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Modal transformation tools in structural dynamics and wind engineering

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Abstract. Structural dynamics usually applies modal transformation rules aimed at de-coupling and/or minimizing the equations of motion. Proper orthogonal decomposition provides mathematical and conceptual tools to define suitable transformed spaces where a multi-variate and/or multi-dimensional random process is represented as a linear combination of one-variate and one-dimensional uncorrelated processes. Double modal transformation is the joint application of modal analysis and proper orthogonal decomposition applied to the loading process. By adopting this method the structural response is expressed as a double series expansion in which structural and loading mode contributions are superimposed. The simultaneous use of the structural modal truncation, the loading modal truncation and the cross-modal orthogonality property leads to efficient solutions that take into account only a few structural and loading modes. In addition the physical mechanisms of the dynamic response are clarified and interpreted.

Key words: double modal transformation; modal analysis; proper orthogonal decomposition; structural dynamics; wind engineering.

1. Introduction

The dynamic response of linear structural systems is usually evaluated by transforming the equations of motion from the initial Lagrangian space into a space characterized by suitable properties. Using Classical Modal Analysis (CMA) (Hurty and Rubinstain 1964) the problem is solved in the principal space where, under conditions depending on damping (Caughey and O'Kelly 1965), the equations of motion are de-coupled. Re-writing the equations of motion in state space, a complex transformation exists which de-couples the equations of motion independently of damping properties (Foss 1958, Veletsos and Ventura 1986, Argyris and Mlejnek 1991). The new equations of motion, of the first order instead of the second, are complex. In both cases, the solution of a limited number of modal equations is usually enough to express the structural response. The substructure synthesis regards the structure as an assemblage of sub-structures whose motion is represented by a linear combination of admissible shapes: component-mode synthesis, branch-mode analysis and component-mode substitution (Meirovitch 1980) are special synthesis techniques which give up the de-coupling of the equations of motion by pursuing the aim of minimizing the number of equations to be solved. All the above methods apply transformation rules based on structural modal shapes. External forces passively follow these transformations assuming, in the new space, a purely

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mathematical meaning lacking in physical properties.

Karhunen-Loeve expansion (Loeve 1955, Papoulis 1965), also known as the Proper Orthogonal Decomposition (POD), provides mathematical and conceptual tools to extend many concepts traditionally used in the mechanical sector to random processes. Proposed around the mid 40's by several independent sources (Kosambi 1943, Loeve 1945, Karhunen 1946, Kac and Siegert 1947), POD expresses a generalized multi-variate/multi-dimensional stochastic process by a series of orthonormal vectors/functions, the eigenvectors/eigenfunctions of the covariance matrix/function, whose coefficients are reciprocally uncorrelated. It owes its popularity to the attractive properties that only few terms of the series are usually needed to reproduce the actual process and a link often exists between each dominant term of the series and the different main mechanisms that contribute to the overall physical phenomenon.

POD was first applied in meteorology (Lorenz 1959, Holmstrom 1963, Freiberger and Grenander 1967), where is also known as the Empirical Orthogonal Function (EOF) expansion (Obukhov 1960), for mapping meteorological fields (Obled and Creutin 1986). Lumley (1967, 1970) introduced POD in fluid mechanics to extract flow organized structures, such as dominant eddies, from the stochastic turbulent field, and represent these by deterministic functions; initially restrained by the lack of the necessary experimental data (Bakewell and Lumley 1967), its use has become ever more usual (Aubry *et al.* 1988, Moin and Moser 1989, Berkooz *et al.* 1993, Holmes *et al.* 1996) in recent years. Armitt (1968) pioneered the application of POD in bluff body aerodynamics by analyzing the wind pressure field on a cooling tower; in subsequent years many researchers (Lee 1975, Best and Holmes 1983, Kareem and Cermak 1984, Kareem *et al*. 1989, Holmes 1990, MacDonald *et al.* 1990, Letchford and Mehta 1993, Bienkiewicz *et al.* 1993, 1995, Tamura *et al.* 1997, Kikuchi *et al.* 1997, Holmes *et al.* 1997, Uematsu *et al.* 1997, Kareem and Cheng 1999, Tamura *et al.* 1999, Baker 2000) followed his example by using POD to represent and understand wind tunnel and full-scale pressure measurements on a great variety of buildings. Analogous methods have been applied by Tumer *et al*. (2000) to represent hydrodynamic actions on slender cylinders in oscillating flows. Applications of POD in data compaction and reduction and in pattern recognition and image processing are reported by Ahmed and Rao (1975) and Devijver and Kittler (1982), respectively. POD has been also applied to formulate a spectral stochastic finite-element technique (Spanos and Ghanem 1989, Ghanem and Spanos 1990, 1991a, 1991b, Ghanem and Brzakala 1996). Ghanem and Spanos (1993) applied POD in the solution of nonlinear vibration problems for representing the stochastic loading term; in this context, POD was also applied as a tool for investigating the motion of nonlinear mechanical systems simulated by numerical (Kreuzer and Kust 1996, Fenni and Kappugantu 1998, Georgiou and Schwartz 1999) and physical models (Benedettini and Rega 1997, Alaggio and Rega 2000), pointing out bifurcation conditions and transition to chaos. The use of POD as a spectral decomposition (Lumley 1970) embedded in a Monte Carlo simulation procedure was proposed by Li and Kareem (1989, 1991); methods for the simulation of wind, wave and earthquake fields have been developed by Li and Kareem (1991, 1993), Caddemi and Di Paola (1994), Di Paola and Pisano (1996), Di Paola (1998) and Carassale and Solari (2000a,b). Spectral modes have been recently applied to determine the wind-excited response of structures (Gullo *et al*. 1998, Benfratello *et al*. 1998, Carassale *et al*. 1998, 1999a,b, Kareem 1999, Carassale and Solari 1999, 2000a,b) and the seismic response of multi-supported structural systems (Carassale *et al*. 2000, Zingales 2000). Masri *et al*. (1998) applied POD to represent the nonstationary seismic motion through the eigensolutions of the covariance function.

The joint expansion of the Lagrangian motion coordinates by CMA and of the loading random process by POD using covariance and/or spectral modes is called Double Modal Transformation (DMT) (Carassale *et al*. 1998, 1999a,b, Carassale and Solari 1999) and provides a vast range of operative possibilities whose limits are probably still unknown. Through DMT the dynamic response is expressed by a double linear combination of structural and loading modes weighted by Structural Principal Coordinates (SPCs) and Loading Principal Components (LPCs), respectively. In principle, each SPC is excited by each LPC. Actually, due to Structural Modal Truncation (SMT), only few SPCs contribute to the dynamic response. Similarly, due to Loading Modal Truncation (LMT), only few LPCs contribute to the loading process. Cross-Modal Orthogonality (CMO) properties often exist which further simplify the solution by making several SPCs unexcited by given LPCs. It follows that the dynamic response of structures to multi-variate and/or multi-dimensional loading processes can be generally expressed by retaining only a few structural and loading modes. The case in which one structural mode and one loading mode fully represent the dynamic response is not unusual.

This paper provides a general framework and some critical remarks about CMA and POD. The basic elements of these methods are used to present and discuss DMT as a new tool to determine the dynamic response of multi-degree-of-freedom/continuous linear structures excited by multi-variate/multi-dimensional weakly-stationary Gaussian processes. Attention is focused on the main properties of the covariance and spectral eigensolutions, on the peculiarities of time domain and frequency domain approaches, and on the different aspects of discrete and continuous modeling. The conclusions deal with the application field of these criteria and with their advantages and disadvantages. Some prospects for future developments are also presented.

2. Classical modal analysis of MDOF linear systems

Consider an M-Degrees-Of-Freedom (MDOF) linear structure whose equation of motion is given by :

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boldsymbol{C}\dot{\boldsymbol{q}}(t) + \boldsymbol{K}\boldsymbol{q}(t) = \boldsymbol{A}\boldsymbol{v}(t) \tag{1}$$

where $\boldsymbol{q}(t) = \{q_1(t) \,..\, q_M(t)\}^T$ is the Lagrangian displacement vector, $\dot{\boldsymbol{q}}(t)$ and $\ddot{\boldsymbol{q}}(t)$ are the vectors of the structural velocities and accelerations; $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the mass, viscous damping and stiffness matrices of the structure; $\boldsymbol{f}(t) = \boldsymbol{A}\boldsymbol{v}(t)$ is the Lagrangian loading vector where $\boldsymbol{v}(t) = \{v_1(t) \,..\, v_N(t)\}^T$ is a weakly-stationary N-variate nil mean Gaussian process and $\boldsymbol{A}$ is an $M \times N$ deterministic matrix. The hypothesis that $\boldsymbol{v}(t)$ is nil mean does not involve any restriction in the linear field.

Let $\omega_1^2, \,..\, \omega_M^2$ be the structural eigenvalues corresponding to the squared natural circular frequencies sorted in increasing order; $\boldsymbol{\psi}_1, \,..\, \boldsymbol{\psi}_M$ are the corresponding structural eigenvectors. They are the non-trivial solutions of the homogeneous linear algebraic system :

$$(\boldsymbol{K} - \omega_j^2 \boldsymbol{M})\boldsymbol{\psi}_j = \boldsymbol{0} \quad (j = 1, \,..\, M) \tag{2}$$

Since $\boldsymbol{K}$ and $\boldsymbol{M}$ are real, symmetric and positive definite matrices, their eigenvalues are real and positive; their eigenvectors are real and enjoy the orthonormality conditions :

$$\boldsymbol{\Psi}^T\boldsymbol{M}\boldsymbol{\Psi} = \boldsymbol{I}\,; \qquad \boldsymbol{\Psi}^T\boldsymbol{K}\boldsymbol{\Psi} = \boldsymbol{\Omega} \tag{3}$$

where $\boldsymbol{\Psi} = [\boldsymbol{\psi}_1 \,..\, \boldsymbol{\psi}_M]$ is the $M \times M$ non-singular structural modal matrix; $\boldsymbol{I}$ is the identity matrix;

$\boldsymbol{\Omega} = \mathbf{diag}\,\{\omega_1^2, .. \,\omega_M^2\}$ is the diagonal matrix of the structural eigenvalues.

Eq. (1) is usually solved by applying the principal transformation rule :

$$\boldsymbol{q}(t) = \boldsymbol{\Psi}\boldsymbol{p}(t) = \sum_1^M {}_j\,\boldsymbol{\psi}_j p_j(t) \tag{4}$$

where $\boldsymbol{p}(t) = \{p_1(t) .. \,p_M(t)\}^T$ is the vector of the structural principal coordinates, i.e., the image of $\boldsymbol{q}(t)$ in the principal space. If the structure has classical vibration modes (Caughey and O'Kelly 1965), the substitution of Eq. (4) into Eq. (1) de-couples the equations of motion in the principal space :

$$\ddot{p}_j(t) + 2\xi_j\omega_j\dot{p}_j(t) + \omega_j^2 p_j(t) = \boldsymbol{\psi}_j^T\boldsymbol{A}\boldsymbol{v}(t) \quad (j = 1, ..M) \tag{5}$$

ξ_j being the j-th damping ratio. Structural modal truncation consists in expressing the structural response (Eq. 4) by considering only a limited number $M_t < M$ of the structural modes.

3. Discrete proper orthogonal decomposition

Discrete POD is the expansion of a multi-variate random process into a series of orthogonal vectors whose coefficients are mono-variate uncorrelated random processes. It is called Covariance Proper Transformation (CPT) or Spectral Proper Transformation (SPT) according to whether the orthogonal vectors are the eigenvectors of the covariance matrix or of the spectral density matrix of the process. CPT and SPT are linked by noteworthy relationships.

3.1. Covariance proper transformation

Let $\boldsymbol{C}_v = E\,[\boldsymbol{v}(t)\,\boldsymbol{v}^T(t)]$ be the covariance matrix of $\boldsymbol{v}(t)$ at the zero time lag, where $E[\bullet]$ is the statistic average operator. Let $\lambda_1, .. \,\lambda_N$ be the eigenvalues of $\boldsymbol{C}_v$, called covariance eigenvalues; $\boldsymbol{\phi}_1, .. \,\boldsymbol{\phi}_N$ are the corresponding covariance eigenvectors. They are the non-trivial solutions of the linear homogeneous algebraic system :

$$(\boldsymbol{C}_v - \lambda_k\boldsymbol{I})\,\boldsymbol{\phi}_k = \boldsymbol{0} \quad (k = 1, .. \,N) \tag{6}$$

Since $\boldsymbol{C}_v$ is a real, symmetric and positive definite matrix, its eigenvalues are real and positive; its eigenvectors are real and enjoy the orthonormality conditions :

$$\boldsymbol{\Phi}^T\boldsymbol{\Phi} = \boldsymbol{I}\,; \qquad \boldsymbol{\Phi}^T\boldsymbol{C}_v\boldsymbol{\Phi} = \boldsymbol{\Lambda} \tag{7}$$

where $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1 .. \,\boldsymbol{\phi}_N]$ is the $N \times N$ non-singular covariance modal matrix; $\boldsymbol{\Lambda} = \mathbf{diag}\,\{\lambda_1, .. \,\lambda_N\}$ is the diagonal matrix of the covariance eigenvalues. Due to Eq. (7) $\boldsymbol{C}_v$ has the spectral decomposition :

$$\boldsymbol{C}_v = \boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^T = \sum_1^N {}_k\,\boldsymbol{\phi}_k\boldsymbol{\phi}_k^T\lambda_k \tag{8}$$

Using Karhunen–Loeve expansion (Loeve 1955), the Covariance Proper Transformation (CPT) is defined by :

$$\boldsymbol{v}(t) = \boldsymbol{\Phi}\boldsymbol{x}(t) = \sum_1^N {}_k\,\boldsymbol{\phi}_k x_k(t) \tag{9}$$

where $\boldsymbol{x}(t) = \{x_1(t) \,..\, x_N(t)\}^T$ is the N-variate random process representing the image of $\boldsymbol{v}(t)$ in the covariance principal space; $x_k(t)$ is the k-th covariance principal component. The joint application of Eqs. (8) and (9) provides :

$$\boldsymbol{C}_x = \boldsymbol{\Lambda} \tag{10}$$

where $\boldsymbol{C_x} = E\,[\boldsymbol{x}(t)\boldsymbol{x}^T(t)]$ is the covariance matrix of $\boldsymbol{x}(t)$ at the zero time lag. Since $\boldsymbol{\Lambda}$ is diagonal, then $\boldsymbol{x}(t)$ is a vector of N processes uncorrelated at the zero time lag. Their variances are the covariance eigenvalues.

CPT admits modal truncation rules similar to CMA. By sorting covariance eigenvalues in decreasing order, $\boldsymbol{v}(t)$ may be usually approximated by a limited number $N_c < N$ of covariance modes. Evidence of this fact in wind engineering was pointed out by simulating measured pressure fields on cooling towers (Armitt 1968), square cylinders in two-dimensional flows (Lee 1975), low-rise buildings (Best and Holmes 1983, Holmes 1990, Letchford and Mehta 1993, Bienkiewicz *et al.* 1993, 1995, Tamura *et al.* 1997, Holmes *et al.* 1997), circular storage bins, silos and tanks (MacDonald *et al.* 1990), circular cylinders of finite height (Kareem *et al.* 1989, Kareem and Cheng 1999), tall buildings (Kareem and Cermak 1984, Kikuchi *et al.* 1997, Tamura *et al.* 1999) and latticed domes (Uematsu *et al.* 1997). Analogous properties apply to turbulence and vortex wake representations by theoretical models (Carassale *et al.* 1998, 1999a).

It is also worthy to note that CPT often establishes links between different covariance modes and different physical phenomena (Holmes *et al.* 1997, Baker 2000). Main covariance modes of the wind pressure field on low-rise buildings, for instance, tendentially correspond to the separate contributions of longitudinal, lateral and vertical turbulent fluctuations (Holmes 1990, Tamura *et al.* 1997). Similarly, alongwind forces, crosswind forces and torsional moments on tall buildings may be associated to different modes each dominated by the distinct effects of atmospheric turbulence and vortex shedding (Kareem and Cermak 1984, Kikuchi *et al.* 1997).

3.2. Spectral proper transformation

Consider the power spectral density matrix (psdm) $\boldsymbol{S_v}(\omega)$ of $\boldsymbol{v}(t)$, ω being the circular frequency. It is normalized by the relationship :

$$\boldsymbol{C_v} = \int_{-\infty}^{\infty} \boldsymbol{S}_v(\omega)\,d\omega \tag{11}$$

Let $\gamma_1(\omega), \,..\, \gamma_N(\omega)$ be the eigenvalues of $\boldsymbol{S_v}(\omega)$, called spectral eigenvalues; $\boldsymbol{\theta}_1(\omega)\,, \,..\, \boldsymbol{\theta}_N(\omega)$ are the corresponding spectral eigenvectors. They are the non-trivial solutions of the linear homogeneous algebraic system :

$$[\boldsymbol{S}_v(\omega) - \gamma_k(\omega)\,\boldsymbol{I}\,]\boldsymbol{\theta}_k(\omega) = \boldsymbol{0} \qquad (k = 1, \,..\, N) \tag{12}$$

Since $\boldsymbol{S_v}$ is Hermitian and semi-positive definite, its eigenvalues are real and non-negative; its eigenvectors are in general complex and enjoy the orthonormality conditions :

$$\boldsymbol{\Theta}^{*T}(\omega)\Theta(\omega) = \boldsymbol{I}\;; \qquad \boldsymbol{\Theta}^{*T}(\omega)\boldsymbol{S}_v(\omega)\boldsymbol{\Theta}(\omega) = \boldsymbol{\Gamma}(\omega) \tag{13}$$

where $\boldsymbol{\Theta}(\omega) = [\boldsymbol{\theta}_1(\omega) \,..\, \boldsymbol{\theta}_N(\omega)]$ is the $N \times N$ non-singular spectral modal matrix ; $\boldsymbol{\Theta}^*(\omega)$ is the complex conjugate of $\boldsymbol{\Theta}(\omega)$; $\boldsymbol{\Gamma}(\omega) = \textbf{diag}\,\{\gamma_1(\omega), \,..\, \gamma_N(\omega)\}$ is the diagonal matrix of the spectral eigenvalues. Due to Eq. (13) $\boldsymbol{S_v}(\omega)$ has the spectral decomposition :

$$\boldsymbol{S_v}(\omega) = \boldsymbol{\Theta}(\omega)\boldsymbol{\Gamma}(\omega)\boldsymbol{\Theta}^{*T}(\omega) = \sum_{1}^{N}{}_k \boldsymbol{\theta}_k(\omega)\boldsymbol{\theta}_k^{*T}(\omega)\gamma_k(\omega) \tag{14}$$

Together with Cholesky's decomposition (Meirovitch 1980), Eq. (14) belongs to the class of the infinite possible decompositions of spectral matrices (Li and Kareem 1995, Di Paola 1998, Kareem 1999).

Since $\boldsymbol{v}(t)$ is a weakly-stationary random process, the classical Fourier transform cannot be applied (Lin 1967). However, using the theory of generalized functions, the Spectral Proper Transformation (SPT) may be formally defined as (Lumley 1970) :

$$\boldsymbol{V}(\omega) = \boldsymbol{\Theta}(\omega)\boldsymbol{Y}(\omega) = \sum_{1}^{N}{}_k \boldsymbol{\theta}_k(\omega)Y_k(\omega) \tag{15}$$

where $\boldsymbol{V}(\omega)$ is the generalized Fourier transform of $\boldsymbol{v}(t)$; $\boldsymbol{Y}(\omega) = \{Y_1(\omega) \,..\, Y_N(\omega)\}^T$ is the generalized Fourier transform of the N-variate random process $\boldsymbol{y}(t) = \{y_1(t) \,..\, y_N(t)\}^T$ representing the image of $\boldsymbol{v}(t)$ in the spectral principal space; $y_k(t)$ is the k-th spectral principal component; $Y_k(\omega)$ is the generalized Fourier transform of $y_k(t)$. The joint application of Eqs. (14) and (15) provides :

$$\boldsymbol{S_y}(\omega) = \boldsymbol{\Gamma}(\omega) \tag{16}$$

where $\boldsymbol{S_y}(\omega)$ is the psdm of $\boldsymbol{y}(t)$. Since $\boldsymbol{\Gamma}(\omega)$ is diagonal, $\boldsymbol{y}(t)$ is a vector of N one-variate independent processes whose power spectral density functions are the spectral eigenvalues. Alternative expressions can be used, based on Fourier-Stieltjes integrals and spectral distribution matrices (Priestley 1981).

Calculating the inverse generalized Fourier transform of Eq. (15), SPT can be rewritten by the relationship (Carassale *et al.* 1999a) :

$$\boldsymbol{v}(t) = \boldsymbol{L}[\boldsymbol{y}(t)] = \sum_{1}^{N}{}_k \boldsymbol{l}_k[y_k(t)] \tag{17}$$

in which $\boldsymbol{L} = [\boldsymbol{l}_1 \,..\, \boldsymbol{l}_N]$ is a linear matrix operator, $\boldsymbol{l}_1, .. \boldsymbol{l}_N$ being linear vector operators such that (Carassale and Solari 1999) :

$$\boldsymbol{L}[\,\bullet\,] = \boldsymbol{G}(t)*[\,\bullet\,];\ \boldsymbol{l}_k[\,\circ\,] = \boldsymbol{g}_k(t)*[\,\circ\,] \quad (k = 1, .. N) \tag{18}$$

where $\boldsymbol{G}(t) = [\boldsymbol{g}_1(t) \,..\, \boldsymbol{g}_N(t)]$, $\boldsymbol{G}(t)$ and $\boldsymbol{g}_k(t)$ being the inverse Fourier transforms of $\boldsymbol{\Theta}(\omega)$ and $\boldsymbol{\theta}_k(\omega)$; symbol * denotes the convolution product. Basic criteria for realizing a system of stochastic differential equations corresponding to Eq. (17) are discussed by Kailath (1980).

Likewise CPT, also SPT usually allows one to express $\boldsymbol{v}(t)$ by a limited number $N_s < N$ of spectral modes Eqs. (15) and (17), by sorting spectral eigenvalues in decreasing order. However, differently from previous case, the ordering of the eigenvalues and the number of modes to be retained generally depend on the frequency; this calls for evaluations to be carried out case by case. Proofs of this property have been pointed out by Di Paola (1998) and Carassale and Solari (1999) simulating digitally multi-variate wind velocity fields and by Carassale *et al*. (1999a) analyzing theoretical turbulence and vortex shedding models.

Although no specific analysis has been yet carried out to point out the existence of links between different spectral modes and different physical phenomena, it is to be expected that, where such links are established by CPT, these are confirmed and clarified by SPT.

3.3. Relationships linking CPT and SPT

Replacing Eqs. (8) and (14) into Eq. (11) provides the following formula linking CPT and SPT (Carassale *et al.* 1999a) :

$$\boldsymbol{\Phi\Lambda\Phi}^T = \int_{-\infty}^{\infty} \boldsymbol{\Theta}(\omega)\boldsymbol{\Gamma}(\omega)\boldsymbol{\Theta}^{*T}(\omega)d\omega \tag{19}$$

The problem considerably simplifies when spectral eigenvectors are independent of frequency, i.e., $\boldsymbol{\Theta}(\omega) = \overline{\boldsymbol{\Theta}}$, $\boldsymbol{G}(t) = \overline{\boldsymbol{\Theta}}\delta(t)$, $\delta(t)$ being Dirac's function. In this case covariance eigenvectors and spectral eigenvectors coincide, while covariance eigenvalues are the frequency integrals of spectral eigenvalues (Carassale and Solari 1999) :

$$\overline{\boldsymbol{\Theta}} = \boldsymbol{\Phi}\ ; \quad \boldsymbol{\Lambda} = \int_{-\infty}^{\infty} \boldsymbol{\Gamma}(\omega)d\omega \tag{20}$$

Then SPT (Eq. 15) coincides with CPT Eq. (9) :

$$\boldsymbol{y}(t) = \boldsymbol{x}(t)\ ;\ \boldsymbol{L}[\,\bullet\,] = \boldsymbol{\Phi}\ ;\ \boldsymbol{l}_k[\,\circ\,] = \boldsymbol{\phi}_k \quad (k = 1, ..\, N) \tag{21}$$

which means that CPT makes the covariance principal components uncorrelated for any frequency and any time lag.

4. Double modal transformation in discrete modeling

DMT is the joint application of structural CMA and loading POD through CPT and SPT. The use of this technique in the time domain (Carassale and Solari 1999) and in the frequency domain (Carassale *et al*. 1999a) offers a broad view of its most relevant properties.

4.1. Time domain solution

The time domain application of DMT implies the joint solution of Eqs. (5) and (17). They form an $(M+N)$ system of linear equations whose solution involves the digital simulation of M independent processes $y_k(t)$ $(k = 1, ..\, N)$. Under suitable conditions concerning $\boldsymbol{L}[\,\bullet\,]$, the above system becomes differential.

In the case in which the spectral eigenvectors are independent of frequency, the problem drastically simplifies and assumes noteworthy analytical and conceptual properties. In this particular case CPT and SPT coincide Eq. (21) and the substitution of Eq. (9) into Eq. (5) provides :

$$\ddot{p}_j(t) + 2\xi_j\omega_j\dot{p}_j(t) + \omega_j^2 p_j(t) = \sum_{1}^{N}{}_k B_{jk}x_k(t) \qquad (j = 1, ..M) \tag{22}$$

where $B_{jk} = \boldsymbol{\psi}_j^T\boldsymbol{A\phi}_k$ quantifies the influence of the k-th loading covariance mode on the j-th structural mode. It is the j, k-th term of the $M \times N$ cross-modal participation covariance matrix :

$$\boldsymbol{B} = \boldsymbol{\Psi}^T \boldsymbol{A} \boldsymbol{\Phi} \tag{23}$$

Let us consider the $M \times N$ differential equations (Carassale and Solari 1999) :

$$\ddot{Z}_{jk}(t) + 2\xi_j \omega_j \dot{Z}_{jk}(t) + \omega_j^2 Z_{jk}(t) = x_k(t) \qquad (j = 1,..M;\ k = 1,..N) \tag{24}$$

whose solutions $Z_{jk}(t)$ are called partial principal coordinates. The global principal coordinates are obtained through the linear combination :

$$p_j(t) = \sum_1^N {}_k B_{jk} Z_{jk}(t) \qquad (j = 1,..M) \tag{25}$$

Replacing Eq. (25) into Eq. (4), the Lagrangian displacement vector is given by a double linear combination of structural modes and loading modes :

$$\boldsymbol{q}(t) = \sum_1^M {}_j \sum_1^M {}_k \boldsymbol{q}^{(jk)}(t) \tag{26}$$

$$\boldsymbol{q}^{(jk)}(t) = \boldsymbol{\psi}_j B_{jk} Z_{jk}(t) \qquad (j = 1, ..\ M;\ k = 1, ..\ N) \tag{27}$$

where $\boldsymbol{q}^{(jk)}(t) = \{q_1^{(jk)}(t)\ ..\ q_M^{(jk)}(t)\}^T$ is the j, k-th component term of $\boldsymbol{q}(t)$ due to the j-th structural mode and the k-th loading mode.

Matrix $\boldsymbol{B}$ contains many coefficients that are negligible or rigorously null. Due to structural modal truncation, only $M_t < M$ structural principal coordinates contribute to the response. Due to covariance modal truncation, only $N_c < N$ covariance principal components contribute to the excitation. Due to the reciprocal shape of structural and loading covariance eigenvectors, it often happens that the j-th structural mode is weakly influenced by the k-th loading covariance mode; in this case the eigenvectors are said to be quasi-orthogonal with respect to $\boldsymbol{A}$ ($B_{jk} = \boldsymbol{\psi}_j^T \boldsymbol{A} \boldsymbol{\phi}_k \simeq 0$) ; $\boldsymbol{\psi}_j$ is said to be orthogonal to $\boldsymbol{\phi}_k$ with respect to $\boldsymbol{A}$ when $B_{jk} = 0$. It follows that structural response to multi-variate loading processes can be generally expressed by a double linear combination of few structural modes and few loading modes.

Fig. 1 illustrates some results of a time-domain application of DMT to determine the dynamic response of an $M = 3$ DOF system subjected to N=3 loading components (Carassale and Solari 1999). Fig. 1(a) shows the 9 component terms $q_2^{(jk)}$ ($j, k = 1,2,3$), Eq. (27) of $q_2(t)$. Fig. 1(b) shows the composition of $q_2(t)$ (Eq. 26), using all the 9 terms (solid line) and the 3 terms corresponding to ($j = k = 1$), ($j = k = 2$), ($j = 2$, $k = 3$) (dashed line).

Eq. (22) has two particular cases of noteworthy importance.

1. $\boldsymbol{A}$ is an $M \times M$ square matrix, i.e., the number N of the loading components is equal to the number M of the structural coordinates. When, moreover, $\boldsymbol{B}$ is diagonal ($\boldsymbol{\psi}_j^T \boldsymbol{A} \boldsymbol{\phi}_k = 0$ for every $j \neq k$), the cross-modal orthogonality property applies, i.e., the j-th principal coordinate is the dynamic response of a single-degree-of-freedom system excited by the j-th loading component :

$$\ddot{p}_j(t) + 2\xi_j \omega_j \dot{p}_j(t) + \omega_j^2 p_j(t) = B_{jj} x_j(t) \qquad (j = 1,..M) \tag{28}$$

The example shown in Fig. 2(a) (Di Paola 1998) is enlightening. The structural flexural mode (Fig. 2b) is excited by the first wind loading mode (Fig. 2d) which represents an alongwind force

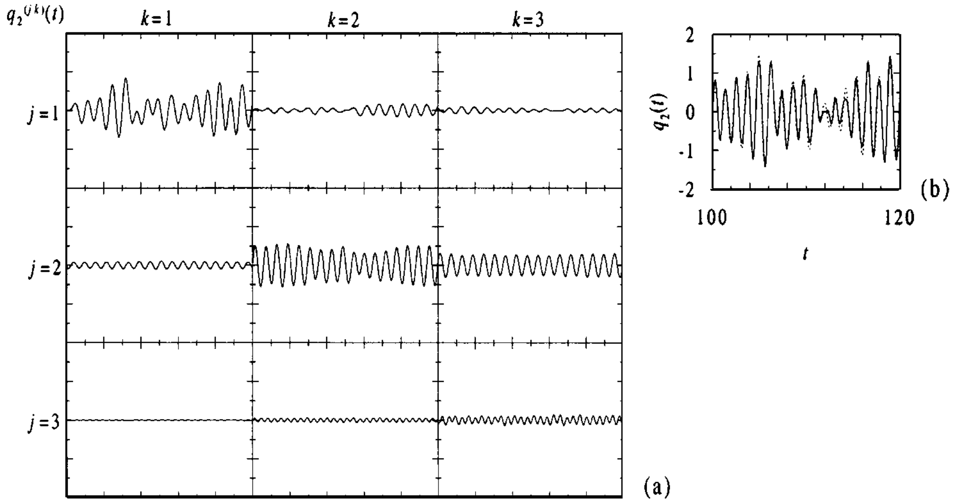

Fig. 1 (a) Component terms $q_2^{(jk)}$ of q_2; (b) composition of q_2 by all 9 terms (solid line) and by the 3 main terms (dashed line)

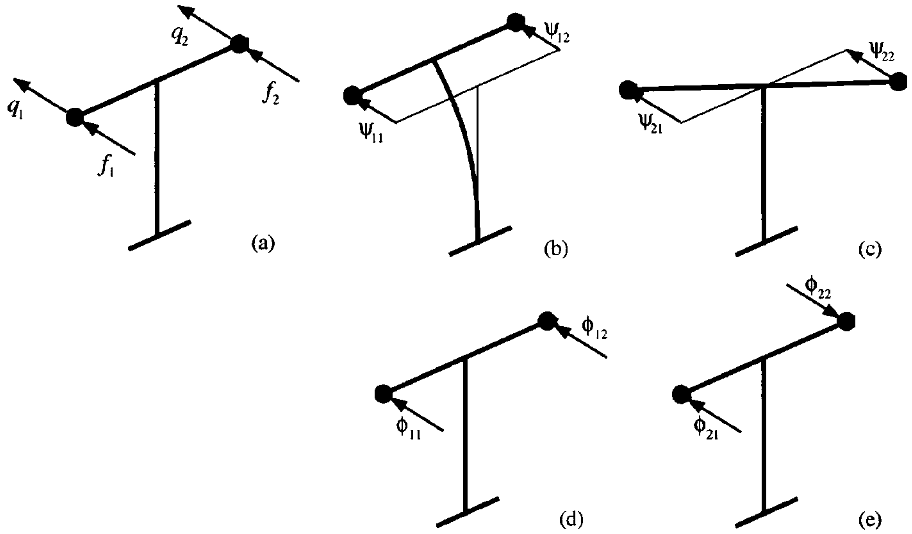

Fig. 2 (a) 2 DOF system excited by a 2 component wind loading; (b) structural flexural mode; (c) first wind loading mode; (d) structural torsional mode; (e) second wind loading mode

accounting for the uncorrelation of f_1 and f_2. The structural torsional mode (Fig. 2c) is excited by the second wind loading mode (Fig. 2e) which schematizes the torsional action due to the uncorrelation of f_1 and f_2.

The example shown in Fig. 3(a) points out analogous concepts with reference to the multi-support seismic excitation of a single story shear-type building (Carassale *et al.* 2000). The first structural mode (Fig. 3b) denotes a skew-symmetric vibration excited by the first seismic mode (Fig. 3d) which represents a uniform ground motion ($u_1 = u_2$). The second structural mode (Fig. 3c) is a symmetric vibration excited by the second seismic mode (Fig. 3e) which involves a motion of

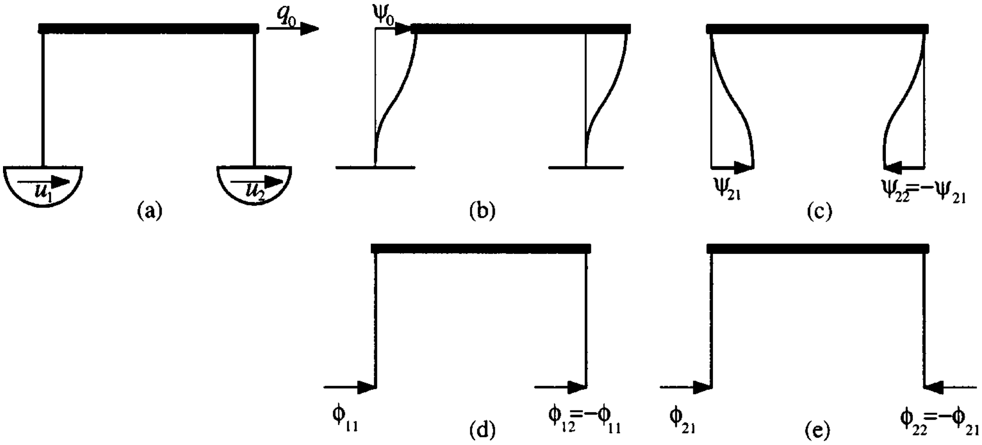

Fig. 3 (a) Single story shear building; (b) skew-symmetric first structural mode; (c) first seismic loading mode; (d) symmetric second structural mode; (e) second seismic loading mode

supports such as $u_1 = -u_2$.

2. $\boldsymbol{A} = \boldsymbol{a}$ is a column vector of M components, i.e., $\boldsymbol{f}(t) = \boldsymbol{a}v(t)$, where $v(t)$ is a one-variate (N=1) random process. In this case $\boldsymbol{B} = \boldsymbol{b}$ is a column vector whose j-th component $b_j = \boldsymbol{\psi}_j^T\boldsymbol{a}$ is a classical modal participation coefficient and $x_1(t) = v(t)$. DMT thus coincides with CMA and Eq. (22) becomes:

$$\ddot{p}_j(t) + 2\xi_j\omega_j\dot{p}_j(t) + \omega_j^2 p_j(t) = b_j v(t) \qquad (j = 1,..M) \tag{29}$$

which is the classical modal equation for structures excited by a mono-variate seismic motion.

4.2. Frequency domain solution

In contrast to using DMT in the time domain, the frequency domain approach does not imply any relevant difference between the cases in which the spectral eigenvectors depend or do not depend on the frequency. The joint use of Eqs. (5) and (14) provides the following expression of the psdm of the principal coordinates :

$$\boldsymbol{S_p}(\omega) = \boldsymbol{H}(\omega)\boldsymbol{D}(\omega)\boldsymbol{\Gamma}(\omega)\boldsymbol{D}^{*T}(\omega)\boldsymbol{H}^*(\omega) \tag{30}$$

where $\boldsymbol{H}(\omega) = \mathbf{diag}\,\{H_1(\omega), .. H_M(\omega)\}$; $H_j(\omega)$ is the complex frequency response function related to the j-th principal coordinate :

$$H_j(\omega) = \frac{1}{\omega_j^2 - \omega^2 + 2i\xi_j\omega\omega_j} \tag{31}$$

$\boldsymbol{D}(\omega)$ is the cross-modal participation spectral matrix :

$$\boldsymbol{D}(\omega) = \boldsymbol{\Psi}^T\boldsymbol{A}\boldsymbol{\Theta}(\omega) \tag{32}$$

i being imaginary unit. Therefore $\boldsymbol{D}(\omega) = \boldsymbol{B}$ for $\boldsymbol{\Theta}(\omega) = \boldsymbol{\Phi}$ (Eq. 23).

The variance of $p_j(t)$ is given by :

$$\sigma_{pj}^2 = \int_{-\infty}^{\infty} |H_j(\omega)|^2 \sum_{1}^{N}{}_k |D_{jk}(\omega)|^2 \gamma_k(\omega) d(\omega) \qquad (j = 1,..M) \tag{33}$$

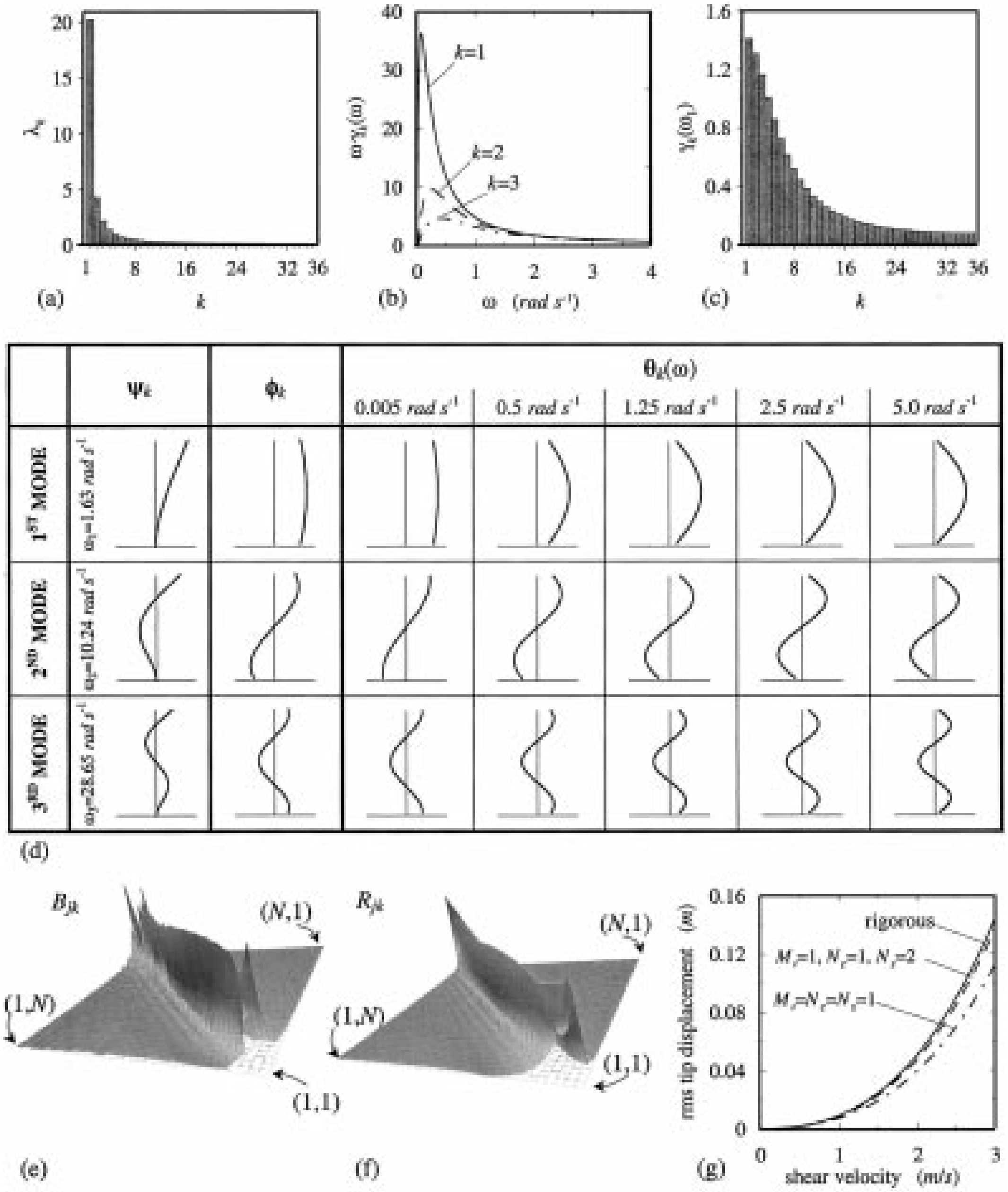

Fig. 4 Dynamic alongwind response of a chimney by DMT: (a,b,c) covariance and spectral turbulence eigenvalues; (d) structural eigenvectors, covariance and spectral turbulence eigenvectors; (e) matrix $\boldsymbol{B}$; (f) matrix $\boldsymbol{R}$; (e) root mean square (rms) value of tip displacement

where $D_{jk}(\omega) = \boldsymbol{\psi}_j^T \boldsymbol{A} \boldsymbol{\theta}_k(\omega)$ is the j, k-th term of the matrix $\boldsymbol{D}(\omega)$. Likewise B_{jk}, D_{jk} quantifies the influence of the k-th loading spectral mode on the j-th structural mode.

The solution of Eq. (33) can be simplified by examining loading processes with particular harmonic contents. Two special cases exist :

1. The loading process has a spectral content that quickly decreases on increasing the frequency as is typical of turbulence excitation. Assuming that structure is lightly damped and natural frequencies are well separated, Eq. (33) is suitably approximated by (Carassale *et al*. 1999a) :

$$\sigma_{pj}^2 = \frac{1}{\omega_j^2}\sum_1^N {}_k B_{jk}^2 \lambda_k + \frac{1}{8\,\omega_j^3 \xi_j}\sum_1^N {}_k R_{jk}^2 \gamma_k(\omega_j) \qquad (j = 1,..M) \tag{34}$$

where:

$$R_{jk} = |\, D_{jk}(\omega_j)\,| \tag{35}$$

R_{jk} being the j, k-th term of the matrix $\boldsymbol{R}$.

The first and the second terms on the right hand side of Eq. (34) correspond to the background and to the resonant parts of the response, respectively. Eq. (34) provides an algebraic solution of Eq. (33) involving only the eigensolutions of structure and loading process. Likewise Eq. (22), also Eq. (34) simplifies significantly when $\boldsymbol{B}$ and $\boldsymbol{R}$ are square diagonal matrices :

$$\sigma_{pj}^2 = \frac{1}{\omega_j^2} B_{jj}^2 \lambda_j + \frac{1}{8\,\omega_j^3 \xi_j} R_{jj}^2 \gamma_j(\omega_j) \qquad (j = 1,..M) \tag{36}$$

2. The loading process has a narrow band frequency harmonic content as in the case of vortex shedding. When the harmonic content is much lower than the fundamental structural frequency, the response is quasi-static and Eq. (34) holds neglecting the second term in the right hand side. When, otherwise, the excitation is resonant with the k-th natural frequency, then Eq. (34) usually holds neglecting the first term in the right hand side and assuming $\sigma_{pj}^2 = 0$ for $j \neq k$.

Fig. 4 summarizes the main results of a frequency domain application of DMT to determine the dynamic alongwind response of a chimney modeled by $M = N = 36$. Figs. 4(a)-(c) show the covariance and spectral eigenvalues of longitudinal turbulence; Fig. 4(d) points out the deep analogies between the structural eigenvectors and the covariance and spectral turbulence eigenvectors; Figs. 4(e),(f) demonstrate that, at least in this case, the matrices $\boldsymbol{B}$ and $\boldsymbol{R}$ are almost diagonal; Fig. 4(g) confirms that the structural response is accurately reproduced by few structural and loading modes. Full details concerning this study and other analyses of the alongwind and crosswind response of slender structures are given by Carassale *et al.* (1999a).

5. Continuous modeling

The discrete use of DMT involves formal and conceptual aspects characterized by great elegance, physical significance and a wide range of applications in structural dynamics and wind engineering. Nevertheless, DMT computational burden is quite similar to that required by classical solutions. On one hand it avoids a lot of traditional operations but, on the other, besides the evaluation of structural eigenvalues and eigenvectors, it also requires the determination of the eigenvalues and eigenvectors of external loading. This situation changes significantly when structural and/or loading eigensolutions are known in closed form. This is typical of several continuous problems governed by operators with suitable regularity properties.

The following paragraphs provide a general discussion of this matter referring, for simplicity, to structural systems and loading fields defined over the same mono-dimensional domain D. The generalization to multi-dimensional problems does not imply conceptual advances but only relevant formal complications.

5.1. Structural modal analysis

Consider a linear continuous mono-dimensional structure whose motion is governed by the partial differential equation:

$$\mu(z)\ddot{q}(z;t) + C[\dot{q}(z;t)] + K[q(z;t)] = a(z)v(z;t) \tag{37}$$

where z is the coordinate of D; $q(z;t)$, $\dot{q}(z;t)$, $\ddot{q}(z;t)$ are the displacement, velocity and acceleration of structure, respectively; $\mu(z)$ denotes the mass distribution; $C[\bullet]$ and $K[\bullet]$ are viscous damping and stiffness operators; $f(z;t) = a(z)v(z;t)$ is the external force, where $v(z;t)$ is a weakly-stationary nil mean Gaussian random field defined on D and $a(z)$ is a given deterministic function.

Let ω_1^2, ω_2^2, .. be the structural eigenvalues sorted in increasing order; $\psi_1(z)$, $\psi_2(z)$, .. are the corresponding structural eigenfunctions. They are the non-trivial solutions of the linear homogeneous Fredholm integral equation of the second kind :

$$\psi_j(z) = \omega_j^2 \int_D \eta(z, z')\mu(z')\psi_j(z')dz' \qquad (j = 1, 2, ..) \tag{38}$$

where $\eta(z, z')$ is Green's function related to K (Hurty and Rubinstain 1964). It is also known as the structural kernel.

Assuming that K is a real, self-adjoint and positive definite operator, then the eigenvalues are real and positive; the eigenfunctions are real, form a complete set and enjoy the following orthonormality conditions :

$$\int_D \mu(z)\psi_r(z)\psi_s(z)dz = \delta_{rs}; \quad \omega_r^2 \int_D \int_D \eta(z, z')\mu(z)\mu(z')\psi_r(z)\psi_s(z')dzdz' = \delta_{rs} \qquad (r, s = 1, 2, ..) \tag{39}$$

where δ_{rs} is Kronecker's delta. Noteworthy closed form expression of ω_j^2, $\psi_j(z)$ ($j = 1, 2, ..$) are available for uniform mass distributions, simple $K[\bullet]$ operators and particular constraint conditions (Hurty and Rubinstain 1964, Meirovitch 1967).

Eq. (37) is usually solved by applying the principal transformation rule :

$$q(z;t) = \sum_{1}^{\infty}{}_j \psi_j(z)p_j(t) \tag{40}$$

where $p_j(t)$ is the j-th principal coordinate. Under suitable conditions on $\boldsymbol{C}$ operator, the substitution of Eq. (40) into Eq. (37) leads to the following set of infinite independent equations :

$$\ddot{p}_j(t) + 2\xi_j\omega_j\dot{p}_j(t) + \omega_j^2 p_j(t) = \int_D \psi_j(z)a(z)v(z;t)dz \qquad (j = 1, 2, ..) \tag{41}$$

ξ_j being the j-th damping ratio ($j = 1, 2, ..$). Likewise for discrete modeling, also in continuous modeling the structural response can be usually expressed by a limited number M_t of modal terms .

5.2. Covariance proper transformation

Let $C_v(z, z') = E[v(z\,;t)v(z'\,;t)]$ be the covariance function of $v(z\,;t)$ and $v(z'\,;t)$ at the zero time lag; λ_1, λ_2, .. are the covariance eigenvalues; $\phi_1(z)$, $\phi_2(z)$, .. are the corresponding covariance eigenfunctions. They are the non-trivial solutions of the homogeneous Fredholm integral equation of the second kind :

$$\lambda_k \phi_k(z) = \int_D C_v(z, z')\phi_k(z')dz' \qquad (k = 1, 2, ..) \tag{42}$$

The kernel $C_v(z, z')$ of Eq. (42) is bounded, symmetric, real and positive definite. All the eigenvalues are real and positive while eigenfunctions are real, form a complete set and enjoy the orthonormality conditions (Kanwal 1971) :

$$\int_D \phi_r(z)\phi_s(z)dz = \delta_{rs}; \int_D \int_D C_v(z, z')\phi_r(z)\phi_s(z')dzdz' = \lambda_r \delta_{rs} \qquad (r, s = 1, 2, ..) \tag{43}$$

from which it derives :

$$C_v(z, z') = \sum_{1}^{\infty}{}_k \phi_k(z)\phi_k(z')\lambda_k \tag{44}$$

The case in which C_v is a degenerate kernel (Kanwal 1971), i.e., the series of existing eigenvalues and eigenfunctions is limited, does not imply relevant conceptual differences. However, it is not considered here for formal simplicity.

Closed formulae of λ_k, $\phi_k(z)$ $(k = 1, 2, ..)$ are given by Van Trees (1968) and Ghanem and Spanos (1991b) for noteworthy covariance kernels. Preliminary analytical eigensolutions of the covariance kernel of a theoretical turbulence model have been obtained by Carassale *et al.* (1999b).

Likewise Eq. (9), the continuous Covariance Proper Transformation (CPT) is defined by :

$$v(z;t) = \sum_{1}^{\infty}{}_k \phi_k(z)x_k(t) \tag{45}$$

where $x_1(t)$, $x_2(t)$, .. are the so-called covariance principal components. The joint use of Eqs. (44) and (45) provides :

$$C_{x_r x_s} = \lambda_r \delta_{rs} \quad (r, s = 1, 2, ..) \tag{46}$$

$C_{x_r x_s} = E[x_r(t)x_s(t)]$ being the covariance of $x_r(t)$ and $x_s(t)$. Thus, the variance of $x_r(t)$, $\sigma^2_{x_r} = C_{x_r x_r}$, coincides with the r-th eigenvalue λ_r. Furthermore, since $C_{x_r x_s} = 0$ for $r \neq s$, $x_r(t)$ and $x_s(t)$ are uncorrelated processes at the zero time lag.

Likewise for discrete modeling, also in continuous modeling $v(z\,;t)$ can be usually approximated by a limited number N_c of covariance terms.

5.3. Spectral proper transformation

Consider the cross-power spectral density function (cpsdf) $S_v(z, z'\,;\omega)$ of $v(z\,;t)$ and $v(z'\,;t)$. It is normalized by the relationship :

$$C_v(z, z') = \int_{-\infty}^{\infty} S_v(z, z';\omega) d\omega \tag{47}$$

Let $\gamma_1(\omega)$, $\gamma_2(\omega)$, .. be the spectral eigenvalues; $\theta_1(z\,;\omega)$, $\theta_2(z\,;\omega)$, .. are the corresponding spectral eigenfunctions. They are the non-trivial solutions of the homogeneous Fredholm integral equation of the second kind :

$$\gamma_k(\omega)\theta_k(z;\omega) = \int_D S_v(z, z';\omega)\theta_k(z';\omega)dz' \qquad (k = 1, 2, ..) \tag{48}$$

Since $S_v\,(z, z'\,;\,\omega)$ is a bounded, Hermitian and semi-positive definite kernel, all the eigenvalues are real and non-negative while the eigenfunctions, in general complex, form a complete set and enjoy the following orthonormality conditions :

$$\int_D \theta_r^*(z;\omega)\theta_s(z;\omega)dz = \delta_{rs}; \int_D\int_D S_v(z, z';\omega)\theta_r^*(z;\omega)\theta_s(z';\omega)dzdz' = \gamma_r(\omega)\delta_{rs} \qquad (r, s = 1, 2, ..) \tag{49}$$

from which it derives :

$$S_v(z, z';\omega) = \sum_{1}^{\infty}{}_k \theta_k(z;\omega)\theta_k^*(z';\omega)\gamma_k(\omega) \tag{50}$$

Likewise for C_v, also the case in which S_v is a degenerate kernel is not considered here for formal simplicity.

Closed form eigensolutions of the spectral kernel of a theoretical turbulence model have been obtained by Carassale *et al.* (1999b) and by Carassale and Solari (2000a,b). The use of these solutions in a Monte Carlo environment aimed at simulating stochastic turbulent fields is demonstrated and discussed by Carassale and Solari (2000a,b).

Likewise Eq. (17), the continuous Spectral Proper Transformation (SPT) is defined by :

$$v(z;t) = \sum_{1}^{\infty}{}_k L_k[y_k(t)] \tag{51}$$

where $y_1(t)$, $y_2(t)$, .. are the so-called spectral principal components; $L_1[\circ]$, $L_2[\circ]$, .. are linear operators such that :

$$S_{y_r y_s}(\omega) = \gamma_r(\omega)\delta_{rs} \quad (r, s = 1, 2, ..) \tag{52}$$

$S_{y_r y_s}(\omega)$ being the cpsdf of $y_r(t)$ and $y_s(t)$. Since $S_{y_r y_s}(\omega) = 0$ for $r \neq s$, $y_r(t)$ and $y_s(t)$ are one-variate independent processes whose psdf are the spectral eigenvalues.

The use of SPT in continuous modeling involves modal truncation rules that are conceptually the same of the corresponding discrete approach.

5.4. Relationships linking CPT and SPT

By replacing Eqs. (44) and (50) into Eq. (47), CPT and SPT are linked by :

$$\sum_{1}^{\infty}{}_{k}\phi(z)\phi_k(z')\lambda_k = \sum_{1}^{\infty}{}_{k}\int_{-\infty}^{\infty}\theta_k(z;\omega)\theta_k^*(z';\omega)\gamma_k(\omega)d\omega \tag{53}$$

Again the problem considerably simplifies when spectral eigenfunctions do not depend on frequency, i.e., $\theta_k(z;\omega) = \bar{\theta}_k(z)$. In this case the covariance and spectral eigenfunctions coincide, while the covariance eigenvalues are the frequency integrals of the spectral eigenvalues :

$$\bar{\theta}_k(z) = \phi_k(z); \quad \lambda_k = \int_{-\infty}^{\infty}\gamma_k(\omega)d\omega \qquad (k = 1, 2, ..) \tag{54}$$

Then SPT (Eq. 51) coincides with CPT (Eq.45) :

$$y_k(t) = x_k(t)\,; \quad L_k[\circ] = \phi_k(z) \quad (k = 1, 2, ..) \tag{55}$$

which means that CPT makes the covariance principal coordinates uncorrelated for any frequency and any time lag.

5.5. Double modal transformation

Continuous DMT does not involve significant conceptual differences with respect to the discrete approach.

Using DMT in the time domain implies the joint solution of Eqs. (37) and (51). They form a linear system whose solution involves the digital simulation of the independent processes $y_k(t)$ ($k = 1, 2, ..$) (Carassale and Solari 2000).

Fig. 5 shows some results of a time domain application of DMT to determine the dynamic alongwind response of a cantilever vertical beam (Carassale and Solari 2000a). Figs. 5(a) and (b) show a Monte Carlo simulation of two longitudinal turbulence histories (z being the height over ground and l the total structural height) by using one to five spectral turbulence modes. Fig. 5(c) shows the corresponding recomposition of the first principal structural coordinate.

Likewise in the discrete case, the problem simplifies if the spectral eigenfunctions are independent of frequency. In this case CPT and SPT coincide and the substitution of Eq. (45) into Eq. (41) leads to Eq. (22), provided that M and N are replaced by infinity and B_{jk} is defined as :

$$B_{jk} = \int_D a(z)\psi_j(z)\varphi_k(z)dz \qquad (j, k = 1, 2, ..) \tag{56}$$

Also Eqs. (28) and (29) can be extended to continuous modeling by a simple analogy with the discrete solution; all physical concepts remain unchanged.

Similarly, the frequency domain application of DMT does not imply relevant differences between the cases in which the spectral eigenfunctions depend or not on the frequency. The joint use of Eqs. (41) and (50) leads to Eq. (33), provided that M and N are replaced by infinity and D_{jk} is defined as :

$$D_{jk}(\omega) = \int_D a(z)\psi_j(z)\theta_k(z;\omega)dz \qquad (j, k = 1, 2, ..) \tag{57}$$

Also Eqs. (34)~(36) can be extended to continuous modeling by a simple analogy with the discrete solution; all physical concepts remain unchanged.

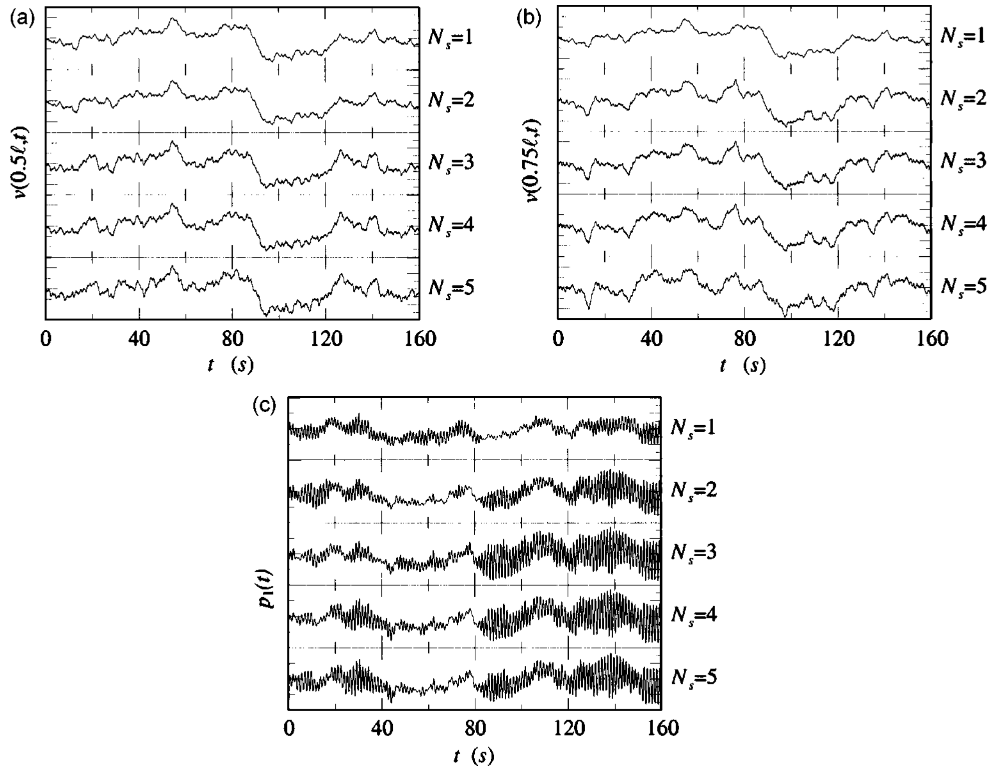

Fig. 5 Dynamic alongwind response of a vertical contilever beam: (a, b) POD recomposition of two longitudinal turbulence histories at different heights; (c) POD recomposition of the first principal structural coordinate

Noteworthy applications of this method have been developed by Carassale *et al.* (1999b) to determine the gust-excited alongwind response of slender structures. It is shown in particular that the knowledge of closed form eigensolutions makes the calculation of the dynamic response integrally analytical.

6. Conclusions

Modal transformation rules in classical structural dynamics are based on structural modal shapes. External forces passively follow these transformations assuming, in the new space, a purely mathematical meaning lacking in physical properties. Proper orthogonal decomposition provides mathematical and conceptual tools to extend most of these rules to stochastic loading processes.

Double modal transformation is the joint expansion of Lagrangian motion coordinates into a series of normal modes and of loading random process by POD technique using covariance and/or spectral modes. Using this method the dynamic response can be expressed as a double series in which few structural and loading modes are needed. This implies formal and conceptual aspects characterized by great elegance, physical significance and a wide range of applications. Nevertheless DMT computational burden is quite similar to that required by classical solutions. On one hand it avoids a lot of traditional operations but, on the other, besides the evaluation of structural eigensolutions, it also requires the determination of loading eigensolutions.

This situation changes significantly when structural and/or loading eigensolutions are known in closed form. This is typically the case of several continuous problems governed by suitable regularity properties. The closed form evaluation of the eigensolutions of continuous structural systems is a well-known field of structural dynamics. The closed form evaluation of the eigensolutions of the loading processes opens the door to a new research field aimed at defining the load through its eigensolutions instead of the classical spectral equations.

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