Advances in Aircraft and Spacecraft Science, *Vol. 7, No. 1 (2020) 19-40*
DOI: https://doi.org/10.12989/aas.2020.7.1.019

Analysis of laminated and sandwich spherical shells using a new higher-order theory

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(Received February 23, 2019, Revised May 2, 2019, Accepted May 30, 2019)

Abstract. In the present study, a fifth-order shear and normal deformation theory using a polynomial function in the displacement field is developed and employed for the static analysis of laminated composite and sandwich simply supported spherical shells subjected to sinusoidal load. The significant feature of the present theory is that it considers the effect of transverse normal strain in the displacement field which is eliminated in classical, first-order and many higher-order shell theories, while predicting the bending behavior of the shell. The present theory satisfies the zero transverse shear stress conditions at the top and bottom surfaces of the shell. The governing equations and boundary conditions are derived using the principle of virtual work. To solve the governing equations, the Navier solution procedure is employed. The obtained results are compared with Reddy's and Mindlin's theory for the validation of the present theory.

Keywords: shear deformation; transverse normal strain; spherical shell; laminated composite; sandwich; static analysis

1. Introduction

Advanced composite materials are becoming more popular and essential class of material for modern technological development all over the world. Due to attractive properties of composite materials, like high stiffness-to-weight ratio, strength-to-weight, light in weight, high specific strength and high abrasion resistance. The use of laminated composite and sandwich shell panels of cylindrical and spherical surfaces being increased in aerospace engineering in the last decades. Some of the typical applications of shell structures in aerospace engineering are fuselage, silos, pipelines, space station for habitation modules, rocket stages, energy absorbers, junction elements, etc. It requires to carry out the analysis of shells for various sensitive modes of failure like bending, buckling, and vibration. Therefore, analysis of spherical shells subjected to transverse loads becomes an active area of research.

The well-known classical shell theory (CST) available in the literature does not suitable for the analysis of thick laminated composite shells due to neglect of shear deformation. Hence, Mindlin (1951) has developed the first-order shear deformation theory (FSDT) in which the transverse

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http://www.techno-press.org/?journal=aas&subpage=7 ISSN: 2287-528X (Print), 2287-5271 (Online)

shear stresses are considered but found constant through the thickness, i.e. it does not satisfy the shear stress boundary conditions at top and bottom surfaces of the shell. These limitations of CST and FSDT force the researchers to develop higher-order shear deformation theories (HSDTs) for the analysis of laminated composite spherical shells by using different analytical and numerical methods. Several review articles are available in the literature which cover the various studies on beams, plates and shells such as Sayyad and Ghugal (2017, 2015), Qatu (2002a, 2002b), Asadi *et al.* (2012), etc. Reddy (1984a, b), has developed a third-order shear deformation theory for the bending and free vibration analysis of laminated composite shells and plates. This theory is further extended by the many researchers for various problems of beams, plates and shells. Liew and Lim (2011) have presented a new HSDT for the free vibration analysis of doubly curved shallow shells. Lee and Reddy (2004) have applied HSDT for the vibration suppression analysis of laminated shells. Carrera *et al.* (2010, 2013, 2017, 2008, 2009) have presented different studies on analysis of laminated and sandwich shells subjected to different types of loads using Carrera's unified solution (CUF). Neves *et al.* (2013) have proposed the free vibration analysis of functionally graded shells by HSDT based on radial basis function. Mantari and Soares (2012a, 2012b, 2014) have introduced a new generalized HSDT for the bending and free vibration analysis of multilayered and functionally graded shells. Matsunaga (2007) has applied a HSDT for the free vibration and buckling analysis of cross ply laminated shells considering the effects of transverse shear and normal deformations. Oktem *et al.* (2012) have presented a static response of functionally graded doubly curved shells using HSDT. Finite element formulations for the static and free vibration analysis of spherical shells are presented by Pradyumana and Bandyopadhyay (2008). Effect of twenty one types of boundary conditions on the vibration of the doubly curved shallow shell is presented by Qatu and Asadi (2012). Tornabene (2011a, 2011b, 2011c, 2009, 2012) have applied a generalized differential quadrature method (GDQ) for the free vibration analysis of doubly curved shells of revolution with and without resting on elastic foundation using FSDT. Further GDQ method is extended by Tornabene *et al.* (2013, 2016) for the static analysis of laminated doubly curved shells subjected to different loading conditions. The static behavior of doubly curved shells is investigated by Tornabene *et al.* (2014) using carrera's unified approach (CUF), differential geometry and GDQ method. Based on the CUF, Tornabene *et al.* (2013, 2015) also presented layerwise theories for free vibration of a doubly curved shell and panels. Viola *et al.* (2013) have developed 2D higher-order shear deformation theory for the static analysis of doubly curved shells and panels. Recently, Sayyad and Ghugal (2019) have presented static and free vibration analysis of laminated and sandwich spherical shells using various higher-order shell theories. Semi-analytical solutions using Navier's technique are obtained.

The present study is based on some of the important observations and recommendations of the Carrera. The objectives of the present study are listed below:

1) It is recommended by Carrera that to predict accurate bending behaviour of thick laminated shells, effect of transverse normal strain cannot be neglected. Refinement in classical theories should be done by taking the effects of transverse shear and normal deformations into consideration. Therefore, the present theory considers the effects of both transverse shear and normal deformations.

2) It is also suggested by Carrera that the expansion of polynomial shape function up to third order is not sufficient to capture the bending behaviour of thick composite shells subjected to mechanical and environmental loads. Therefore, to capture the accurate bending behaviour of laminated and sandwich spherical shells, the present polynomial shape function is expanded up to fifth-order.

3) In the present study, all displacements and stresses in isotropic, laminated composite and sandwich spherical shells are presented in one place which will be more suitable for future researchers and readers.

4) Transverse shear stresses are calculated using three-dimensional equations of equilibrium.

To achieve the above objectives, the present theory, formulated using the principle of virtual work. Nine variationally consistent governing differential equations are derived. Closed-form solutions are obtained using Navier's solution technique. Finally, non-dimensional displacements and stresses are obtained with the help of a computer program in MATLAB 8.5 (R2015a) .

2. Methodology

2.1 Spherical shell under consideration

A laminated composite spherical shell with width ***a*** along *x*-direction, breadth ***b*** in *y*-direction, thickness ***h*** in z-direction and radii of curvature R_1 and R_2 is considered for the present theory. The geometry and coordinate system of a spherical shell element is shown in Fig. 1. Shell has ***N*** number of layers made up of orthotropic material and assumed to be perfectly bonded together. Layers are stacked symmetrically or anti-symmetrically. Only cross-ply laminated spherical shell is considered in the present study. The shell is subjected to transverse mechanical load sinusoidal in nature.

2.2 Displacement field

Displacement field of the present theory is selected based on certain assumptions. 1) The present theory is displacement based shear deformation theory 2) The in-plane displacements (*u* and *v*) includes extension, bending and shear components. 3) The transverse displacement (*w*) considers the effect of shear and normal deformations 4) polynomial type shape functions are used to consider the effects of transverse shear and normal deformations. 5) Three dimensional Hooke's law is used to determine stresses. The displacement field assumed for the present fifth-order shear and normal deformation theory is as follows.

$$\begin{aligned} u(x,y,z) &= \left(1+\frac{z}{R_1}\right)u_0(x,y) - z\frac{\partial w_0}{\partial x} + \left(z-\frac{4z^3}{3h^2}\right)\phi_x(x,y) + \left(z-\frac{16z^5}{5h^4}\right)\psi_x(x,y) \\ v(x,y,z) &= \left(1+\frac{z}{R_2}\right)v_0(x,y) - z\frac{\partial w_0}{\partial y} + \left(z-\frac{4z^3}{3h^2}\right)\phi_y(x,y) + \left(z-\frac{16z^5}{5h^4}\right)\psi_y(x,y) \\ w(x,y,z) &= w_0(x,y) + \left(1-\frac{4z^2}{h^2}\right)\phi_z(x,y) + \left(1-\frac{16z^4}{h^4}\right)\psi_z(x,y) \end{aligned} \tag{1}$$

where, *u, v, w* are the displacements in *x, y, z* directions, respectively. u_0, v_0, w_0 are the mid-plane displacements in *x, y, z* directions, respectively. ϕ_x, ϕ_y, ϕ_z, ψ_x, ψ_y, ψ_z are the shear slopes. Therefore, the present theory has nine unknowns.

2.3 Strain-displacement relationship

Using the linear theory of elasticity, the normal and shear strains associated with the present displacement field are obtained.

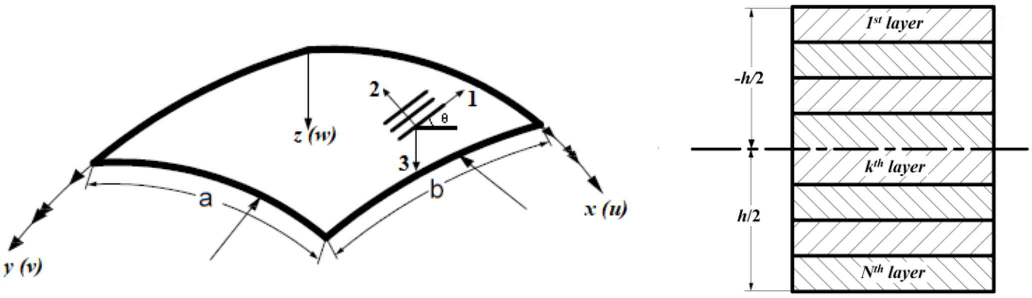


Fig. 1 Geometry and layer numbering system of laminated spherical shell

$$\begin{aligned}
\varepsilon_x &= \left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R_1}\right) - z\frac{\partial^2 w_0}{\partial x^2} + f_1(z)\frac{\partial \phi_x}{\partial x} + f_2(z)\frac{\partial \psi_x}{\partial x} + \frac{f_1'(z)}{R_1}\phi_z + \frac{f_2'(z)}{R_1}\psi_z \\
\varepsilon_y &= \left(\frac{\partial v_0}{\partial y} + \frac{w_0}{R_2}\right) - z\frac{\partial^2 w_0}{\partial y^2} + f_1(z)\frac{\partial \phi_y}{\partial y} + f_2(z)\frac{\partial \psi_y}{\partial y} + \frac{f_1'(z)}{R_2}\phi_z + \frac{f_2'(z)}{R_2}\psi_z \\
\varepsilon_z &= f_1''(z)\phi_z + f_2''(z)\psi_z \\
\gamma_{xy} &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w_0}{\partial x \partial y} + f_1(z)\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + f_2(z)\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) \\
\gamma_{xz} &= f_1'(z)\phi_x + f_2'(z)\psi_x + f_1'(z)\frac{\partial \phi_z}{\partial x} + f_2'(z)\frac{\partial \psi_z}{\partial x} \\
\gamma_{yz} &= f_1'(z)\phi_y + f_2'(z)\psi_y + f_1'(z)\frac{\partial \phi_z}{\partial y} + f_2'(z)\frac{\partial \psi_z}{\partial y}
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
& f_1(z) = \left(z - \frac{4z^3}{3h^2}\right),\ f_2(z) = \left(z - \frac{16z^5}{5h^4}\right),\ f_1'(z) = \left(1 - \frac{4z^2}{h^2}\right), \\
& f_2'(z) = \left(1 - \frac{16z^4}{h^4}\right),\ f_1''(z) = -\frac{8z}{h^2},\ f_2''(z) = -\frac{64z^3}{h^4}
\end{aligned} \tag{3}$$

The prime $(\ )'$ indicates differentiation of the shape function with respect to coordinate z.

2.4 Stress-strain relationship

Using the generalized Hooke's law, the stress-strain relationship for the k^{th} layer of laminated composite shell can be written as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^k = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{44} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^k \tag{4}$$

$$\bar{\sigma} = \bar{Q}_{ij}\bar{\varepsilon}$$

where, $\bar{\sigma}$ represents stress vector, $\bar{\varepsilon}$ represents strain vector and $\overline{Q_{ij}}$ are the stiffness coefficients. 1, 2, 3 are the material reference axes, and x, y, z are the laminate reference axes (see Fig. 1). θ is the angle made by the fibres with respect to positive x axis. The principal material axes of lamina (1, 2, 3) will coincide with the reference axes of the laminate (x, y, z) in case of orthotropic cross-ply laminates. It is therefore necessary to transform the constitutive relations from lamina fiber axes to laminate reference axes.

2.5 Principle of virtual work

The principle of virtual work is used to derive nine variationally consistent governing differential equations associated with the present fifth-order shear and normal deformation theory. An analytical form of the principle of virtual work is as follows:

$$\int_0^a\int_0^b\int_{-h/2}^{+h/2}\left(\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\sigma_z\delta\varepsilon_z+\tau_{xy}\delta\gamma_{xy}+\tau_{xz}\delta\gamma_{xz}+\tau_{yz}\delta\gamma_{yz}\right)dz\,dy\,dx-\int_0^a\int_0^b q(x,y)\,\delta w\,dy\,dx=0 \tag{5}$$

where δ is the variational operator. Using fundamental lemma of calculus, the following governing equations are derived for the present theory.

$$\begin{aligned}
&\boldsymbol{\delta u_0}:\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0\\
&\boldsymbol{\delta v_0}:\frac{\partial N_y}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0\\
&\boldsymbol{\delta w_0}:\frac{\partial^2 M_x^b}{\partial x^2}+\frac{\partial^2 M_y^b}{\partial y^2}+2\frac{\partial^2 M_{xy}^b}{\partial x\,\partial y}-\frac{N_x}{R_1}-\frac{N_y}{R_2}+q=0\\
&\boldsymbol{\delta\phi_x}:\frac{\partial M_x^{S1}}{\partial x}+\frac{\partial M_{xy}^{S1}}{\partial y}-Q_x^{S1}=0\\
&\boldsymbol{\delta\psi_x}:\frac{\partial M_x^{S2}}{\partial x}+\frac{\partial M_{xy}^{S2}}{\partial y}-Q_x^{S2}=0\\
&\boldsymbol{\delta\phi_{y:}}:\frac{\partial M_y^{S1}}{\partial y}+\frac{\partial M_{xy}^{S1}}{\partial x}-Q_y^{S1}=0\\
&\boldsymbol{\delta\psi_y}:\frac{\partial M_y^{S2}}{\partial y}+\frac{\partial M_{xy}^{S2}}{\partial x}-Q_y^{S2}=0\\
&\boldsymbol{\delta\phi_z}:\frac{\partial Q_x^{S1}}{\partial x}+\frac{\partial Q_y^{S1}}{\partial y}-\frac{S_{xs1}}{R_1}-\frac{S_{ys1}}{R_2}-S^{S1}=0\\
&\boldsymbol{\delta\psi_z}:\frac{\partial Q_x^{S2}}{\partial x}+\frac{\partial Q_y^{S2}}{\partial y}-\frac{S_{xs2}}{R_1}-\frac{S_{ys2}}{R_2}-S^{S2}=0
\end{aligned} \tag{6}$$

where, expressions for stress resultants can be derived from following relations.

$$\begin{aligned}
&\left(N_x,N_y,N_{xy},M_x^b,M_y^b,M_{xy}^b\right)=\int_{-h/2}^{h/2}\left[\sigma_x,\sigma_y,\tau_{xy},z\sigma_x,z\sigma_y,z\tau_{xy}\right]dz\\
&\left(M_x^{S_1},M_y^{S_1},M_{xy}^{S_1},M_x^{S_2},M_y^{S_2}M_{xy}^{S_2}\right)=\int_{-h/2}^{h/2}\left\{\left[f_1(z)\left(\sigma_x,\sigma_y,\tau_{xy}\right)\right],\left[f_2(z)\left(\sigma_x,\sigma_y,\tau_{xy}\right)\right]\right\}dz
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\left(Q_x^{S_1}, Q_y^{S_1}, Q_x^{S_2}, Q_y^{S_2}\right) = \int_{-h/2}^{h/2} \left\{ \left[f_1'(z)\left(\tau_{xz}, \tau_{yz}\right)\right], \left[f_2'(z)\left(\tau_{xz}, \tau_{yz}\right)\right] \right\} dz \\
&\left(S^{S_1}, S^{S_2}\right) = \int_{-h/2}^{h/2} \left\{ \sigma_z \left[f_1''(z), f_2''(z) \right] \right\} dz \\
&\left(S_{xS1}, S_{xS2}\right) = \int_{-h/2}^{h/2} \left\{ \sigma_x \left[f_1'(z), f_2'(z) \right] \right\} dz \\
&\left(S_{yS1}, S_{yS2}\right) = \int_{-h/2}^{h/2} \left\{ \sigma_y \left[f_1'(z), f_2'(z) \right] \right\} dz
\end{aligned} \tag{7}$$

Set of governing equations stated in Eq. (6) can be written in-terms of nine unknown variables involved in the displacement field as follows:

$$\begin{aligned}
\delta u_0 :\ & A_{11}\frac{\partial^2 u_0}{\partial x^2} + \frac{A_{11}}{R_1}\frac{\partial w_0}{\partial x} - B_{11}\frac{\partial^3 w_0}{\partial x^3} + A_{S1_{11}}\frac{\partial^2 \phi_x}{\partial x^2} + A_{S2_{11}}\frac{\partial^2 \psi_x}{\partial x^2} + A_{12}\frac{\partial^2 v_0}{\partial x \partial y} + \frac{A_{12}}{R_2}\frac{\partial w_0}{\partial x} - B_{12}\frac{\partial^2 w_0}{\partial x\,\partial y^2} \\
& + A_{S1_{12}}\frac{\partial^2 \phi_y}{\partial x \partial y} + A_{S2_{12}}\frac{\partial^2 \psi_y}{\partial x \partial y} + E_{13}\frac{\partial \phi_z}{\partial x} + F_{13}\frac{\partial \psi_z}{\partial x} + A_{66}\frac{\partial^2 u_0}{\partial y^2} + A_{66}\frac{\partial^2 v_0}{\partial x \partial y} - 2B_{66}\frac{\partial^3 w_0}{\partial x \partial y^2} + A_{S1_{66}}\frac{\partial^2 \phi_x}{\partial y^2} \\
& + A_{S1_{66}}\frac{\partial^2 \phi_y}{\partial x \partial y} + A_{S2_{66}}\frac{\partial^2 \psi_x}{\partial y^2} + A_{S2_{66}}\frac{\partial^2 \psi_y}{\partial x \partial y} + \left(\frac{Q_{111}}{R_1} + \frac{Q_{112}}{R_2}\right)\frac{\partial \phi_z}{\partial x} + \left(\frac{Q_{211}}{R_1} + \frac{Q_{212}}{R_2}\right)\frac{\partial \psi_z}{\partial x} = 0
\end{aligned} \tag{8}$$

$$\begin{aligned}
\delta v_0 :\ & A_{12}\frac{\partial^2 u_0}{\partial x \partial y} + \frac{A_{12}}{R_1}\frac{\partial w_0}{\partial y} - B_{12}\frac{\partial^3 w_0}{\partial x^2 \partial y} + A_{S1_{12}}\frac{\partial^2 \phi_x}{\partial x \partial y} + A_{S2_{12}}\frac{\partial^2 \psi_x}{\partial x \partial y} + A_{22}\frac{\partial^2 v_0}{\partial y^2} + \frac{A_{22}}{R_2}\frac{\partial w_0}{\partial y} - B_{22}\frac{\partial^3 w_0}{\partial y^3} \\
& + A_{S1_{22}}\frac{\partial^2 \phi_y}{\partial y^2} + A_{S2_{22}}\frac{\partial^2 \psi_y}{\partial y^2} + E_{23}\frac{\partial \phi_z}{\partial y} + F_{23}\frac{\partial \psi_z}{\partial y} + A_{66}\frac{\partial^2 u_0}{\partial x \partial y} + A_{66}\frac{\partial^2 v_0}{\partial x^2} - 2B_{66}\frac{\partial^3 w_0}{\partial x^2 \partial y} + A_{S1_{66}}\frac{\partial^2 \phi_x}{\partial x \partial y} \\
& + A_{S1_{66}}\frac{\partial^2 \phi_y}{\partial x^2} + A_{S2_{66}}\frac{\partial^2 \psi_x}{\partial x \partial y} + A_{S2_{66}}\frac{\partial^2 \psi_y}{\partial x^2} + \left(\frac{Q_{112}}{R_1} + \frac{Q_{122}}{R_2}\right)\frac{\partial \phi_z}{\partial y} + \left(\frac{Q_{212}}{R_1} + \frac{Q_{222}}{R_2}\right)\frac{\partial \psi_z}{\partial y} = 0
\end{aligned} \tag{9}$$

$$\begin{aligned}
\delta w_0 :\ & B_{11}\left(\frac{\partial^3 u_0}{\partial x^3} + \frac{1}{R_1}\frac{\partial^2 w_0}{\partial x^2}\right) - D_{11}\frac{\partial^4 w_0}{\partial x^4} + B_{S1_{11}}\frac{\partial^3 \phi_x}{\partial x^3} + B_{S2_{11}}\frac{\partial^3 \psi_x}{\partial x^3} + B_{12}\left(\frac{\partial^3 v_0}{\partial x^2 \partial y} + \frac{1}{R_2}\frac{\partial^2 w_0}{\partial x^2}\right) \\
& - D_{12}\frac{\partial^3 w_0}{\partial x^2\,\partial y^2} + B_{S1_{12}}\left(\frac{\partial^3 \phi_y}{\partial x^2 \partial y} + \frac{\partial^3 \psi_y}{\partial x^2 \partial y}\right) + J_{13}\frac{\partial^3 \phi_z}{\partial x^3} + K_{13}\frac{\partial^3 \psi_z}{\partial x^3} + B_{12}\left(\frac{\partial^3 u_0}{\partial x \partial y^2} + \frac{1}{R_1}\frac{\partial^2 w_0}{\partial y^2}\right) \\
& - D_{12}\frac{\partial^3 w_0}{\partial x^2\,\partial y^2} + B_{S1_{12}}\left(\frac{\partial^3 \phi_x}{\partial x \partial y^2} + \frac{\partial^3 \psi_x}{\partial x \partial y^2}\right) + B_{22}\left(\frac{\partial^3 v_0}{\partial y^3} + \frac{1}{R_2}\frac{\partial^2 w_0}{\partial y^2}\right) - D_{22}\frac{\partial^4 w_0}{\partial y^4} + B_{S1_{22}}\frac{\partial^3 \phi_y}{\partial y^3} \\
& + B_{S2_{22}}\frac{\partial^3 \psi_y}{\partial y^3} + J_{23}\frac{\partial^2 \phi_z}{\partial y^2} + K_{23}\frac{\partial^2 \psi_z}{\partial y^2} + 2B_{66}\left(\frac{\partial^3 u_0}{\partial x\,\partial y^2} + \frac{\partial^3 v_0}{\partial x^2 \partial y}\right) - 4D_{66}\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + 2B_{S1_{66}}\left(\frac{\partial^3 \phi_x}{\partial x\,\partial y^2} + \frac{\partial^3 \phi_y}{\partial x^2 \partial y}\right) \\
& + 2B_{S2_{66}}\left(\frac{\partial^3 \psi_x}{\partial x\,\partial y^2} + \frac{\partial^3 \psi_y}{\partial x^2 \partial y}\right) - \frac{A_{11}}{R_1}\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R_1}\right) + \frac{B_{11}}{R_1}\frac{\partial^2 w_0}{\partial x^2} - \frac{A_{S1_{11}}}{R_1}\frac{\partial \phi_x}{\partial x} - \frac{A_{S2_{11}}}{R_1}\frac{\partial \psi_x}{\partial x} - \frac{A_{12}}{R_1}\left(\frac{\partial v_0}{\partial y} + \frac{w_0}{R_2}\right) \\
& + \frac{B_{12}}{R_1}\frac{\partial^2 w_0}{\partial y^2} - \frac{A_{S1_{12}}}{R_1}\frac{\partial \phi_y}{\partial y} - \frac{A_{S2_{12}}}{R_1}\frac{\partial \psi_y}{\partial y} - \frac{E_{13}}{R_1}\phi_z - \frac{F_{13}}{R_1}\psi_z - \frac{A_{12}}{R_2}\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R_1}\right) + \frac{B_{12}}{R_2}\frac{\partial^2 w_0}{\partial x^2} - \frac{A_{S1_{12}}}{R_2}\frac{\partial \phi_x}{\partial x} \\
& - \frac{A_{S2_{12}}}{R_2}\frac{\partial \psi_x}{\partial x} - \frac{A_{22}}{R_2}\left(\frac{\partial v_0}{\partial y} + \frac{w_0}{R_2}\right) + \frac{B_{22}}{R_2}\frac{\partial^2 w_0}{\partial y^2} - \frac{A_{S1_{22}}}{R_2}\frac{\partial \phi_y}{\partial y} - \frac{A_{S2_{22}}}{R_2}\frac{\partial \psi_y}{\partial y} - \frac{E_{23}}{R_2}\phi_z - \frac{F_{23}}{R_2}\psi_z - \\
& \left(\frac{Q_{311}}{R_1} + \frac{Q_{312}}{R_2}\right)\frac{\partial^2 \phi_z}{\partial x^2} - \left(\frac{Q_{411}}{R_1} + \frac{Q_{412}}{R_2}\right)\frac{\partial^2 \psi_z}{\partial x^2} - \left(\frac{Q_{312}}{R_1} + \frac{Q_{322}}{R_2}\right)\frac{\partial^2 \phi_z}{\partial y^2} - \left(\frac{Q_{412}}{R_1} + \frac{Q_{422}}{R_2}\right)\frac{\partial^2 \psi_z}{\partial y^2} \\
& + \left(\frac{Q_{111}}{R_1^2} + \frac{Q_{112}}{R_1 R_2}\right)\phi_z + \left(\frac{Q_{211}}{R_1^2} + \frac{Q_{212}}{R_1 R_2}\right)\psi_z + \left(\frac{Q_{112}}{R_1 R_2} + \frac{Q_{122}}{R_2^2}\right)\phi_z + \left(\frac{Q_{212}}{R_1 R_2} + \frac{Q_{222}}{R_2^2}\right)\psi_z = -q
\end{aligned} \tag{10}$$

$$
\begin{aligned}
\delta\phi_x :\ & A_{S1_{11}}\frac{\partial^2 u_0}{\partial x^2}+\frac{A_{S1_{11}}}{R_1}\frac{\partial w_0}{\partial x}-B_{S1_{11}}\frac{\partial^3 w_0}{\partial x^3}+A_{SS1_{11}}\frac{\partial^2 \phi_x}{\partial x^2}+C_{11}\frac{\partial^2 \psi_x}{\partial x^2}+A_{S1_{12}}\frac{\partial^2 v_0}{\partial x\partial y}+\frac{A_{S1_{12}}}{R_2}\frac{\partial w_0}{\partial x}\\
&-B_{S1_{12}}\frac{\partial^3 w_0}{\partial x\partial y^2}+A_{SS1_{12}}\frac{\partial^2 \phi_y}{\partial x\partial y}+C_{12}\frac{\partial^2 \psi_y}{\partial x\partial y}+L_{1_{13}}\frac{\partial \phi_z}{\partial x}+L_{2_{13}}\frac{\partial \psi_z}{\partial x}+A_{S1_{66}}\frac{\partial^2 u_0}{\partial y^2}+A_{S1_{66}}\frac{\partial^2 v_0}{\partial x\partial y}\\
&-2B_{S1_{66}}\frac{\partial^3 w_0}{\partial x\partial y^2}+A_{SS1_{66}}\frac{\partial^2 \phi_x}{\partial y^2}+A_{SS1_{66}}\frac{\partial^2 \phi_y}{\partial x\partial y}+C_{66}\frac{\partial^2 \psi_x}{\partial y^2}+C_{66}\frac{\partial^2 \psi_y}{\partial x\partial y}-G_{55}\phi_x-I_{55}\psi_x\\
&-G_{55}\frac{\partial \phi_z}{\partial x}-I_{55}\frac{\partial \psi_z}{\partial x}+\left(\frac{Q_{511}}{R_1}+\frac{Q_{512}}{R_2}\right)\phi_z+\left(\frac{Q_{611}}{R_1}+\frac{Q_{612}}{R_2}\right)\psi_z=0
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\delta\psi_x :\ & A_{S2_{11}}\frac{\partial^2 u_0}{\partial x^2}+\frac{A_{S2_{11}}}{R_1}\frac{\partial w_0}{\partial x}-B_{S2_{11}}\frac{\partial^3 w_0}{\partial x^3}+C_{11}\frac{\partial^2 \phi_x}{\partial x^2}+A_{SS2_{11}}\frac{\partial^2 \psi_x}{\partial x^2}+A_{S2_{12}}\frac{\partial^2 v_0}{\partial x\partial y}+\frac{A_{S2_{12}}}{R_2}\frac{\partial w_0}{\partial x}\\
&-B_{S2_{12}}\frac{\partial^3 w_0}{\partial x\partial y^2}+C_{12}\frac{\partial^2 \phi_y}{\partial x\partial y}+A_{SS2_{12}}\frac{\partial^2 \psi_y}{\partial x\partial y}+M_{1_{13}}\frac{\partial \phi_z}{\partial x}+M_{2_{13}}\frac{\partial \psi_z}{\partial x}+A_{S2_{66}}\frac{\partial^2 u_0}{\partial y^2}+A_{S2_{66}}\frac{\partial^2 v_0}{\partial x\partial y}\\
&-2B_{S2_{66}}\frac{\partial^3 w_0}{\partial x\partial y^2}+C_{66}\frac{\partial^2 \phi_x}{\partial y^2}+C_{66}\frac{\partial^2 \phi_y}{\partial x\partial y}+A_{SS2_{66}}\frac{\partial^2 \psi_x}{\partial y^2}+A_{SS2_{66}}\frac{\partial^2 \psi_y}{\partial x\partial y}-I_{55}\phi_x-H_{55}\psi_x\\
&-I_{55}\frac{\partial \phi_z}{\partial x}-H_{55}\frac{\partial \psi_z}{\partial x}+\left(\frac{Q_{711}}{R_1}+\frac{Q_{712}}{R_2}\right)\phi_z+\left(\frac{Q_{811}}{R_1}+\frac{Q_{812}}{R_2}\right)\psi_z=0
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\delta\phi_y :\ & A_{S1_{12}}\frac{\partial^2 u_0}{\partial x\partial y}+\frac{A_{S1_{12}}}{R_1}\frac{\partial w_0}{\partial y}-B_{S1_{12}}\frac{\partial^3 w_0}{\partial x^2\partial y}+A_{SS1_{12}}\frac{\partial^2 \phi_x}{\partial x\partial y}+C_{12}\frac{\partial^2 \psi_x}{\partial x\partial y}+A_{S1_{22}}\frac{\partial^2 v_0}{\partial y^2}+\frac{A_{S1_{22}}}{R_2}\frac{\partial w_0}{\partial y}\\
&-B_{S1_{22}}\frac{\partial^3 w_0}{\partial y^3}+A_{SS1_{22}}\frac{\partial^2 \phi_y}{\partial y^2}+C_{22}\frac{\partial^2 \psi_y}{\partial y^2}+L_{1_{23}}\frac{\partial \phi_z}{\partial y}+L_{2_{23}}\frac{\partial \psi_z}{\partial y}+A_{S1_{66}}\frac{\partial^2 u_0}{\partial x\partial y}+A_{S1_{66}}\frac{\partial^2 v_0}{\partial x^2}\\
&-2B_{S1_{66}}\frac{\partial^3 w_0}{\partial x^2\partial y}+A_{SS1_{66}}\frac{\partial^2 \phi_x}{\partial x\partial y}+A_{SS1_{66}}\frac{\partial^2 \phi_y}{\partial x^2}+C_{66}\frac{\partial^2 \psi_x}{\partial x\partial y}+C_{66}\frac{\partial^2 \psi_y}{\partial x^2}-G_{44}\phi_y-I_{44}\psi_y\\
&-G_{44}\frac{\partial \phi_z}{\partial y}-I_{44}\frac{\partial \psi_z}{\partial y}+\left(\frac{Q_{512}}{R_1}+\frac{Q_{522}}{R_2}\right)\phi_z+\left(\frac{Q_{612}}{R_1}+\frac{Q_{622}}{R_2}\right)\psi_z=0
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\delta\psi_y :\ & A_{S2_{12}}\frac{\partial^2 u_0}{\partial x\partial y}+\frac{A_{S2_{12}}}{R_1}\frac{\partial w_0}{\partial y}-B_{S2_{12}}\frac{\partial^3 w_0}{\partial x^2\partial y}+C_{12}\frac{\partial^2 \phi_x}{\partial x\partial y}+A_{SS2_{12}}\frac{\partial^2 \psi_x}{\partial x\partial y}+A_{S2_{22}}\frac{\partial^2 v_0}{\partial y^2}+\frac{A_{S2_{22}}}{R_2}\frac{\partial w_0}{\partial y}\\
&-B_{S2_{22}}\frac{\partial^3 w_0}{\partial y^3}+C_{22}\frac{\partial^2 \phi_y}{\partial y^2}+A_{SS2_{22}}\frac{\partial^2 \psi_y}{\partial y^2}+M_{1_{23}}\frac{\partial \phi_z}{\partial y}+M_{2_{23}}\frac{\partial \psi_z}{\partial y}+A_{S2_{66}}\frac{\partial^2 u_0}{\partial x\partial y}+A_{S2_{66}}\frac{\partial^2 v_0}{\partial x^2}\\
&-2B_{S2_{66}}\frac{\partial^3 w_0}{\partial x^2\partial y}+C_{66}\frac{\partial^2 \phi_x}{\partial x\partial y}+C_{66}\frac{\partial^2 \phi_y}{\partial x^2}+A_{SS2_{66}}\frac{\partial^2 \psi_x}{\partial x\partial y}+A_{SS2_{66}}\frac{\partial^2 \psi_y}{\partial x^2}-I_{44}\phi_y-H_{44}\psi_y\\
&-I_{44}\frac{\partial \phi_z}{\partial y}-H_{44}\frac{\partial \psi_z}{\partial y}+\left(\frac{Q_{712}}{R_1}+\frac{Q_{422}}{R_2}\right)\phi_z+\left(\frac{Q_{812}}{R_1}+\frac{Q_{822}}{R_2}\right)\psi_z=0
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\delta\phi_z :\ & G_{55}\frac{\partial \phi_x}{\partial x}+I_{55}\frac{\partial \psi_x}{\partial x}+G_{55}\frac{\partial^2 \phi_z}{\partial x^2}+I_{55}\frac{\partial^2 \psi_z}{\partial x^2}+G_{44}\frac{\partial \phi_y}{\partial y}+I_{44}\frac{\partial \psi_y}{\partial y}+G_{44}\frac{\partial^2 \phi_z}{\partial y^2}+I_{44}\frac{\partial^2 \psi_z}{\partial y^2}\\
&-E_{13}\frac{\partial u_0}{\partial x}-\frac{E_{13}}{R_1}w_0+J_{13}\frac{\partial^2 w_0}{\partial x^2}-L_{1_{13}}\frac{\partial \phi_x}{\partial x}-M_{1_{13}}\frac{\partial \psi_x}{\partial x}-E_{23}\frac{\partial v_0}{\partial y}-\frac{E_{23}}{R_2}w_0+J_{23}\frac{\partial^2 w_0}{\partial y^2}-L_{1_{23}}\frac{\partial \phi_y}{\partial y}\\
&-M_{1_{23}}\frac{\partial \psi_y}{\partial y}+N_{1_{33}}\phi_z+N_{3_{33}}\psi_z-\left(\frac{Q_{1613}}{R_1}+\frac{Q_{1623}}{R_2}\right)\phi_z-\left(\frac{Q_{1813}}{R_1}+\frac{Q_{1823}}{R_2}\right)\psi_z+\frac{Q_{111}}{R_1}\left(\frac{\partial u_0}{\partial x}+\frac{w_0}{R_1}\right)
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&+\frac{Q_{311}}{R_1}\frac{\partial^2 w_0}{\partial x^2}+\frac{Q_{511}}{R_1}\frac{\partial \phi_x}{\partial x}+\frac{Q_{711}}{R_1}\frac{\partial \psi_x}{\partial x}-\frac{Q_{1311}}{R_1^2}\phi_z-\frac{Q_{1411}}{R_1^2}\psi_z+\left[\frac{Q_{112}}{R_1}\left(\frac{\partial v_0}{\partial y}+\frac{w_0}{R_2}\right)\right]+\frac{Q_{312}}{R_1}\frac{\partial^2 w_0}{\partial y^2}\\
&+\frac{Q_{512}}{R_1}\frac{\partial \phi_y}{\partial y}+\frac{Q_{712}}{R_1}\frac{\partial \psi_y}{\partial y}-\frac{Q_{1312}}{R_1R_2}\phi_z-\frac{Q_{1412}}{R_1R_2}\psi_z-\frac{Q_{1613}}{R_1}\phi_z-\frac{Q_{1713}}{R_1}\psi_z+\left[\frac{Q_{112}}{R_2}\left(\frac{\partial u_0}{\partial x}+\frac{w_0}{R_1}\right)\right]\\
&-\frac{Q_{312}}{R_2}\frac{\partial^2 w_0}{\partial x^2}+\frac{Q_{512}}{R_2}\frac{\partial \phi_x}{\partial x}+\frac{Q_{712}}{R_2}\frac{\partial \psi_x}{\partial x}-\frac{Q_{1312}}{R_1R_2}\phi_z-\frac{Q_{1412}}{R_1R_2}\psi_z+\left[\frac{Q_{122}}{R_2}\left(\frac{\partial v_0}{\partial y}+\frac{w_0}{R_2}\right)\right]-\frac{Q_{322}}{R_2}\frac{\partial^2 w_0}{\partial y^2}\\
&+\frac{Q_{522}}{R_2}\frac{\partial \phi_y}{\partial y}+\frac{Q_{722}}{R_2}\frac{\partial \psi_y}{\partial y}-\frac{Q_{1322}}{R_2^2}\phi_z-\frac{Q_{1422}}{R_2^2}\psi_z-\frac{Q_{1623}}{R_2}\phi_z-\frac{Q_{1723}}{R_2}\psi_z=0
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&\delta\psi_z: I_{55}\frac{\partial \phi_x}{\partial x}+H_{55}\frac{\partial \psi_x}{\partial x}+I_{55}\frac{\partial^2 \phi_z}{\partial x^2}+H_{55}\frac{\partial^2 \psi_z}{\partial x^2}+I_{44}\frac{\partial \phi_y}{\partial y}+H_{44}\frac{\partial \psi_y}{\partial y}+I_{44}\frac{\partial^2 \phi_z}{\partial y^2}+H_{44}\frac{\partial^2 \psi_z}{\partial y^2}\\
&-F_{13}\frac{\partial u_0}{\partial x}-\frac{F_{13}}{R_1}w_0+O_{13}\frac{\partial^2 w_0}{\partial x^2}-L_{2_{13}}\frac{\partial \phi_x}{\partial x}-M_{2_{13}}\frac{\partial \psi_x}{\partial x}-F_{23}\frac{\partial v_0}{\partial y}+\frac{F_{23}}{R_2}w_0+O_{23}\frac{\partial^2 w_0}{\partial y^2}-L_{2_{23}}\frac{\partial \phi_y}{\partial y}\\
&-M_{2_{23}}\frac{\partial \psi_y}{\partial y}+N_{3_{33}}\phi_z+N_{2_{33}}\psi_z-\left(\frac{Q_{1713}}{R_1}+\frac{Q_{1723}}{R_2}\right)\phi_z-\left(\frac{Q_{1913}}{R_1}+\frac{Q_{1923}}{R_2}\right)\psi_z+\frac{Q_{211}}{R_1}\left(\frac{\partial u_0}{\partial x}+\frac{w_0}{R_1}\right)\\
&+\frac{Q_{411}}{R_1}\frac{\partial^2 w_0}{\partial x^2}+\frac{Q_{611}}{R_1}\frac{\partial \phi_x}{\partial x}+\frac{Q_{811}}{R_1}\frac{\partial \psi_x}{\partial x}-\frac{Q_{1411}}{R_1^2}\phi_z-\frac{Q_{1511}}{R_1^2}\psi_z+\left[\frac{Q_{212}}{R_1}\left(\frac{\partial v_0}{\partial y}+\frac{w_0}{R_2}\right)\right]+\frac{Q_{412}}{R_1}\frac{\partial^2 w_0}{\partial y^2}\\
&+\frac{Q_{612}}{R_1}\frac{\partial \phi_y}{\partial y}+\frac{Q_{812}}{R_1}\frac{\partial \psi_y}{\partial y}-\frac{Q_{1412}}{R_1R_2}\phi_z-\frac{Q_{1512}}{R_1R_2}\psi_z-\frac{Q_{1813}}{R_1}\phi_z-\frac{Q_{1913}}{R_1}\psi_z+\left[\frac{Q_{212}}{R_2}\left(\frac{\partial u_0}{\partial x}+\frac{w_0}{R_1}\right)\right]\\
&+\frac{Q_{412}}{R_2}\frac{\partial^2 w_0}{\partial x^2}+\frac{Q_{612}}{R_2}\frac{\partial \phi_x}{\partial x}+\frac{Q_{812}}{R_2}\frac{\partial \psi_x}{\partial x}-\frac{Q_{1412}}{R_1R_2}\phi_z-\frac{Q_{1512}}{R_1R_2}\psi_z+\left[\frac{Q_{222}}{R_2}\left(\frac{\partial v_0}{\partial y}+\frac{w_0}{R_2}\right)\right]-\frac{Q_{422}}{R_2}\frac{\partial^2 w_0}{\partial y^2}\\
&+\frac{Q_{622}}{R_2}\frac{\partial \phi_y}{\partial y}+\frac{Q_{822}}{R_2}\frac{\partial \psi_y}{\partial y}-\frac{Q_{1422}}{R_2^2}\phi_z-\frac{Q_{1522}}{R_2^2}\psi_z-\frac{Q_{1823}}{R_2}\phi_z-\frac{Q_{1923}}{R_2}\psi_z=0
\end{aligned}
\tag{16}
$$

Along the edges $x=0$ and $x=a$,

Either $u_0=0$ or N_x is prescribed

Either $v_0=0$ or N_{xy} is prescribed

Either $w_0=0$ or M_x^b is prescribed

Either $\frac{\partial w_0}{\partial x}=0$ or M_{xy}^b is prescribed

Either $\phi_x=0$ or M_x^{S1} is prescribed (17)

Either $\psi_x=0$ or M_x^{S2} is prescribed

Either $\phi_y=0$ or M_{xy}^{S1} is prescribed

Either $\psi_y=0$ or M_{xy}^{S2} is prescribed

Either $\phi_z=0$ or Q_x^{S1} is prescribed

Either $\psi_z=0$ or Q_{xz}^{S2} is prescribed

Along the edges y =0 and $y=b$,

Either $u_0=0$ or N_{xy} is prescribed

Either $v_0=0$ or N_x is prescribed

Either $w_0=0$ or M_y^b is prescribed (18)

Either $\frac{\partial w_0}{\partial y}=0$ or M_{xy}^b is prescribed

Either $\phi_x=0$ or M_{xy}^{S1} is prescribed

$$\begin{gathered}\text{Either } \psi_x=0 \text{ or } M_{xy}^{S2} \text{ is prescribed}\\ \text{Either } \phi_y=0 \text{ or } M_y^{S1} \text{ is prescribed}\\ \text{Either } \psi_y=0 \text{ or } M_y^{S2} \text{ is prescribed}\\ \text{Either } \phi_z=0 \text{ or } Q_y^{S1} \text{ is prescribed}\\ \text{Either } \psi_z=0 \text{ or } Q_{yz}^{S2} \text{ is prescribed}\end{gathered} \tag{18}$$

where,

$$\begin{aligned}
&\left(A_{ij}, B_{ij}, D_{ij}, A_{S1_{ij}}, A_{S2_{ij}}, B_{S1_{ij}}, B_{S2_{ij}}\right)=Q_{ij}\int_{-h/2}^{h/2}\left[1.0,\, z, z^2, f_1(z), f_2(z), zf_1(z), zf_2(z)\right]dz\\
&\left(Q_{1_{ij}}, Q_{2_{ij}}, Q_{3_{ij}}, Q_{4_{ij}}\right)=Q_{ij}\int_{-h/2}^{h/2}\left[f_1'(z), f_2'(z), zf_1'(z), zf_2'(z)\right]dz\\
&\left(A_{SS1_{ij}}, A_{SS2_{ij}}, C_{ij}\right)=Q_{ij}\int_{-h/2}^{h/2}\left\{\left[f_1(z)\right]^2, \left[f_2(z)\right]^2, \left[f_1(z)\, f_2(z)\right]\right\}dz\\
&\left(Q_{5_{ij}}, Q_{6_{ij}}, Q_{7_{ij}}, Q_{8_{ij}}\right)=Q_{ij}\int_{-h/2}^{h/2}\left\{f_1(z)\left[f_1'(z), f_2'(z)\right], f_2(z)\left[f_1'(z), f_2'(z)\right]\right\}dz\\
&\left(Q_{13_{ij}}, Q_{15_{ij}}, Q_{14_{ij}}\right)=Q_{ij}\int_{-h/2}^{h/2}\left\{\left[f_1'(z)\right]^2, \left[f_2'(z)\right]^2, \left[f_1'(z)\, f_2'(z)\right]\right\}dz\\
&\left(G_{ij}, H_{ij}, I_{ij}\right)=Q_{ij}\int_{-h/2}^{h/2}\left\{\left[f_1'(z)\right]^2, \left[f_2'(z)\right]^2, \left[f_1'(z)\, f_2'(z)\right]\right\}dz\\
&\left(E_{ij}, F_{ij}, N_{1_{ij}}, N_{2_{ij}}, N_{3_{ij}}\right)=Q_{ij}\int_{-h/2}^{h/2}\left\{\left[f_1''(z)\right], \left[f_2''(z)\right], \left[f_1''(z)\right]^2, \left[f_2''(z)\right]^2, \left[f_1''(z)\, f_2'(z)\right]\right\}dz\\
&\left(Q_{16_{ij}}, Q_{17_{ij}}, Q_{18_{ij}}, Q_{19_{ij}}\right)=Q_{ij}\int_{-h/2}^{h/2}\left\{f_1'(z)\left[f_1''(z), f_2''(z)\right], f_2(z)\left[f_1''(z), f_2''(z)\right],\right\}dz\\
&\left(J_{ij}, L_{1_{ij}}, L_{2_{ij}}\right)=Q_{ij}\int_{-h/2}^{h/2} f_1''(z)\left[z, f_1(z), f_2(z)\right]dz\\
&\left(O_{ij}, M_{1_{ij}}, M_{2_{ij}}\right)=Q_{ij}\int_{-h/2}^{h/2} f_2''(z)\left[z, f_1(z), f_2(z)\right]dz
\end{aligned} \tag{19}$$

2.6 Navier type closed-form solution for spherical shell

According to Navier's solution procedure the following solution form for the unknown variables is assumed which satisfying the simply supported boundary conditions exactly.

$$\begin{aligned}
\left(u_0, \phi_x, \psi_x\right)&=\left(u_1, \phi_{x1}, \psi_{x1}\right)\cos\alpha x\,\sin\beta y\\
\left(v_0, \phi_y, \psi_y\right)&=\left(v_1, \phi_{y1}, \psi_{y1}\right)\sin\alpha x\,\cos\beta y\\
\left(w_0, \phi_z, \psi_z\right)&=\left(w_1, \phi_{z1}, \psi_{z1}\right)\sin\alpha x\,\sin\beta y
\end{aligned} \tag{20}$$

where, $\alpha=\pi/a$, $\beta=\pi/b$; u_1, ϕ_{x1}, ψ_{x1}, v_1, ϕ_{y1}, ψ_{y2}, w_1, ϕ_{z1}, ψ_{z1} are the unknown coefficients to be determined. The expression for the transverse sinusoidal load acting on the top surface of the laminated shell is expressed as follows

$$q(x,y)=q_0\sin\alpha x\sin\beta y \tag{21}$$

where, q_0 is the maximum intensity of the load. By using the foregoing expressions of displacement variables and transverse load in the governing equations (8)-(16), the following equation can be derived:

$$[K]\{\Delta\}=\{f\} \tag{22}$$

where $[K]$ is the stiffness matrix, $\{f\}$ is the force vector and $\{\Delta\}$ is the vector of unknowns. Elements of these matrices are mentioned in Appendix.

3. Numerical result and discussion

In this section, an efficacy and validity of the present theory are proved by applying it to the bending analysis of laminated composite and sandwich spherical shells made up of the following materials.

$$\begin{aligned}
&\text{MAT1:} \quad E = 210\ \text{GPa}, \mu = 0.3\\
&\text{MAT2:} \quad E_1 = E_2 = 0.04,\ E_3 = 0.5, G_{13} = G_{23} = 0.06, G_{12} = 0.016,\ \mu_{12} = \mu_{32} = \mu_{31} = 0.25\\
&\text{MAT3:} \quad \frac{E_1}{E_2} = 25, \frac{E_3}{E_2} = 1,\ \frac{G_{12}}{E_2} = \frac{G_{13}}{E_2} = 0.5, \frac{G_{23}}{E_2} = 0.2,\ \mu_{12} = \mu_{13} = \mu_{23} = 0.25
\end{aligned} \tag{23}$$

Table 1 Non-dimensional displacement and stresses in isotropic spherical shell under the sinusoidal mechanical load (*a/h*=10)

R/a	Theory	$\bar{u}$	$\bar{w}$	$\overline{\sigma_x}$	$\overline{\sigma_y}$	$\overline{\tau_{xy}}$	$\overline{\tau_{xz}}$	$\overline{\tau_{yz}}$
5	**Present**	**0.0530**	**2.7244**	**0.1484**	**0.1484**	**0.1280**	**0.2111**	**0.2111**
	Reddy (1984b)	0.0506	2.6472	0.1519	0.1519	0.1225	0.2130	0.2130
	Mindlin (1951)	0.0502	2.6262	0.1506	0.1506	0.1230	0.2137	0.2137
10	**Present**	**0.0515**	**2.9902**	**0.1794**	**0.1794**	**0.1244**	**0.2308**	**0.2308**
	Reddy (1984b)	0.0491	2.8756	0.1793	0.1793	0.1187	0.2313	0.2313
	Mindlin (1951)	0.0486	2.8508	0.1777	0.1777	0.1192	0.2319	0.2319
20	**Present**	**0.0493**	**3.0649**	**0.1924**	**0.1924**	**0.1192**	**0.2363**	**0.2363**
	Reddy (1984b)	0.0471	2.9389	0.1906	0.1906	0.1139	0.2364	0.2364
	Mindlin (1951)	0.0467	2.9131	0.1889	0.1889	0.1145	0.2370	0.2370
50	**Present**	**0.0476**	**3.0536**	**0.1989**	**0.1989**	**0.1151**	**0.2379**	**0.2379**
	Reddy (1984b)	0.0456	2.9572	0.1962	0.1962	0.1102	0.2379	0.2379
	Mindlin (1951)	0.0452	2.9310	0.1944	0.1944	0.1109	0.2385	0.2385
100	**Present**	**0.0470**	**3.0896**	**0.2008**	**0.2008**	**0.1135**	**0.2381**	**0.2381**
	Reddy (1984b)	0.0450	2.9598	0.1979	0.1979	0.1088	0.2381	0.2381
	Mindlin (1951)	0.0446	2.9336	0.1961	0.1961	0.1095	0.2387	0.2387
Plate	**Present**	**0.0441**	**2.9445**	**0.2001**	**0.2001**	**0.1065**	**0.2382**	**0.2382**
	Reddy (1984b)	0.0444	2.9607	0.1994	0.1994	0.1074	0.2382	0.2382
	Mindlin (1951)	0.0440	2.9345	0.1976	0.1976	0.1080	0.2387	0.2387
	Pagano (1970)	0.0443	2.9425	0.1988	0.1988	---	0.2383	0.2383

Table 2 Non-dimensional displacement and stresses in two layer (0°/90°) laminated spherical shell under the sinusoidal mechanical load (*a/h*=10)

R/a	Theory	$\bar{u}$	$\bar{w}$	$\overline{\sigma_x}$	$\overline{\sigma_y}$	$\overline{\tau_{xy}}$	$\overline{\tau_{xz}}$	$\overline{\tau_{yz}}$
5	**Present**	0.0151	1.1271	0.6461	0.0752	0.0695	0.0841	0.1391
	Reddy (1984b)	0.0151	1.1164	0.6530	0.0754	0.0694	0.0823	0.1382
	Mindlin (1951)	0.0148	1.1096	0.6262	0.0747	0.0686	0.0839	0.1402
10	**Present**	0.0126	1.2006	0.7076	0.0819	0.0631	0.1043	0.1335
	Reddy (1984b)	0.0126	1.1894	0.7130	0.0818	0.0630	0.1026	0.1324
	Mindlin (1951)	0.0122	1.1819	0.6835	0.0811	0.0622	0.1044	0.1344
20	**Present**	0.0109	1.2205	0.7292	0.0841	0.0586	0.1134	0.1283
	Reddy (1984b)	0.0110	1.2091	0.7337	0.0839	0.0586	0.1119	0.1270
	Mindlin (1951)	0.0106	1.2014	0.7033	0.0831	0.0577	0.1137	0.1290
50	**Present**	0.0099	1.2262	0.7385	0.0851	0.0556	0.1184	0.1244
	Reddy (1984b)	0.0100	1.2148	0.7424	0.0847	0.0555	0.1170	0.1230
	Mindlin (1951)	0.0096	1.2070	0.7116	0.0840	0.0546	0.1189	0.1250
100	**Present**	0.0096	1.2270	0.7410	0.0853	0.0545	0.1200	0.1230
	Reddy (1984b)	0.0096	1.2156	0.7447	0.0850	0.0545	0.1186	0.1216
	Mindlin (1951)	0.0092	1.2078	0.7138	0.0842	0.0536	0.1205	0.1235
Plate	**Present**	0.0091	1.2197	0.7398	0.0860	0.0531	0.1215	0.1215
	Reddy (1984b)	0.0092	1.2158	0.7466	0.0851	0.0534	0.1201	0.1201
	Mindlin (1951)	0.0088	1.2081	0.7156	0.0843	0.0525	0.1220	0.1220
	Pagano (1970)	--	1.2250	0.7302	0.0886	0.0535	0.1210	0.1250

Table 3 Non-dimensional displacement and stresses in three layer (0◦/90◦/0◦) laminated spherical shell under the sinusoidal mechanical load (*a/h*=10)

R/a	Theory	$\bar{u}$	$\bar{w}$	$\overline{\sigma_x}$	$\overline{\sigma_y}$	$\overline{\tau_{xy}}$	$\overline{\tau_{xz}}$	$\overline{\tau_{yz}}$
5	**Present**	0.0112	0.6910	0.5441	0.0364	0.0398	0.3431	0.1133
	Reddy (1984b)	0.0108	0.6769	0.5218	0.0352	0.0388	0.3508	0.1109
	Mindlin (1951)	0.0098	0.6025	0.4780	0.0311	0.0346	0.3658	0.1018
10	**Present**	0.0095	0.7211	0.5757	0.0388	0.0347	0.3568	0.1178
	Reddy (1984b)	0.0091	0.7032	0.5515	0.0374	0.0338	0.3645	0.1152
	Mindlin (1951)	0.0083	0.6233	0.5029	0.0329	0.0301	0.3784	0.1053
20	**Present**	0.0085	0.7283	0.5865	0.0396	0.0317	0.3604	0.1190
	Reddy (1984b)	0.0082	0.7102	0.5617	0.0381	0.0309	0.3681	0.1163
	Mindlin (1951)	0.0074	0.6288	0.5115	0.0335	0.274	0.3817	0.1062
50	**Present**	0.0079	0.7304	0.5912	0.0400	0.0298	0.3614	0.1193
	Reddy (1984b)	0.0075	0.7121	0.5662	0.0385	0.0290	0.3691	0.1167
	Mindlin (1951)	0.0069	0.6303	0.5153	0.0338	0.0257	0.3826	0.1064
100	**Present**	0.0076	0.7307	0.5925	0.0401	0.0292	0.3616	0.1194
	Reddy (1984b)	0.0073	0.7124	0.5674	0.0386	0.0283	0.3692	0.1167

Table 3 Continued

R/a	Theory	$\bar{u}$	$\bar{w}$	$\overline{\sigma_x}$	$\overline{\sigma_y}$	$\overline{\tau_{xy}}$	$\overline{\tau_{xz}}$	$\overline{\tau_{yz}}$
100	Mindlin (1951)	0.0067	0.6305	0.5163	0.0339	0.0252	0.3828	0.1064
Plate	**Present**	0.0074	0.7285	0.5925	0.0407	0.0284	0.3616	0.1194
	Reddy (1984b)	0.0071	0.7125	0.5684	0.0387	0.0277	0.3693	0.1168
	Mindlin (1951)	0.0065	0.6306	0.5172	0.0340	0.0246	0.3828	0.1065
	Pagano (1970)	--	0.7528	0.5898	0.0418	0.0289	0.3570	0.1200

Table 4 Non-dimensional displacement and stresses in three layer (0◦/core/0◦) sandwich spherical shell under the sinusoidal mechanical load (*a/h*=10)

R/a	Theory	$\bar{u}$	$\bar{w}$	$\overline{\sigma_x}$	$\overline{\sigma_y}$	$\overline{\tau_{xy}}$	$\overline{\tau_{xz}}$	$\overline{\tau_{yz}}$
5	**Present**	0.0133	1.0198	1.0745	0.0671	0.0948	0.2957	0.0483
	Reddy (1984b)	0.0131	1.0063	1.0733	0.0745	0.0932	0.2956	0.0486
	Mindlin (1951)	0.0109	0.7122	1.0147	0.0607	0.0715	0.3096	0.0384
10	**Present**	0.0103	1.0416	1.1111	0.0821	0.0824	0.3014	0.0492
	Reddy (1984b)	0.0102	1.0250	1.1081	0.0891	0.0812	0.3011	0.0495
	Mindlin (1951)	0.0088	0.7215	1.0385	0.0708	0.0628	0.3137	0.0389
20	**Present**	0.0087	1.0466	1.1244	0.0894	0.0757	0.3029	0.0495
	Reddy (1984b)	0.0086	1.0298	1.1207	0.0962	0.0747	0.3025	0.0497
	Mindlin (1951)	0.0077	0.7238	1.0471	0.0757	0.0582	0.3147	0.0390
50	**Present**	0.0078	1.0481	1.1307	0.0936	0.0715	0.3033	0.0495
	Reddy (1984b)	0.0077	1.0312	1.1267	0.1003	0.0707	0.3029	0.0498
	Mindlin (1951)	0.0070	0.7245	1.0512	0.0786	0.0553	0.3150	0.0390
100	**Present**	0.0074	1.0483	1.1326	0.0950	0.0700	0.3034	0.0495
	Reddy (1984b)	0.0073	1.0314	1.1284	0.1017	0.0693	0.3029	0.0498
	Mindlin (1951)	0.0068	0.7246	1.0524	0.0795	0.0544	0.3150	0.0390
Plate	**Present**	0.0071	1.0473	1.1335	0.0966	0.0685	0.3034	0.0495
	Reddy (1984b)	0.0070	1.0315	1.1300	0.1030	0.0679	0.3029	0.0498
	Mindlin (1951)	0.0066	0.7246	1.0535	0.0805	0.0534	0.3151	0.0390
	Pagano (1970)	0.0071	1.1002	1.1518	0.1098	0.0706	0.2997	0.0526

For the comparison purpose the numerical results are presented in the following non-dimensional form.

$$\begin{gathered}\bar{u}\left(0,\frac{b}{2},-\frac{h}{2}\right)=\frac{h^2E_3}{q_0a^3}u,\quad \bar{w}\left(\frac{a}{2},\frac{b}{2},0\right)=\frac{100h^3E_3}{q_0a^4}w,\\ \left(\bar{\sigma}_x,\bar{\sigma}_y\right)\left(\frac{a}{2},\frac{b}{2},-\frac{h}{2}\right)=\frac{h^2}{q_0a^2}\left(\sigma_x,\sigma_y\right),\ \bar{\tau}_{xy}\left(0,0,-\frac{h}{2}\right)=\frac{h^2}{q_0a^2}\tau_{xy}\\ \bar{\tau}_{xz}\left(0,\frac{b}{2},\frac{z}{h}\right)=\frac{h}{q_0a}\tau_{xz},\ \bar{\tau}_{yz}\left(\frac{a}{2},0,\frac{z}{h}\right)=\frac{h}{q_0a}\tau_{yz}\end{gathered} \tag{24}$$

where, E_3 is of middle layer.

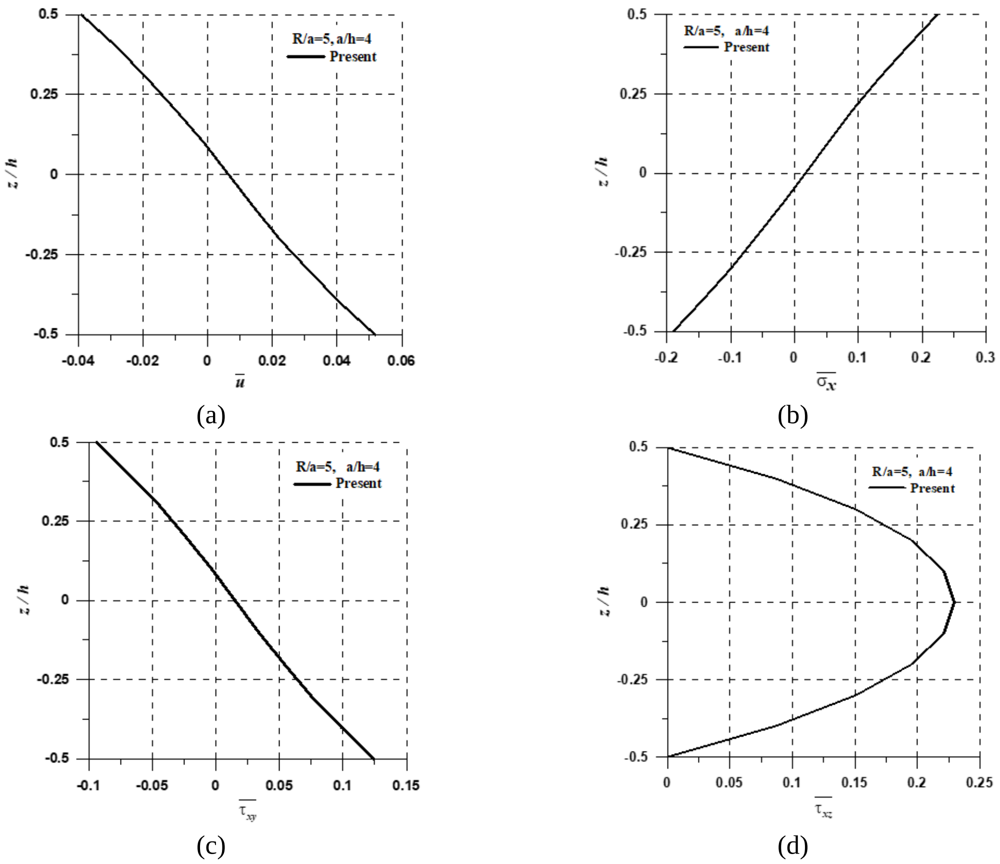


Fig. 2 Through-the-thickness variations of (a) in-plane displacement ($\bar{u}$), (b) in-plane normal stress ($\overline{\sigma_x}$), (c) in-plane shear stress ($\overline{\tau_{xy}}$) and (d) transverse shear stress ($\overline{\tau_{xz}}$) for isotropic spherical shell subjected to sinusoidal load

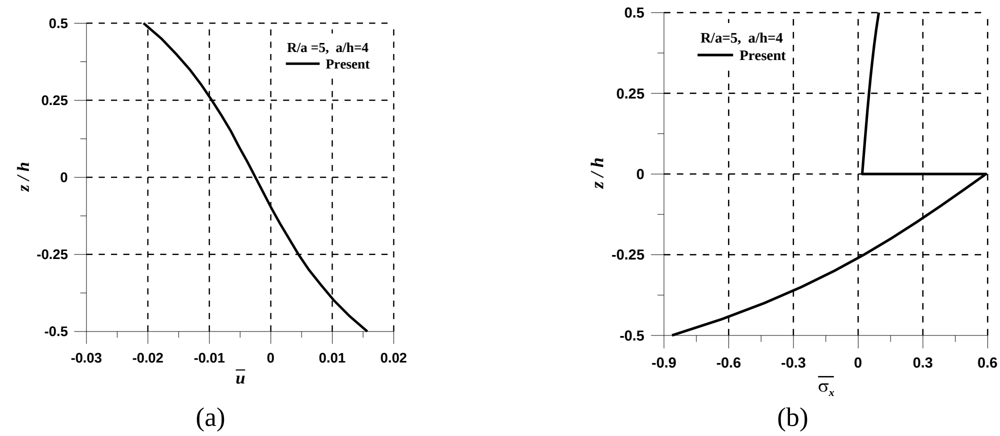
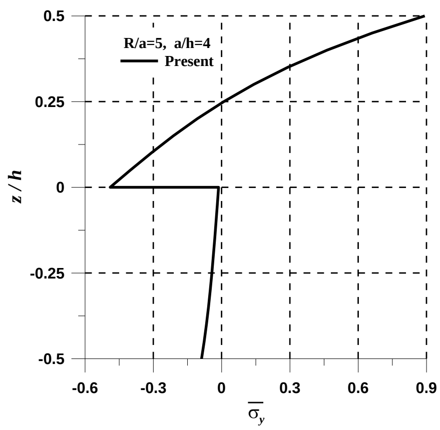
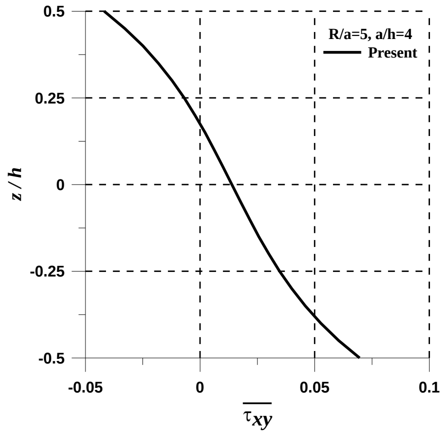
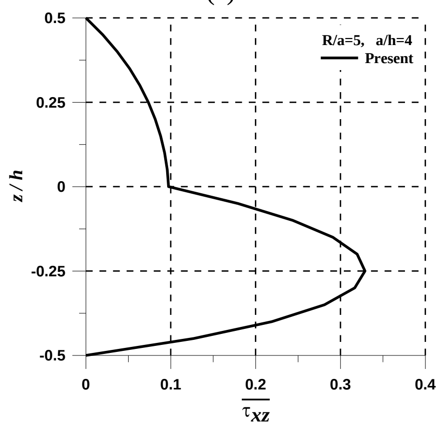


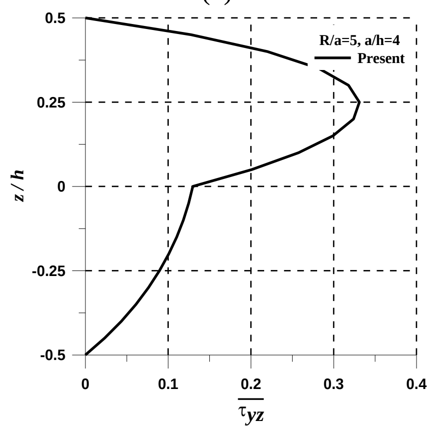
Fig. 3 Through-the-thickness variations of displacement and stresses for two layered ($0^0/90^0$) laminated spherical shell





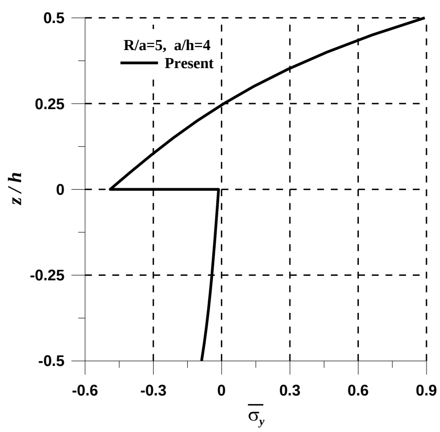
(c) (d)

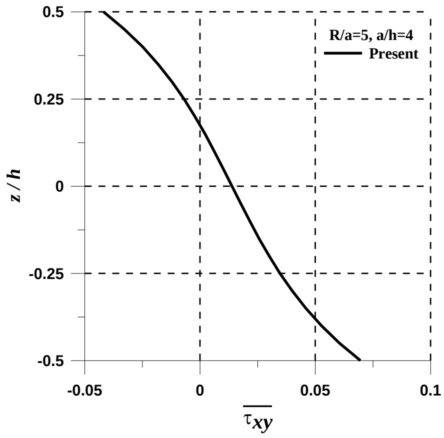




(e) (f)

Fig. 3 Continued





(a) (b)

Fig. 4 Through-the-thickness variations of displacements and stresses for three layered (0°/90°/0°) laminated spherical shell

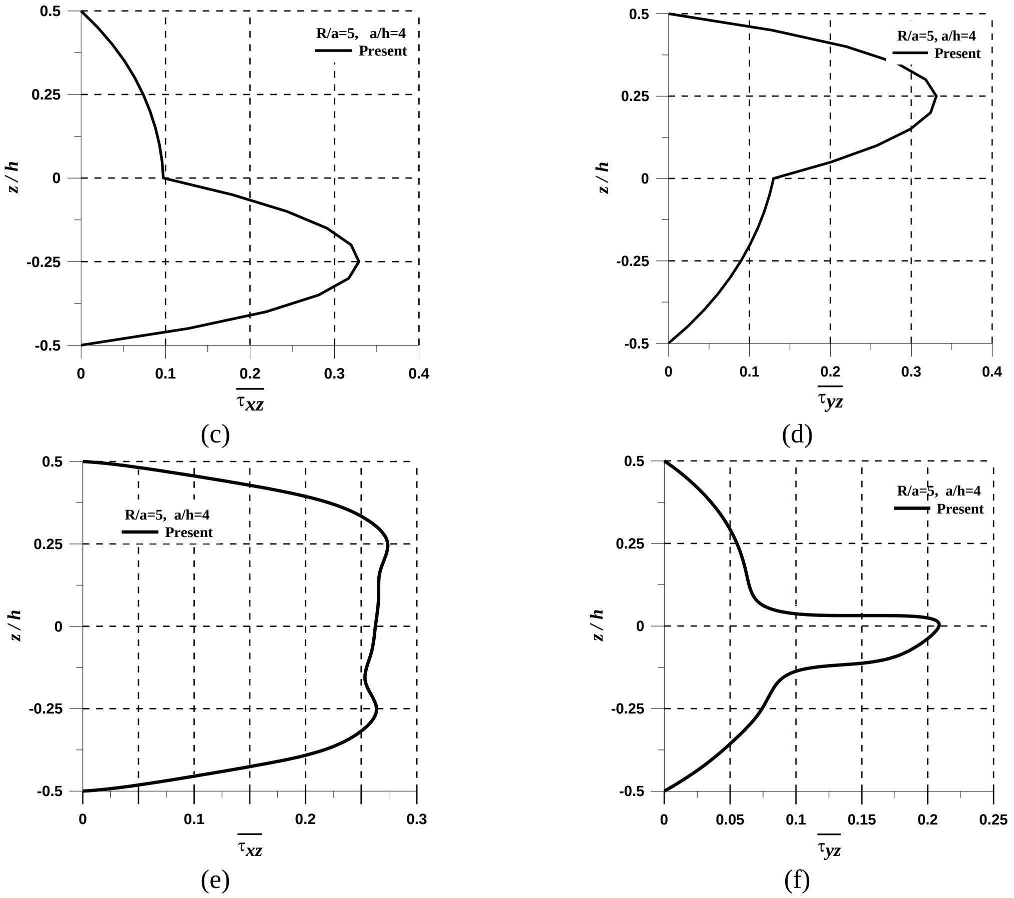


Fig. 4 Continued

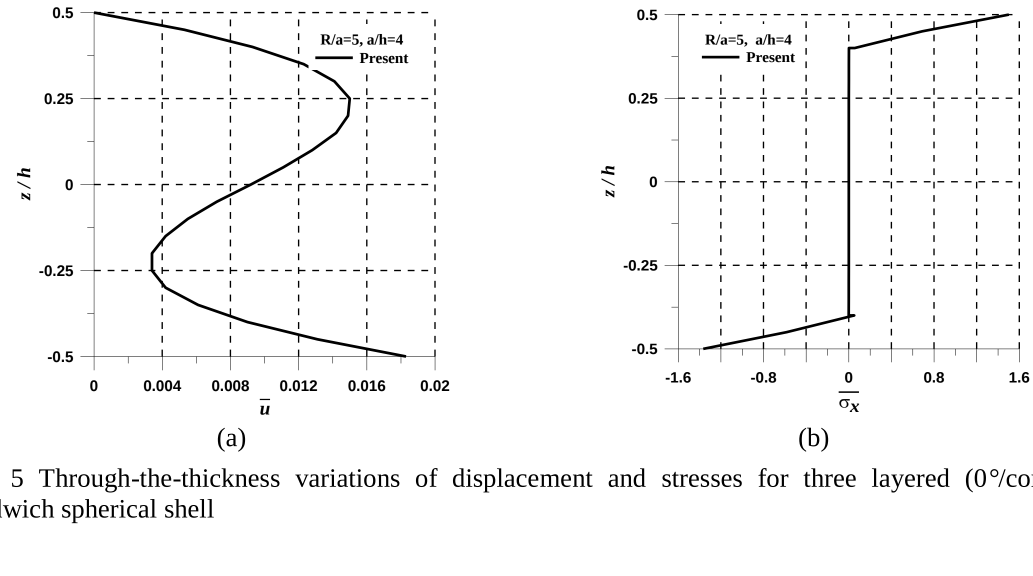


Fig. 5 Through-the-thickness variations of displacement and stresses for three layered (0°/core/0°) sandwich spherical shell

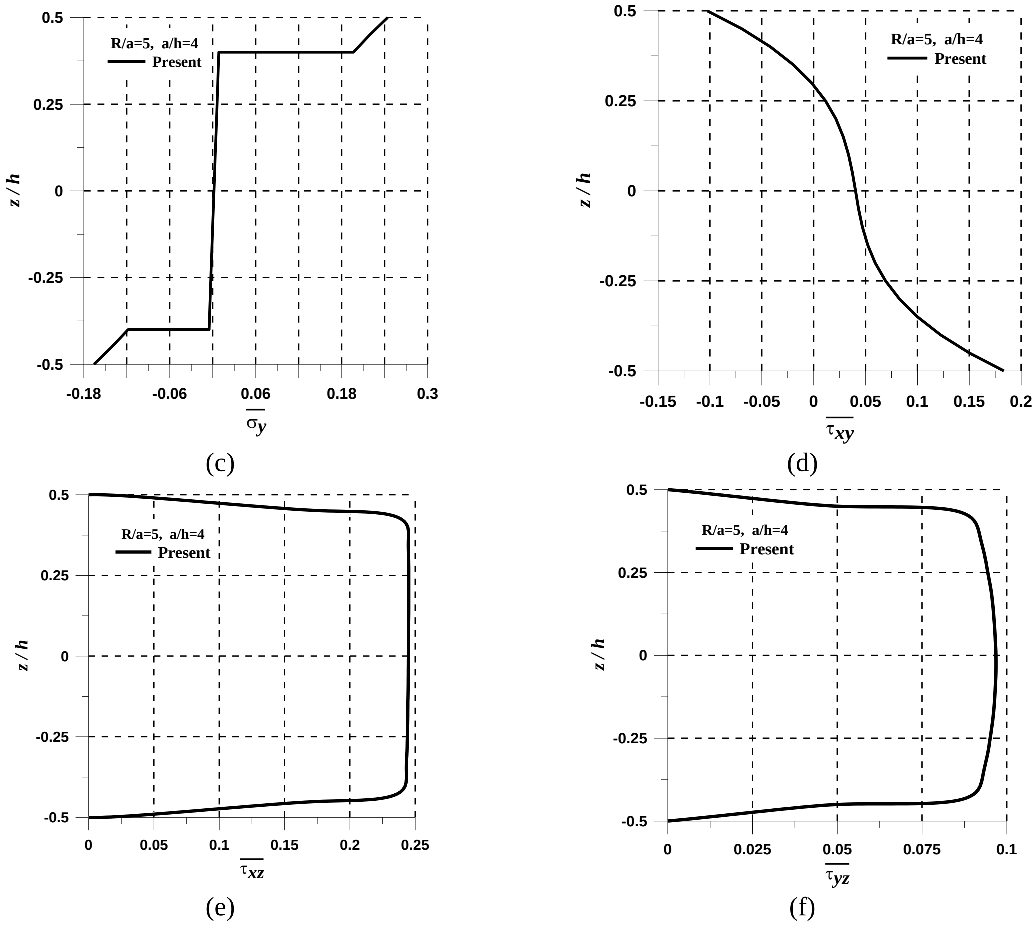


Fig. 5 Continued

3.1 Discussion on isotropic spherical shell

The non-dimensional displacements and stresses in the isotropic spherical shell subjected to sinusoidal load using the present theory are shown in Table 1 for various (R/ a) ratios and a/h=10. The shell is made up of material MAT1. 3D elasticity solution of spherical shell is not available in the literature. Therefore, the present results are compared with parabolic shear deformation theory (PSDT) of Reddy and first-order shear deformation theory (FSDT) of Mindlin. Also, the exact elasticity solution of Pagano is used to compare the displacements and stresses in the isotropic plate ($R = \infty$). The examination of Table 1 reveals that the present theory predicts excellent results for the plate compared with 3D elasticity solution of Pagano. Displacements and stresses obtained using the present theory for spherical shells are in excellent agreement with those presented by Reddy and Mindlin. Through-the-thickness variations of displacement and stresses in isotropic spherical shells are plotted in Fig. 2.

3.2 Discussion on laminated spherical shell

The effects of transverse shear and normal deformations are more pronounced in thick laminated composite shells. Therefore, the present theory is applied for the bending analysis of symmetric and anti-symmetric laminated composite spherical shells. The present results of displacements and stresses for anti-symmetric (0°/90°) and symmetric (0°/90°/0°) laminated composite spherical shells are summarized in Tables 3 and 4 respectively. Both the types of shells are made up of material MAT3 and all the layers are of equal thickness. From Tables 3 and 4 it is observed that the present results are in excellent agreement with 3D exact elasticity solution for laminated plate given by Pagano. This is in fact, due to inclusion of fifth-order term in terms of thickness coordinate in the displacement field and the effect of normal deformation. It is pointed out that the in-plane displacement and stresses in laminated shells are increased with increase in R/a ratio, whereas the in-plane shear stresses decrease due to increase in radii of curvature. The transverse shear stresses are obtained using the 3D equations of equilibrium to ascertain continuity at the layer interface. Through-the-thickness variations of displacements and stresses for (0°/90°) and (0°/90°/0°) laminated spherical shells are shown in Figs. 3 and 4 respectively.

In case of two layered (0°/90°) anti-symmetric laminated spherical shell, from Fig. 3(b) it is observed that the in-plane normal stress (σ_x) has a maximum value at the top surface ($z = -h/2$) of the shell, i.e. 0^0 layer and minimum at the bottom surface ($z = h/2$) i.e., 90° layer. Exactly opposite trend is observed from Fig. 3(c). It is also observed that the in-plane normal stresses changes its sign at z = -0.26h. Figs. 3(e) and 3f show variations of transverse shear stresses (τ_{xz}, τ_{yz}) through-the-thickness of shell is found maximum at $z = -0.25h$ whereas is found maximum at z = 0.25h. For three layered ($0^0/90^0/0^0$) symmetric laminated spherical shell, it is to be noted that for laminated plate (R_1=R_2=∞) stresses are symmetrically distributed over the thickness but for laminated spherical shell these stresses are not symmetric due to curvature effect. In-plane normal stress is maximum at $z = h/2$ whereas is maximum at $z = -h/2$. Similarly, transverse shear stress is maximum at $z = 0.25h$ and is maximum at $z = 0$.

3.3 Discussion on sandwich spherical shell

Effect of thickness stretching is more pronounced in the sandwich shell with soft core. Therefore, the present theory is also applied for the bending analysis of sandwich shell subjected to sinusoidal loading. The shell has top and bottom face sheets made up of fibrous composite material, whereas core at the center is made up of transversely isotropic material. Thickness of each face sheet is 0.1h and that of core is 0.8h. Face sheets are made up of MAT3 and core is made up of MAT2. Table 4 shows a comparison of displacement and stresses developed in sandwich spherical shells subjected to sinusoidal load for various R/a ratios. From Table 4, it is observed that as R/a ratio increases the in-plane normal stresses increase whereas in-plane shear stresses decreases. Fig. 5 shows through-the-thickness variations of in-plane displacement and stresses in sandwich spherical shells.

In case of (0°/core/0°) sandwich spherical shell, from Fig. 5b and Fig. 5c it is observed that the in-plane normal stresses (σ_x) have the maximum value at the bottom surface, i.e., ($z= h/2$). Figs. 5e and 5f show variations of transverse shear stresses (τ_{xz}, τ_{yz}) through-the-thickness of the shell and observed to be maximum at ($z = 0$). Also, in case of sandwich spherical shell, it is seen that the in-plane normal stresses and transverse shear stresses are varying in face sheets and almost constant in the core.

4. Conclusions

In the present study a fifth-order shear and normal deformation theory is developed for the static analysis of spherical laminated composite and sandwich spherical shells. The theory accounts for both transverse shear and normal deformation effects. The non-dimensional displacements and stresses are obtained for isotropic, laminated and sandwich spherical shells. The present results are compared with previous results. Based on comparison of numerical results and discussion, it is concluded that the present theory gives excellent results for laminated plates compared to the exact elasticity solution given by Pagano. Also the present results are in close agreement with the theory of Reddy when applied for laminated composite and sandwich spherical shells. Interlaminar shear stresses are obtained using equations of equilibrium by satisfying continuity at the layer interface. Due to the inclusion of transverse shear and normal deformations, the present theory is strongly recommended for the analysis of laminated and sandwich spherical shells.

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CC

Appendix

$$
\begin{aligned}
&K_{11}=-A_{11}\alpha^2-A_{66}\beta^2,\quad K_{12}=K_{21}=-A_{12}\alpha\beta-A_{66}\alpha\beta,\\
&K_{13}=K_{31}=\frac{A_{11}}{R_1}\alpha+\frac{A_{12}}{R_R}\beta+B_{11}\alpha^3+B_{12}\alpha\beta^2+2B_{66}\alpha\beta^2,\\
&K_{14}=K_{41}=-A_{S1_{11}}\alpha^2-A_{S2_{66}}\beta^2,\ K_{15}=K_{51}=-A_{S2_{11}}\alpha^2-A_{S2_{66}}\beta^2,\\
&K_{16}=K_{61}=-A_{S1_{12}}\alpha\beta-A_{S1_{66}}\alpha\beta,\quad K_{17}=K_{71}=-A_{S2_{12}}\alpha\beta-A_{S2_{66}}\alpha\beta,\\
&K_{18}=K_{81}=\left(E_{13}-\frac{Q_{111}}{R_1}-\frac{Q_{112}}{R_2}\right)\alpha,\ K_{19}=K_{91}=\left(F_{13}-\frac{Q_{211}}{R_1}-\frac{Q_{212}}{R_2}\right)\alpha,\\
&K_{22}=-A_{22}\beta^2-A_{66}\alpha^2,\\
&K_{23}=K_{32}=\left(\frac{A_{12}}{R_1}+\frac{A_{22}}{R_2}\right)\beta+B_{12}\alpha^2\beta+B_{22}\beta^3+2B_{66}\alpha^2\beta,\\
&K_{24}=K_{42}=-A_{S1_{12}}\alpha\beta-A_{S1_{66}}\alpha\beta,\ K_{25}=K_{52}=-A_{S2_{12}}\alpha\beta-A_{S2_{66}}\alpha\beta,\\
&K_{26}=K_{62}=-A_{S1_{22}}\beta^2-A_{S1_{66}}\alpha^2,\ K_{27}=K_{72}=-A_{S2_{22}}\beta^2-A_{S2_{66}}\alpha^2,\\
&K_{28}=K_{82}=\left(E_{23}-\frac{Q_{112}}{R_1}-\frac{Q_{122}}{R_2}\right)\beta,\ K_{29}=K_{92}=\left(F_{23}-\frac{Q_{212}}{R_1}-\frac{Q_{222}}{R_2}\right)\beta\\
&K_{33}=-\left(D_{11}\alpha^4+D_{22}\beta^4\right)-2\alpha^2\beta^2\left(D_{12}+2D_{66}\right)-2\alpha^2\left(\frac{B_{11}}{R_1}+\frac{B_{12}}{R_2}\right)\\
&\qquad-2\beta^2\left(\frac{B_{12}}{R_1}+\frac{B_{22}}{R_2}\right)-\left(\frac{A_{11}}{R_1^2}+\frac{A_{22}}{R_2^2}+\frac{2A_{12}}{R_1R_2}\right),\\
&K_{34}=K_{43}=B_{S1_{11}}\alpha^3+B_{S1_{12}}\alpha\beta^2+2B_{S1_{66}}\alpha\beta^2+\frac{A_{S1_{11}}}{R_1}\alpha+\frac{A_{S1_{12}}}{R_2}\alpha,\\
&K_{35}=K_{53}=B_{S2_{11}}\alpha^3+B_{S2_{12}}\alpha\beta^2+2B_{S2_{66}}\alpha\beta^2+\frac{A_{S2_{11}}}{R_1}\alpha+\frac{A_{S2_{12}}}{R_2}\alpha,\\
&K_{36}=K_{63}=B_{S1_{12}}\alpha^2\beta+B_{S1_{22}}\beta^3+2B_{S1_{66}}\alpha^2\beta+\frac{A_{S1_{12}}}{R_1}\beta+\frac{A_{S1_{22}}}{R_2}\beta,\\
&K_{37}=K_{73}=B_{S2_{12}}\alpha^2\beta+B_{S2_{22}}\beta^3+2B_{S2_{66}}\alpha^2\beta+\frac{A_{S2_{12}}}{R_1}\beta+\frac{A_{S2_{22}}}{R_2}\beta,\\
&K_{38}=K_{83}=-J_{13}\alpha^2-J_{23}\beta^2-\frac{E_{13}}{R_1}-\frac{E_{23}}{R_2}+\left(\frac{Q_{311}}{R_1}+\frac{Q_{312}}{R_2}\right)\alpha^2+\left(\frac{Q_{312}}{R_1}+\frac{Q_{322}}{R_2}\right)\beta^2\\
&\qquad+\left(\frac{Q_{111}}{R_1^2}+\frac{Q_{112}}{R_1R_2}\right)+\left(\frac{Q_{112}}{R_1R_2}+\frac{Q_{122}}{R_2^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
&K_{39}=K_{93}=-O_{13}\alpha^2-O_{23}\beta^2-\frac{F_{13}}{R_1}-\frac{F_{23}}{R_2}+\left(\frac{Q_{411}}{R_1}+\frac{Q_{412}}{R_2}\right)\alpha^2+\left(\frac{Q_{412}}{R_1}+\frac{Q_{422}}{R_2}\right)\beta^2\\
&\qquad+\left(\frac{Q_{211}}{R_1^2}+\frac{Q_{212}}{R_1R_2}\right)+\left(\frac{Q_{212}}{R_1R_2}+\frac{Q_{222}}{R_2^2}\right),\\
&K_{44}=-A_{SS1_{11}}\alpha^2-A_{SS1_{66}}\beta^2-G_{55},\ K_{45}=K_{54}=-C_{11}\alpha^2-C_{66}\beta^2-I_{55},\\
&K_{46}=K_{64}=-A_{SS1_{12}}\alpha\beta-A_{SS1_{66}}\alpha\beta,\quad K_{47}=K_{74}=-C_{12}\alpha\beta-C_{66}\alpha\beta,\\
&K_{48}=K_{84}=\left(L_{1_{13}}\alpha-G_{55}\alpha-\frac{Q_{511}}{R_1}-\frac{Q_{512}}{R_2}\right),\ K_{49}=K_{94}=\left(L_{2_{13}}\alpha-I_{55}\alpha-\frac{Q_{611}}{R_1}-\frac{Q_{612}}{R_2},\right)\\
&K_{55}=-A_{SS2_{11}}\alpha^2-A_{SS2_{66}}\beta^2-H_{55},K_{56}=K_{65}=-C_{12}\alpha\beta-C_{66}\alpha\beta,\\
&K_{57}=K_{75}=-A_{S2_{12}}\alpha\beta-A_{SS2_{66}}\alpha\beta,\ K_{58}=K_{85}=-I_{55}\alpha+M_{1_{13}}\alpha-\frac{Q_{711}}{R_1}-\frac{Q_{712}}{R_2},\\
&K_{59}=K_{95}=-H_{55}\alpha+M_{2_{13}}\alpha-\frac{Q_{811}}{R_1}-\frac{Q_{812}}{R_2},\ K_{66}=-A_{SS1_{22}}\beta^2-A_{SS1_{66}}\alpha^2-G_{44},\\
&K_{67}=K_{76}=-C_{22}\beta^2-C_{66}\alpha^2-I_{44},K_{68}=K_{86}=L_{1_{23}}\beta-G_{44}\beta-\frac{Q_{512}}{R_1}-\frac{Q_{522}}{R_2},\\
&K_{69}=K_{96}=L_{2_{23}}\beta-I_{44}\beta-\frac{Q_{612}}{R_1}-\frac{Q_{622}}{R_2},\ K_{77}=-A_{SS2_{22}}\beta^2-A_{SS2_{66}}\alpha^2-H_{44},\\
&K_{78}=K_{87}=M_{1_{23}}\beta-I_{44}\beta-\frac{Q_{712}}{R_1}-\frac{Q_{722}}{R_2},\ K_{79}=K_{97}=M_{2_{23}}\beta-H_{44}\beta-\frac{Q_{812}}{R_1}-\frac{Q_{822}}{R_2},\\
&K_{88}=-G_{55}\alpha^2-G_{44}\beta^2-N_{1_{33}}+2\frac{Q_{1613}}{R_1}+2\frac{Q_{1623}}{R_2}+\frac{Q_{1311}}{R_1^2}+2\frac{Q_{1312}}{R_1R_2}+\frac{Q_{1322}}{R_2^2},\\
&K_{89}=K_{98}=-I_{55}\alpha^2-I_{44}\beta^2-N_{3_{33}}+2\frac{Q_{1613}}{R_1}+2\frac{Q_{1623}}{R_2}+\frac{Q_{1311}}{R_1^2}+2\frac{Q_{1312}}{R_1R_2}+\frac{Q_{1322}}{R_2^2},\\
&K_{99}=-H_{55}\alpha^2-H_{44}\beta^2-N_{2_{33}}
\end{aligned}
\tag{A1}
$$

$$F = \{0, 0, -q_0, 0, 0, 0, 0, 0, 0\}^T \tag{A2}$$

$$\Delta = \{u_1, \phi_{x1}, \psi_{x1}, v_1, \phi_{y1}, \psi_{y1}, w_1, \phi_{z1}, \psi_{z1}\}^T \tag{A3}$$