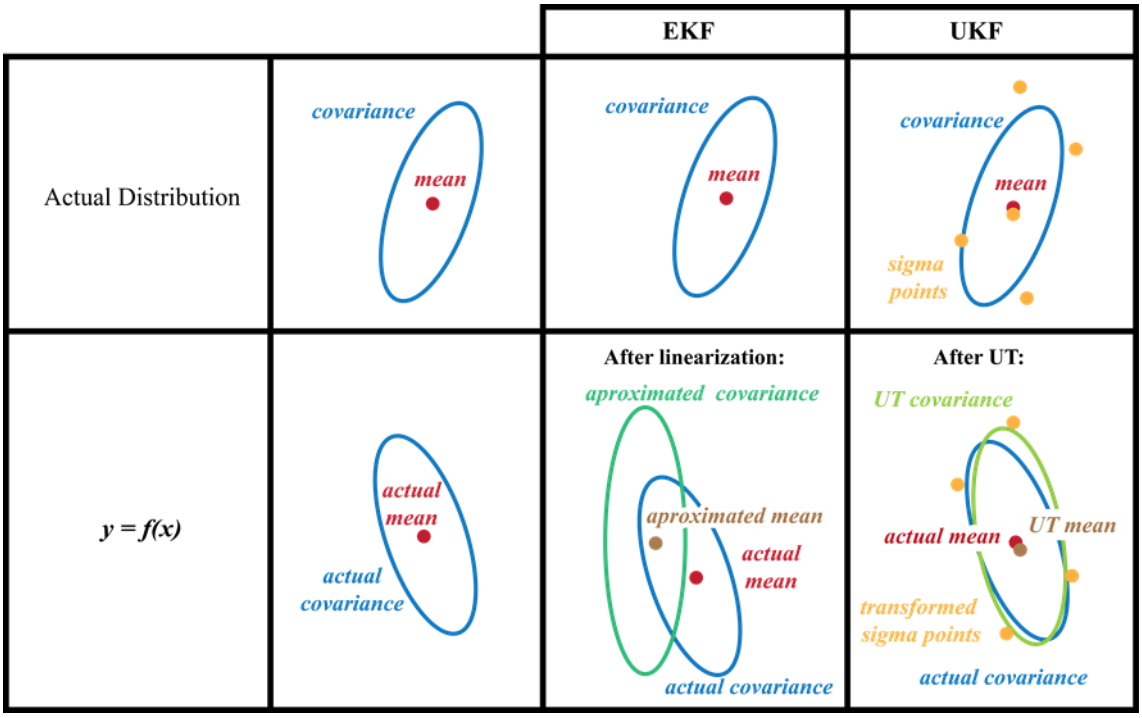

Fig. 4 Simplified comparison between the EKF and UKF transform

nominate some to facilitate future choices.

Starting with the extended Kalman filter (EKF), the main advantage is the algorithm simplicity, because the linearization is a common procedure, i.e., very easy to understand and apply. When dealing with systems quasi-linear, it has the ability to maintain the computational efficiency of the classic Kalman filter, it presents good performance (the analyses error covariance is updated at each time-step). And although the popularity of the EKF, the algorithm presents several limitations, especially when dealing with highly nonlinear systems. This happens because the EKF is based on local approximations, so when applied to highly nonlinear systems, the estimate suffers serious problems, as unstable and quickly divergent behaviours. Also, after the nonlinear transformations, the filter acts as an *ad hoc* estimator that only approximates the optimality of Bayes rule by linearization, which may limit the overall performance. The EKF is restricted to Gaussian noise systems (it presents better results than non-Gaussian). This Gaussian restriction prevents the use of hierarchical models or other models that are described by high non-Gaussian distributions. In addition, the EKF requires the measurement and dynamic models functions to be differentiable, which may be a limitation in some applications. In some cases, the computation of the Jacobian matrix is simply impossible to compute, which renders to impossible the use of the EKF. Even when the matrix exists and can be computed, the actual computation and programming can be quite error-prone, hard to debug, and time-consuming, a huge limitation for time-critical applications (as aircraft tracking and navigation) where the algorithm shall be fast and efficient.

Regarding the Unscented Kalman filter (UKF), one advantage over the previous EKF is the use of sigma points when approximating the nonlinearity (since it applies the unscented transform), so it can capture higher-order moments (caused by the nonlinear transformation) better than the

Taylor series approximations (EKF, which is based on linear approximation at a single point), as Figure 4 shows. Unlike the EKF, the measurement and dynamic model function are not required to be differentiable, neither does it needs to compute the Jacobians matrices, which in some cases is an advantage. The mean estimate is exact for polynomials up to order 3, which contributes for better accuracy however, the covariance computation is only exact for polynomials up to first order.

Another advantage of the UKF is that it does not require the computation of any expected valued (in the closed-form), only requires the evaluations of the dynamic and measurement models. However, the UKF accuracy cannot be expected to be as good as it could be because, in the approximation area, the algorithm uses a fixed number of sigma points (2n+1). It requires slightly more computational operations (for example, when compared with the EKF). The UKF also requires the update of the covariance matrix at each time step, which contributes to more computational load. It is parameter-dependent, which means, the description of the parameters will have a direct impact on the final performance.

The CKF can be compared to UKF in the sense that avoids the linearization by using a set of points to predict the state vector and the covariance matrix. However, CKF has a strictly theoretical derivation based on Bayesian and spherical-radial cubature principles while UKF does not. A CKF advantage is that the set of points are acquired by integration, which leads to equal and positive values of all weights, while the UKF weights are easily negative when facing high dimensional systems, which reduce the filter accuracy and stability. Also similar to UKF, the CKF uses a fixed number of points when calculating the approximation area and updates the covariance matrix at each time step (which, can be translated as an increment on the computational complexity but also more accuracy).

Some of the notable advantages of the CKF are: it is derivative-free and it is the closest known approximation to the Bayesian filter that could be designed for nonlinear Gaussian filtering problem (due to the maximum entropy principle) when given the second-order statistic of the state and innovations process. Unfortunately, in most of the cases, the posterior density is intractable and this can happen because when the CKF method is applied to multi-dimensional systems, it is necessary to compute multi-dimensional integrals and even after they are computed it may be difficult to propagate the posterior density through subsequent time steps. Because there is no guarantee that the new posterior density will remain closed with finite summary statistic expressed in terms of (quasi-) moments. As can be seen, the CKF is committed to digital hardware and due to numerical imprecision, it may exhibit a divergent behaviour or even a complete failure.

Regarding the EnKF, the main advantage is the simplicity of the method, because while the EKF requires at each time step the linearization of the state model and measurement model, the EnKF uses the nonlinear model directly, so there is no need to linearize or compute the Jacobian matrices. Although EnKF does not require the linearization of the model, it still computes the perturbations based on the assumption that they evolve linearly, which means there is an assumption of Gaussian perturbations, i.e., perturbations completely represented by their mean and covariance that remain Gaussian within the assimilation window. This characteristic has a negative impact on the EnKF performance with the nonlinear growth of the perturbations because if the model is very high-nonlinear or if the analysis time window is too long, the Gaussian assumption is no more adequate. An advantage of the EnKF is the dynamic propagation and the ensemble members transformation because those are quite easy to implement and computationally inexpensive. Furthermore, given an ensemble size, the EnKF has the ability to provide an optimal initial ensemble perturbation (it represents the analysis error covariance). The EnKF presents

better results than the EKF if the number of ensemble members is larger than the size of the model.

For most of the dimensional problems, an ensemble size between 50 to 100 is sufficient however, the use of a limited number can introduce sampling errors on the background error covariance. This limitation is more apparent at long distances. One advantage of the EnKF method relative to the other sequential Monte-Carlo methods is the use of shifting instead of reweighting the points (update equations), which allows the algorithm to remain stable in high-dimensional problems. It is important to have into account that even in a perfect model and because of the nonlinearities, the EnKF and also the other methods can drift away from the real solution due to an underestimation of the forecast error covariance. Particularly, the EnKF because it is affected directly by an underestimated representation of the background error covariance, especially outside of the subspace defined by the ensemble forecasts.

9. Conclusions

Most of the filtering methods that are presented in this paper have optimal states estimation only when the mathematic model is linear and accurately known, and the system and measurement noises are white and Gaussian. However, in practical applications, these requirements are not easily satisfied, because most of the times, they are inaccurate or incomplete; the covariance matrix is often ill-conditioned and the non-linearities in the equations that describe the physical system (vehicle dynamics) are imprecise. It is also needed to consider random environmental disturbances, that are not easily calculated. Additionally, there are instrumental inaccuracies that are also hard to modulate and have a direct impact on the estimation because the system model and its associated noise are obtained based on the prior knowledge, which has some errors when compared with the true values. The path to achieving better state estimations includes the search for more accurate adaptive solutions. Moreover, the adaptative method needs to be able to balance precision with computational complexity and processing time because nowadays the methods with better precision are also more computationally expensive, which can be translated in more processing time and for time-critical applications precise and fast methods are a demand in the everyday situation. This survey proposes some solutions to improve the overall performance of widely used methods:

1. EKF method only uses the last estimated state vector to compute an improved linearization. If this process is done based on all available data, the results will be more accurate. However, it is important to find the right balance between accuracy and computational complexity (which, in most cases, is translated as more computational time – characteristic to avoid in time-critical aerospace systems).
2. Design an algorithm that is part recursive and part non-recursive will allow more flexibility in the fundamental trade-off between accuracy and computational complexity.
3. Prove a generalization of the Pitman-Koopman-Darmois theorem for smooth non-linear filtering problems with non-zero process noise, as well as for finite but growing dimensional problems and for problems with less smoothness and less regularity.

The Pitman-Koopman-Darmois theorem says that when we have independent and identically distributed (IID) sampling from a statistical model, all distributions in the model have the same support which does not depend on the parameter. And if all distributions in the model are continuous, then there is a sufficient statistic whose dimension does not depend on the parameter, if and only if, the statistical model is an exponential family of distributions. It is important to note

that the theorem does not cover discrete cases, and most applications within the exponential family theory are discrete. Nowadays, there are some additional theorems that cover discrete distributions, but they need extra conditions so the theorem can be proved.

4. The proposal densities, from the exponential family and exponential sums, should be used rather than the Gaussian proposal;

5. Some ideas should be borrowed from physics to solve engineering applications, as relating to Boltzmann's entropy and the second law of thermodynamics to the evolution of uncertainty in nonlinear filters. Recently, Mitter and Newton (2005) give some steps forward on this topic, but still, there is space for more improvements.

The topic of nonlinear state estimation is still growing, and a better solution may pass by a robust adaptive filter. Xia *et al.* (1994) already formulated an adaptive solution for linear Kalman filters, however, a nonlinear formulation is more complex, where the variability of environmental disturbances, the uncertainties of the vehicle dynamics and the instrumental inaccuracies has to be taken into account and besides all of these, it is necessary to find the right balance between the processing time and computational complexity. This way, it is possible to have an optimal filter with more accurate and precise results.

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