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Modelling the Leipzig Wind Profile with a (k-ε) model

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Abstract. The Leipzig Wind Profile is generally known as a typical neutral planetary boundary layer flow. But it became clear from the present research that it was not completely neutral but weakly stable. We examined whether we could simulate the Leipzig Wind Profile by using a (k-ε) turbulence model including the equation of potential temperature. By solving analytically the Second Moment Closure Model under the assumption of local equilibrium and under the condition of a stratified flow, we expressed the turbulent diffusion coefficients (both momentum and thermal) as functions of flux Richardson number. Our (k-ε) turbulence model which included the equation of potential temperature and the turbulent diffusion coefficients varying with flux Richardson number reproduced the Leipzig Wind Profile.

Key words: a weakly stable atmospheric boundary layer; Mellor-Yamada model; (k-ε) model; Second Moment Closure Model.

1. Introduction

Our final objective on this theme is to make up a numerical model which estimates a change in an urban climate caused by urbanization. A model for an atmospheric boundary layer flow is necessary as one of the preliminary stages for the achievement of our objective. In simulating an atmospheric boundary layer flow, most meteorologists use models which include either an algebraic turbulence length scale or an equation of the length scale. On the other hand, in the field of engineering, the ε equation (the equation of viscous energy dissipation rate) is mostly used instead of the length scale equation. From the standpoint of engineering, it is desirable to use some model in which the ε equation is used in place of the length scale equation. One of the purposes of this research is to verify the possibility of estimating numerically a neutral (actually weakly stable) atmospheric boundary layer flow by means of a (k-ε) turbulence model. The other purpose is to investigate whether the Mellor-Yamada model (level 2.5), which is popularly used in the field of atmospheric boundary layer meteorology, is able to estimate an atmospheric boundary layer flow or not. The validity of this model has not yet been made sure except the case of a surface layer flow in a stable atmospheric boundary layer.

The Leipzig Wind Profile is generally known as a typical neutral planetary boundary layer flow. If we numerically estimate the wind profile by using a (k-ε) turbulence model under the condition of equi-potential temperature, this (k-ε) model does not reproduce the measured wind profile as shown in Fig. 1. The result from the Mellor-Yamada model (level 2.5) also deviates largely from the wind profile. On the other hand, when we calculate the wind profile under the condition of equi-potential

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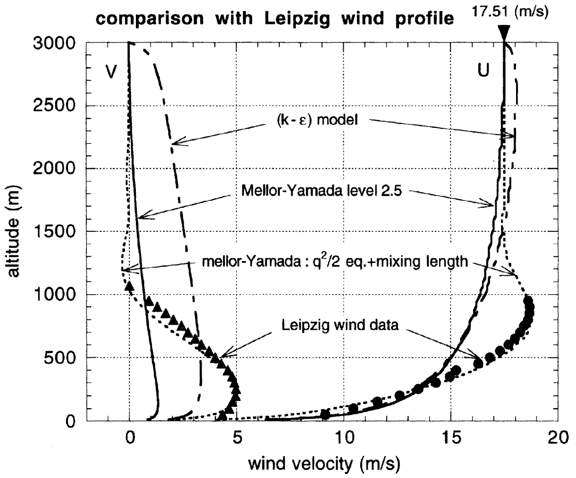

Fig. 1 Estimations of Leipzig Wind Profile by turbulence models without the equation of potential temperature

temperature by combining the turbulent energy equation with the Blackadar length scale (1962), the result accords with the measured profile.

From the above facts, many researchers have been inferred that some modification for the ε equation is necessary so as to estimate a neutral atmospheric boundary layer by using a (k-ε) turbulence model. There are several researches for modifications of the ε equation, for example, Detering and Etling (1985), Duynkerke (1988), Andren (1991), Apsley and Castro (1997), Xu and Taylor (1997, a,b). When we reread in detail the papers on the Leipzig Wind Profile (Lettau 1950, 1962), it is, however, mentioned in the papers that there was an increase in potential temperature of 0.35 degree/ 100 m during the wind profile measurement. This fact makes us infer that the Leipzig Wind Profile was not completely neutral but weakly stable.

Noticing this fact, we examined whether we could simulate the Leipzig Wind Profile by using a (k-ε) turbulence model which included the equation of potential temperature and the modified turbulent coefficients. The turbulent diffusion coefficients of our (k-ε) turbulence model are derived from the Second Moment Closure Model by Launder (1975). By solving analytically the Second Moment Closure Model under the assumption of local equilibrium and under the condition of a stratified flow, we express the turbulent diffusion coefficients (both momentum and thermal) as functions of flux Richardson number.

We investigate in this research whether it is possible to reproduce the Leipzig Wind Profile by means of the present (k-ε) turbulence model without any modification of the ε equation. Moreover, we compare our (k-ε) turbulence model with the Mellor-Yamada model (level 2.5).

2. (k-ε) model and turbulent diffusion coefficients in this research

Table 1 shows the equations of mean wind velocities, mean potential temperature, turbulent energy and viscous energy dissipation rate used in this research. We took the potential temperature difference between the altitude 3000 m and the ground to be 10.5 degree, because the increase in potential temperature of 0.35 degree/100 m was described in Lettau's papers. We pick up the following

Table 1 (k-ε) model used in this research

(k-ε) model in this research

$$\frac{\partial u}{\partial t} = f(V - V_g) + \frac{\partial}{\partial z}\left(\nu_t \frac{\partial U}{\partial z}\right) \quad \text{(Eq. 1)}$$

$$\frac{\partial V}{\partial t} = -f(U - U_g) + \frac{\partial}{\partial z}\left(\nu_t \frac{\partial V}{\partial z}\right) \quad \text{(Eq. 2)}$$

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial z}\left(\nu_\theta \frac{\partial \Theta}{\partial z}\right) \quad \text{(Eq. 3)}$$

$$\frac{\partial k}{\partial t} = P + G - \varepsilon + \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial z}\right) \quad \text{(Eq. 4)}$$

$$\frac{\partial \varepsilon}{\partial t} = \left(\frac{\varepsilon}{k}\right)\{c_{1\varepsilon}(P + G) - c_{2\varepsilon}\varepsilon\} + \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial z}\right) \quad \text{(Eq. 5)}$$

$$P = \nu_t\left\{\left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} \quad \text{(Eq. 6)}$$

$$G = -\beta g \nu_\theta \frac{\partial \Theta}{\partial z} \quad \text{(Eq. 7)}$$

$$\nu_t = C_D(R_f)\left(\frac{k^2}{\varepsilon}\right) \quad \text{(Eq. 8)}$$

$$\nu_\theta = C_\theta(R_f)\left(\frac{k^2}{\varepsilon}\right) \quad \text{(Eq. 9)}$$

$$R_f = -\frac{G}{P} \quad \text{(Eq.10)}$$

parameters : Coriolis parameter : $f = 1.14e\text{-}4$ (1/s), roughness : $z_0 = 0.3$ (m)

boundary conditions :

(1) at altitude 3000 m

$U = Ug = 17.51$ (m/s), $V = Vg = 0$, $\Theta = 293.5$ (K), $dk / dz = 0$, $d\varepsilon / dz = 0$

(2) at altitude 10 m

$\Theta = 283.035$ (K), $\varepsilon = u_*^3 / l_B$ (l_B : Blackadar's length-scale)

(3) at ground

U, V: logarithmic law, $k = u_*^2 / \sqrt{C_D}$ (u_* : friction velocity)

Table 2 Several kinds of turbulent diffusion coefficients

Type I	Mellor-Yamada model level 2.5 (Yamada 1983)
Type II	Turbulent diffusion coefficients converted from those of Mellor-Yamada model (level 2.5) for use in (k-ε) model
Type III	Turbulent diffusion coefficients analytically derived from IP model, where the values of coefficients are those of IP model. So, $C_D(0) = 0.115$, Prt = 0.644. case1 : $c_{1\varepsilon} = 1.44$ case2 : $c_{1\varepsilon} = 1.56$, adjusted so as $\{c_{1\varepsilon} = c_{2\varepsilon} - \{\kappa_2 / \sigma_\varepsilon \sqrt{C_D(0)}\ \}$
Type IV	Turbulent diffusion coefficients analytically derived from IP model, adjusted so as $C_D(0) = 0.09$ and $C_\theta(0) = C_D(0)$ / Prt. case1 : Prt = 0.644 (original IP model) case2 : Prt = 0.74 (Garrat 1992) case3 : Prt = 0.83 (Ueda *et al*. 1981)
Type V	Turbulent diffusion coefficients analytically derived from IP model, adjusted so as $C_D(0) = 0.09$ and $R_{fc} = 0.25$ (Sorbjan 1989). Parameters are Prt and R. case1 : Prt = 0.64, R = 0.5 case2 : Prt = 0.64, R = 0.8 case3 : Prt = 0.74, R = 0.5 case4 : Prt = 0.74, R = 0.8 case5 : Prt = 1.0, R = 0.5
Type VI	Turbulent diffusion coefficients measured by Ueda *et al*. (1981)

three kinds of turbulent diffusion coefficient as shown in Table 2.

(a) coefficients converted from the turbulent diffusion coefficients by Yamada (1983) for use in (k-ε) model
(b) coefficients derived analytically from Isotropization of Production model (hereafter IP model) by Launder (1975)
(c) coefficients measured by Ueda *et al*. (1981)

$C_D(R_f)$ and $C_\theta(R_f)$ in case of (a) are expressed as follows,

$$C_D(R_f) = \begin{cases} 0.449\dfrac{(0.1912-R_f)(0.2341-R_f)}{(1-R_f)(0.2231-R_f)} & \text{for} \quad R_f < 0.16 \\ 0.019 & \text{for} \quad R_f \geq 0.16 \end{cases} \tag{1}$$

$$C_\theta(R_f) = \begin{cases} 0.5916\dfrac{(0.1912-R_f)}{(1-R_f)} & \text{for} \quad R_f < 0.16 \\ 0.0218 & \text{for} \quad R_f \geq 0.16 \end{cases} \tag{2}$$

$$R_f = \begin{cases} 0.6588(R_i + 0.1776 - \sqrt{R_i^2 - 0.3221R_i + 0.03156R_i}) & \text{for} \quad R_i < 0.195 \\ 0.191 & \text{for} \quad R_i \geq 0.195 \end{cases} \tag{3}$$

Eq. (1) and Eq. (2) are adjusted so that $C_D(0) = 0.09$ and $P_{rt} = 0.796$. $P_{rt} = 0.796$ is according to Yamada (1983).

In case of (b), the analytical solutions of IP model can be obtained according to the method of Yamada (1975). The coefficients in the turbulent diffusion coefficients are expressed by the following equations. See Appendix on the detail of the derivation.

$$C_D(R_f) = \phi\left(\frac{a_1a_2}{a_3}\right)\frac{(R_{c1}-R_f)(R_{c2}-R_f)}{(1-R_f)(R_{c3}-R_f)} \tag{4}$$

$$C_\theta(R_f) = a_2\frac{R_{c2}-R_f}{1-R_f} \tag{5}$$

$$R_f = \frac{(R_i + cR_{c1}) - \sqrt{(R_i + cR_{c1})^2 - 4cR_{c3}R_i}}{2c} \tag{6}$$

R_{c2} is a critical flux Richardson number at the turbulent diffusion coefficients described above. We investigate which kind of turbulent diffusion coefficients among those of the three types in Table 2 (Type III, IV and V) is the most suitable to reproduce the Leipzig Wind Profile.

The turbulent diffusion coefficients by Ueda *et al*. (1981) are

$$\frac{C_D(R_i)}{C_D(0)} = \begin{cases} (1+2.5R_i)^{-1} & \text{for } R_i \geq 0 \\ (1-25R_i)^{1/3} & \text{for } R_i < 0 \end{cases} \tag{7}$$

$$\frac{C_\theta(R_f)}{C_\theta(0)} = \frac{C_D(R_f)}{C_D(0)} \frac{(1-10R_f)}{(1-R_f)^2} \tag{8}$$

$$R_i = \frac{P_{rt}R_f(1-R_f)^2}{(1-10R_f)} \tag{9}$$

where $P_{rt} = C_D(0) / C_\theta(0) = 0.833$.

3. Result and discussion

The model in Table 1 was computed by the finite difference method (forward difference in time and central difference in space). A time step was decided by the stability analysis of the following equation

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial z^2} \tag{10}$$

The stability analysis of the following simultaneous equations was not done,

$$\frac{\partial U}{\partial t} = f \cdot V + \nu \frac{\partial^2 U}{\partial z^2} \tag{11}$$

$$\frac{\partial V}{\partial t} = -f \cdot U + \nu \frac{\partial^2 V}{\partial z^2} \tag{12}$$

This fact means that the time step was decided by the stability analysis of the equations of potential

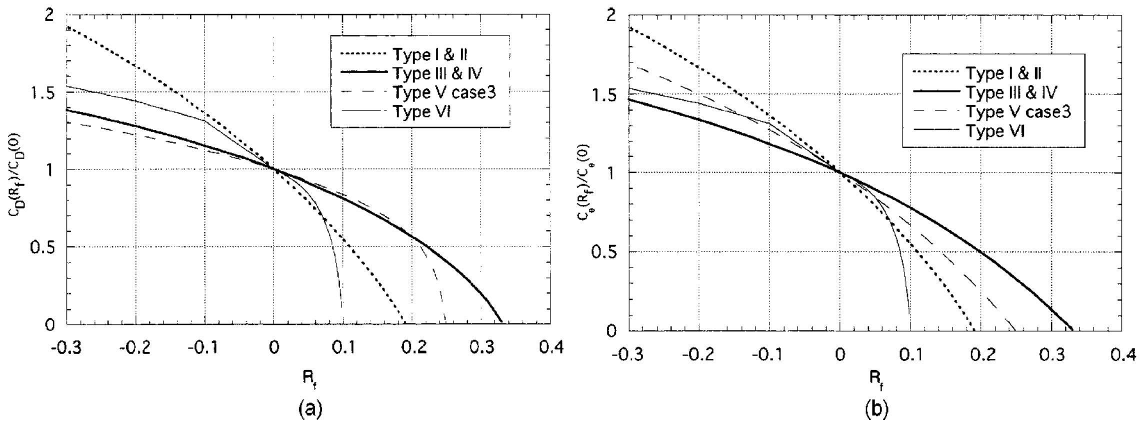

Fig. 2 (a) Variation of C_D according to flux Richardson number (b) Variation of C_θ according to flux Richardson number temperature and turbulent kinetic energy.

We used the results from the isothermal (k-ε) model (refer to Fig. 1) as the initial conditions for U, V, k, and ε. Concerning the potential temperature, we used the following initial condition : $\Theta(z) = 10.465((z-10)/2990)^{0.7} + 283.035$.

Figs. 2a,b show coefficients C_D and C_θ in the turbulent diffusion coefficients used in this research. In the following, we show and discuss the results.

3.1. Results with turbulent diffusion coefficients from the IP model

3.1.1. Type V in Table 2

In the case of Type V, we set $C_D(0) = 0.09$, $R_{fc} = 0.25$(Sorbjan 1989) and took P_{rt} and R as parameters. We decided $C_D(R_f)$ and $C_\theta(R_f)$ by estimating the values of ϕ, $\phi_{1\theta}$ and $\phi_{2\theta}$ using the above parameters. Five cases (case1 to case 5 in Type V) were calculated. The results were not in accord with the measured wind profile as shown in Fig. 3(a).

3.1.2. Type III in Table 2

Fig. 3(b) shows how the adjustment of the coefficient $c_{1\varepsilon}$ according to a change of $C_D(0)$ value had an effect on the result. Though the adjustment had very little effect on the result, it brought a better result than no adjustment. The both results were in accord with the measured data.

3.1.3. Type IV in Table 2

We investigated in Type IV how the results changed according to P_{rt} value. The results did not depend on the change of P_{rt} as is shown in Fig. 3(c). The results were in accord with the measured data.

In consideration of the results from the subsections 3.1.1 to 3.1.3., we chose $C_D(R_f)$ and $C_\theta(R_f)$ of Table IV (case2) as the best analytical solutions of the IP model, though its critical Richardson number (0.33) was larger than the measured values (0.2~0.25).

3.2. Comparison between Type I, Type II, Type IV(case2) and Type VI in Table 2

Fig. 4(a), (b) shows the results from Type I, Type II, Type IV(case2) and Type VI in Table 2. Type I is the result from the Mellor-Yamada model (level 2.5), Type II from a conversion on turbulent diffusion coefficients of the Mellor-Yamada model (level 2.5) for use in (k-ε) model, Type IV (case2) from the most suitable analytical solutions of IP model, and Type VI from the measured data by Ueda *et al.* (1981). Though the data by Ueda *et al*. received no effect of the ground reflection, the calculation result of Type VI did not accord with the measurement. (See Note). Type II and Type IV (case2) were in accord with the measurement. Type I (the Mellor-Yamada model level 2.5) was a little inferior in accuracy to Type II and Type IV (case2).

We decided the time step from the stability analysis of Eq. (10). We did not carry out the stability analysis of the simultaneous Eqs. (11) and (12). It was not always possible to decide a stable time step from the stability analysis of these equations. We assume that this fact produced the oscillations of the mean velocities U and V. However, the oscillations did not occur in the atmospheric boundary layer to which we paid the attention. In practical computation, a computational

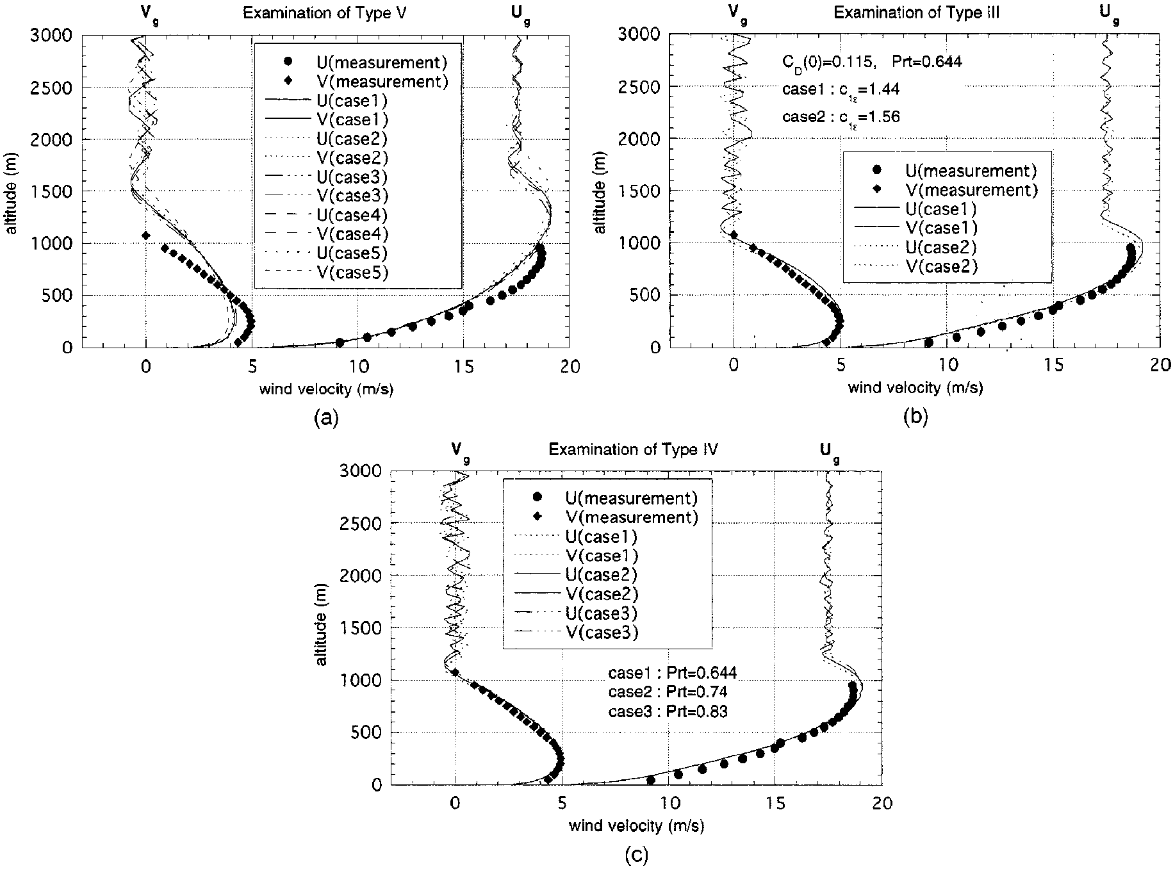

Fig. 3 (a) Examination of Type V (b) Examination of Type III (c) Examination of Type IV

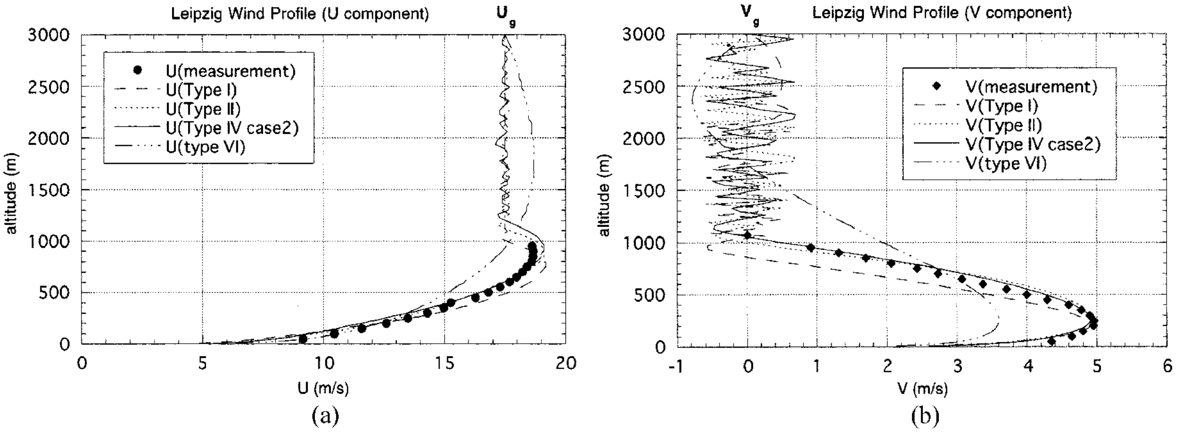

Fig. 4 (a) Comparison on wind velocity between Type I, Type II, Type IV (case 2), and Type VI in Table 2. (U component) (b) Comparison on wind velocity between Type I, Type II, Type IV (case 2), and Type VI in Table 2. (V component)

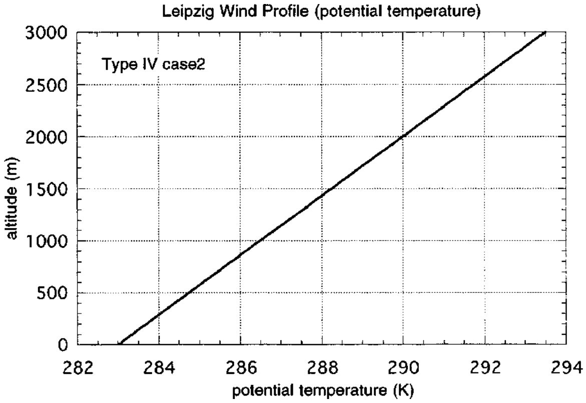

Fig. 5 Computation of potential temperature

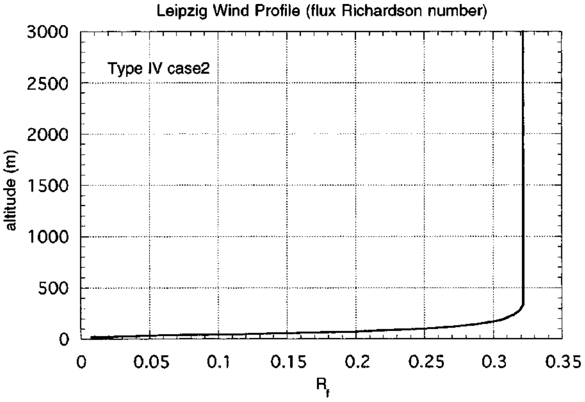

Fig. 6 Estimation of flux Richardson number

domain is three dimensional. In such a case, the time step is decided by the stability analysis of convection-diffusion equation. The time step decided by such a way prevents the oscillations for $z > 1000$ m. From the above reasons, we left the oscillations of velocities for $z > 1000$ m.

3.3. Potential temperature

Fig. 5 shows the calculation result of potential temperature in the case of Type IV (case2) in Table 2. The potential temperature calculated by the present model increases almost linearly with altitude. The rate of increase is 0.35 degree per 100 meters. This is in accord with the measurement. This accordance makes us presume that the effect of radiation was negligible during the measurement.

3.4. Flux Richardson number

Fig. 6 shows the calculation result of the flux Richardson number in the case of Type IV (case2) in Table 2. Flux Richardson number reaches the critical state at an altitude of about 300 meters. Fig. 6 displays that the atmospheric boundary layer flow becomes strongly stable above this height. The fact indicates that any neutral atmospheric boundary layer does not exit actually and that a so-called neutral boundary layer becomes more stable at higher altitude, even if the lower layer is neutral. It also makes us infer that the length scale of Blackadar is not acceptable to a neutral boundary layer but to a weakly stable one. This is discussed by Ueda *et al.* (1981) also.

4. Conclusions

1. The Leipzig Wind profile was not neutral but weakly stable. We indicated that for an estimation of such wind profile it was necessary not to modify the ε equation but to incorporate the equation of potential temperature into the turbulence model.
2. We showed that the turbulent diffusion coefficients (both momentum and thermal) for a stably stratified flow could be derived analytically from the IP model by Launder (1975).

3. The turbulent diffusion coefficients converted from those of Yamada (1983) for use in (k-ε) model also were effective to estimate accurately the Leipzig Wind Profile.
4. We clarified that the present (k-ε) model was possible to estimate a neutral (actually weakly stable) atmospheric boundary layer flow.
5. The Mellor-Yamada model (level 2.5) also was able to estimate a weakly stable atmospheric boundary layer, though it was inferior to the present (k-ε) model in accuracy.

Note

Ueda *et al*. (1981) measured not $C_D(R_f)\,/\,C_D(0)$ but $\nu_t(R_f)\,/\,\nu_t(0)$. They assumed $\nu_t(0) = \kappa u_* z$, where κ is von Karman constant (0.4), u_* friction velocity, and z altitude. Though this difference might be one of the causes leading to the discord with the wind profile, they utilized the following empirical equation by Ellison and Turner (1960).

$$C_\theta(R_f)/C_D(R_f) = (1-R_f/R_{fc})/\{P_{rt}(1-R_f)\}^2 \tag{13}$$

where $P_{rt} = 0.833$ and $R_{fc} = 0.1$ (Ueda *et al*. 1981). When we found out the optimum value of $\phi_{2\theta}$ in the analytical solution of the IP model which best satisfied Eq. (13), the results ($C_D(R_f)\,/\,C_D(0)$ and $C_\theta(R_f)\,/\,C_\theta(0)$) were similar to $\nu_t(R_f)\,/\,\nu_t(0)$ and $\nu_\theta(R_f)\,/\,\nu_\theta(0)$ by Ueda *et al*. (1981). The results also did not reproduce the Leipzig Wind Profile. This fact makes us assume the data by Ueda *et al*. (1981) are unfit for an estimation of an atmospheric boundary layer flow.

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Appendix : Analytical solution of IP model

In case of a stratified flow under the condition of local equilibrium, IP model is expressed as follows,

$$\overline{u_1u_3} = \phi\left(\frac{k}{\varepsilon}\right)\left[-\overline{u_3u_3}\frac{dU}{dz} + \beta g\,\overline{\theta u_1}\right] \tag{ap.1}$$

$$\overline{u_3u_3} = \phi\left(\frac{k}{\varepsilon}\right)\left[\frac{2}{3}\overline{u_1u_3}\frac{dU}{dz} + \frac{4}{3}\beta g\,\overline{\theta u_3}\right] + \frac{2}{3}k \tag{ap.2}$$

$$\overline{\theta u_3} = \left(\frac{k}{\varepsilon}\right)\left[-\phi_{1\theta}\overline{u_3u_3}\frac{d\Theta}{dz} - \phi_{2\theta}(2R)\left(\frac{k}{\varepsilon}\right)\beta g\,\overline{\theta u_3}\frac{d\Theta}{dz}\right] \tag{ap.3}$$

$$\overline{\theta u_1} = -\left(\frac{k}{\varepsilon}\right)\left[\phi_{1\theta}\overline{u_1u_3}\frac{d\Theta}{dz} + \phi_{2\theta}\overline{\theta u_3}\frac{dU}{dz}\right] \tag{ap.4}$$

$$\varepsilon = -\overline{u_1u_3}\frac{dU}{dz} + \beta g\,\overline{\theta u_3} = -\overline{u_1u_3}\frac{dU}{dz}(1-R_f) \tag{ap.5}$$

where $\phi = (1-c_2)/c_1$, $\phi_{1\theta} = 1/c_{1\theta}$, and $\phi_{2\theta} = (1-c_{2\theta})\phi_{1\theta} = (1-c_{2\theta})/c_{1\theta}$. The coefficients c_1, c_2, $c_{1\theta}$ and $c_{2\theta}$ are those in IP model. See Launder (1989).

The above five equations can be solved according to the method of Yamada (1975).

Assuming $\overline{u_1u_3} = -\nu_t\frac{dU}{dz}$, $\nu_t = C_D(R_f)\frac{k^2}{\varepsilon}$ and $\overline{\theta u_3} = -\nu_\theta\frac{d\Theta}{dz}$, $\nu_\theta = C_\theta(R_f)\frac{k^2}{\varepsilon}$, then the coefficients C_D and C_θ are expressed as follows,

$$C_D(R_f) = \phi\left(\frac{a_1a_2}{a_3}\right)\frac{(R_{c1}-R_f)(R_{c2}-R_f)}{(1-R_f)(R_{c3}-R_f)} \tag{ap.6}$$

$$C_\theta(R_f) = a_2 \frac{R_{c2} - R_f}{1 - R_f} \tag{ap.7}$$

$$R_i = \phi \frac{a_1}{a_3} \frac{(R_{c1} - R_f)R_f}{R_{c3} - R_f} \tag{ap.8}$$

$$R_f = \frac{(R_i + cR_{c1}) - \sqrt{(R_i - cR_{c1})^2 - 4cR_{c3}R_i}}{2c} \tag{ap.9}$$

where

$$c = \phi \frac{a_1}{a_3} \tag{ap.10}$$

$$a_1 = \frac{2}{3}(1 + 2\phi) + \phi_{2\theta} \tag{ap.11}$$

$$a_2 = \frac{2}{3}\phi_{1\theta}(1 + 2\phi) + \phi_{2\theta}(2R) \tag{ap.12}$$

$$a_3 = \frac{2}{3}\phi_{1\theta}(1 + \phi) + \phi_{2\theta}(2R) \tag{ap.13}$$

$$R_{c1} = \frac{2}{3}(1 - \phi)/a_1 \tag{ap.14}$$

$$R_{c2} = \frac{2}{3}\phi_{1\theta}(1 - \phi)/a_2 \tag{ap.15}$$

$$R_{c3} = \frac{2}{3}\phi_{1\theta}(1 - \phi)/a_3 \tag{ap.16}$$

When $C_D(0) = 0.09$ and $C_\theta(0) = C_D(0) / P_{rt}$, then $\phi = 0.161$ and $\phi_{1\theta} = 0.161/P_{rt}$.

Notation

U, V : mean velocity
U_g, V_g : geostrophic wind
Θ : mean potential temperature
k : turbulent energy
ε : viscous energy dissipation rate
u_* : friction velocity
f : Coriolis parameter
z_0 : roughness length
β : coefficient of cubical expansion
κ : von Karman constant
R_i : gradient Richardson number

R_f	: flux Richardson number
R_{fc}	: critical R_f
P_{rt}	: turbulent Prandtl number at $R_f = 0$
R	: $((\overline{\theta^2}/(2\varepsilon_\theta))/(k/\varepsilon)$: time ratio(We took R as 0.5.)
$\overline{\theta^2}$	: variance of turbulent potential temperature
ε_θ	: $\alpha\overline{(\partial\theta/\partial x_i)^2}$
α	: thermal diffusivity.

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