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Dynamic response of a bridge deck with one torsional degree of freedom under turbulent wind

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Abstract. Under special conditions of turbulent wind, suspension and cable-stayed bridges could reach instability conditions. In various instances the bridge deck, as like a bluff body, could exhibit single-degree torsional instability. In the present study the turbulent component of flow has been considered as a solution of a differential stochastic linear equation. The input process is represented by a Gaussian zero-mean white noise. In this paper the analytical solution of the dynamic response of the bridge has been determined. The solution has been obtained with a technique of closing on the order of the moments.

Key words: turbulent wind; stochastic analysis; Gaussian processes; bridges.

1. Introduction

The aeroelastic behaviour of the deck in the suspension and cable-stayed bridges is one of the most complex and relevant aspects for the security of the structure. In fact the wind action could cause the collapse of the bridge due to instability phenomena. Many mechanical models are adopted to describe the dynamic behaviour of long-span decks. The most utilised one is the section model with two degrees of freedom. This model has visco-elastic restraints that reproduce, dynamically, the characteristics of the whole structural system. Most complex models consider the whole 3-D structure of the bridge under wind forces taking into account both the tridimensional behaviour of the structure and the spatial distribution of the wind.

Recently, a model with four degree of freedom has been proposed in substitution of the classical section model. It is a non-linear model able to analyse the global vibrational modes of the structure and the modes relative to the cables and the deck. The preliminary study of the dynamic behaviour of a long span bridge subjected to a turbulent wind action is usually developed with a section-model of the deck. If the analysis is performed referring to the instantaneous velocity, the solution of the problem is more complex. In fact the presence of time-depending excitations is described with stochastic models; moreover under special conditions of motion, these sections could reach instability conditions. The classical flutter of bridge decks shows significant differences compared to the one relative to the thin airfoil. The centre of the mass is on the symmetry axis of the section and it is very close to the torsion centre; in this way the inertial coupling is limited and the part of the aeroelastic moment due to the rotation velocity, always negative on the thin airfoil, could change the sign for a deck for a unstreamlined section. As a result a reduction and even the inversion of the total

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damping is obtained.

In this case the dependence of the traslational action induced by the wind on the characteristics of the torsional motion is negligible. Therefore vertical and torsional motions of bridge may be taken as uncoupled. The aerodynamic coupling is of secondary importance especially in those cases where in single-degree torsional instability is manifest (Simiu, Scanlan 1986). In this paper the mathematical dynamic response of a bridge deck with a single torsional degree of freedom under turbulent flow is determined.

In the following Gaussian processes will be utilised; therefore the proposed solution is good only for small displacements. As a consequence the dynamic response under high wind velocity will be more approximate because it is observed that both mean and standard deviation values are 2 or 3 orders of magnitude higher with respect to the ones with lower wind velocity. In this case non-Gaussian character of the response should be adopted. Therefore the evaluation of the critical velocity is beyond the aim of this work. For common sections of bridge decks Scanlan and Tomko (1986) showed that the contribution of the second derivatives of the displacements of the deck model is negligible; at the same time they also studied a method to determine the aeroelastic coefficients.

The following hypotheses are assumed for the input forces: the direction of the wind is constant, the turbulent component $u(t)$ is exclusively in the direction of mean wind.

In the following the process $u(t)$ will be considered as a solution of a differential stochastic linear equation (equation of Langevin), where the input process is represented by a Gaussian zero-mean white noise. Starting from the probabilistic description of the force acting on the system, the problem is to determine the response process, through the temporal moment diagrams. The solution of the equation of motion will be obtained utilising the methods of the Stochastic Differential Calculus; the differential rule of Ito for writing the differential equations and the breaking method on the order of the cumulants (the stochastic equation of the motion is non-linear) will be applied.

2. Equations of motion

The model scheme is shown in Fig. 1. It has only one d.o.f., the torsional rotation (α) along the longitudinal axis of the deck. In fact, the influence of the drag and lift displacements along the direction coincident with the wind one has been neglected. The model is supported by elements with a stiffness and damping that simulate the real bridge behaviour.

The equation of motion of the system is:

$$I[\ddot{\alpha}(t) + 2\xi_\alpha\omega_\alpha\dot{\alpha}(t) + \omega_\alpha^2\alpha(t)] = M(t) \tag{1}$$

where, for a portion of the deck with a unit length:

- I is the mass polar moment;
- ω_α is the natural frequency of the system when the non-linearity is neglected;
- ξ_α is the damping coefficient;
- $M(t)$ represents the force applied to the system as effect of the wind.

The structural response is determined with the techniques from the stochastic differential calculus. In fact the process $u(t)$ is obtained from the following linear stochastic differential equation, (Bartoli,

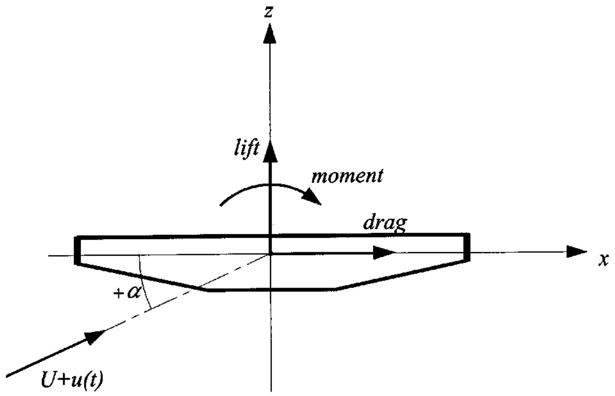

Fig. 1 Scheme of the bridge deck

Borri 1997):

$$\dot{u}(t) = -\theta_1 u(t) + \theta_2 w(t) \tag{2}$$

where *w(t)* is a Gaussian zero-mean white noise; θ_1 and θ_2 "modelise" the spectral density of the input.

In the present case the longitudinal turbulent part of the wind velocity is described with the Davenport spectrum, together with one rational approximation where the coefficient has been evaluated by a least-square procedure. In this case θ_1 and θ_2 are:

$$\begin{cases} \theta_1 = 2\pi U_{10}\dfrac{2.5767}{1200} \\ \theta_2 = 2\pi u^* \sqrt{U_{10}}\sqrt{\dfrac{4.790}{1200}} \end{cases} \tag{3}$$

where: $u^* = 0.4U_{10}\left[\ln\left(\dfrac{z_{10}}{z_0}\right)\right]^{-1}$

U_{10} is the mean velocity of the wind at 10 m of height from the ground; z_0 is the roughness length.

From the linearity characteristics of the process *u(t)* it is possible to state that:

- if the input is a zero-mean process, the response will have an expected value equal to zero;
- if the input is a Gaussian process, the response will have a Gaussian probability distribution;
- if the input is a stationary process, the output will be stationary too.

The aim of the present paper is to determine the dynamic response of decks which exhibit a single degree of freedom flutter. This behaviour is present in bluff and unstreamlined bodies which undergo strongly separated flows. Prominent among these are the decks of suspended-span bridges; they can exhibit single degree torsional instability.

If the turbulent component of the wind is considered, the buffeting and aeroelastic force *M(t)* applied to the system will be:

$$M(t) = 0.5\rho[U+u(t)]^2(2B_d^2)\left[kA_2^*(k)B_d\frac{\dot{\alpha}(t)}{[U+u(t)]} + (k^2A_3^*(k) - c_{2M})\alpha(t) + \frac{C_M(\alpha)}{2}\right] \tag{4}$$

where:

- $A_i^*(k)$ are the aeroelastic derivative;
- $C_M(\alpha) = c_{1M} + c_{2M}\alpha + c_{3M}\alpha^2 + c_{4M}\alpha^3$ is the buffeting coefficient;
- $k = B_d\omega / U$ is the reduced frequency, ω is the actual circular frequency of oscillation;
- B_d is the chord of the deck;
- ρ is air density.

In Eq. (4) the aeroelastic coefficient A_l^* does not appear, as it is negligible in case of bridges.

3. Probabilistic determination of the dynamic response

The equation of motion (1) and the differential Eq. (2), which describes the turbulent component, represent a non-linear system of differential stochastic equations:

$$\begin{cases} I(\ddot{\alpha}(t) + 2\xi_\alpha\omega_\alpha\dot{\alpha}(t) + \omega_\alpha^2\alpha(t)) = 0.5\rho(U+u(t))^2(2B_d^2) \\ \qquad \left[kA_2^*(k)B_d\frac{\dot{\alpha}(t)}{[U+u(t)]} + (k^2A_3^*(k) - c_{2M})\alpha(t) + \frac{C_M(\alpha)}{2}\right] \\ \dot{u}(t) = -\theta_1 u(t) + \theta_2 w(t) \end{cases} \tag{5}$$

System Eq. (5) could be reduced to a system of first order stochastic differential equations through the introduction of an unknown state vector $\overline{Y} \equiv \{Y_1, Y_2, Y_3\}$ where:

$$Y_1 = \alpha(t) \qquad Y_2 = \dot{\alpha}(t) \quad Y_3 = u(t) \tag{6}$$

Introducing the vector $\overline{Y}$ and neglecting the square of turbulence respect to the product of the turbulent component and the mean one, the system can be expressed as:

$$\begin{cases} dY_1 = Y_2 dt \\ dY_2 = -2\xi_\alpha\omega_\alpha Y_2 dt - \omega_\alpha^2 Y_1 dt \\ \quad + \frac{\rho B_d^2}{I}dt\begin{pmatrix} kA_2^*B_dUY_2 + (k^2A_3^* - c_{1M})U^2Y_1 + kA_2^*B_dY_2Y_3 + 2(k^2A_3^* - c_{1M})UY_1Y_3 + \\ \frac{c_{0M}}{2}U^2 + \frac{c_{1M}}{2}U^2Y_1 + \frac{c_{2M}}{2}U^2Y_1^2 + \frac{c_{3M}}{2}U^2Y_1^3 + c_{0M}UY_3 + \\ c_{1M}UY_1Y_3 + c_{2M}UY_1^2Y_3 + c_{3M}UY_1^3Y_3 \end{pmatrix} \\ dY_3 = -\theta_1 Y_3 dt + \theta_2 w dt \end{cases} \tag{7}$$

We introduce a scalar function $\Phi(\overline{Y}) = \prod_{j=1}^{3} \overline{Y}_{\alpha j}^{\beta j}$ with real values, differentiable respect to t and with mix second partial derivatives of the continue components of the state vector, with α_j and β_j (j =1, 2, 3) two set of integer non-negative numbers. The functions $\Phi(\bar{Y})$ can be chosen arbitrarily.

Applying many times the derivation rule of Ito to the functions $\Phi(\bar{Y})$ it is possibly to generate a system of non-linear ordinary differential equations that includes, as unknown, the moments of order up to the third of the stochastic process constituting the state vector.

The assumption of the following vector:

$$\tilde{\Phi}(\bar{Y}) = \{Y_1, Y_2, Y_3, Y_1^2, Y_2^2, Y_3^2, Y_1Y_2, Y_1Y_3, Y_2Y_3\} \tag{8}$$

will lead to a system with nine ordinary differential equations.

The equations have been obtained neglecting the following moments (see Appendix I):

$$E[Y_3w] = 0, \quad E[Y_1w] = 0, \quad E[Y_2w] = 0.$$

As the original stochastic differential system is non-linear, a sequence of coupled equations with an infinite hierarchy is obtained. Such a system is impossible to solve, unless the hierachy is interrupted. Therefore the solution is usually obtained with a technique of closing on the order of the cumulants. Since the Gaussian distribution in the only one completely characterised with the first two cumulants, the closing technique will consist in neglecting the cumulants of order ≥ 3.

The expressions of the third order of moments obtained in function of the moments of lower order by mean of a Gaussian closing on the cumulants to eliminate the infinity hierarchy and the system are shown in Appendix II. This system of nine ordinary differential equations is non-linear. The time evolution of the moments is only obtained with the numerical integration. Approximate evaluations of the moments can be obtained with a Gaussian closing directly on the moments. With the closing on the moments it is possible to obtain a system with ordinary differential linear equations:

$$\frac{d}{dt}E[Y_1] = E[Y_2]$$

$$\frac{d}{dt}E[Y_2] = -2\xi_\alpha\omega_\alpha E[Y_2] - \omega_\alpha^2 E[Y_1]$$

$$+\frac{\rho B_d^2}{I}\left\{\begin{array}{l} kA_2^* B_d U E[Y_2] + \left(k^2 A_3^* U^2 - \dfrac{c_{2M}U^2}{2}\right)E[Y_1] + kA_2^* B_d E[Y_2Y_3] + \\ +(2k^2A_3^*U - c_{2M}U)E[Y_1Y_3] + \dfrac{c_{1M}U^2}{2} \end{array}\right\}$$

$$\frac{d}{dt}E[Y_3] = -\theta_1 E[Y_3] = 0$$

$$\frac{d}{dt}E[Y_1^2] = 2E[Y_1Y_2]$$

$$
\begin{aligned}
\frac{d}{dt}E[Y_2^2] &= -4\xi_\alpha\omega_\alpha E[Y_2^2] - 2\omega_\alpha^2 E[Y_1Y_2] \\
&\quad + 2\frac{\rho B_d^2}{I}\left\{\begin{array}{l} kA_2^* B_d U E[Y_2^2] + \left(k^2 A_3^* U^2 - \dfrac{c_{2M}U^2}{2}\right)E[Y_1Y_2] + \\ + \dfrac{c_{1M}U^2}{2}E[Y_2] + c_{1M}UE[Y_2Y_3] \end{array}\right\} \\
\frac{d}{dt}E[Y_3^2] &= -2\theta_1 E[Y_3^2] + \theta_2^2 \\
\frac{d}{dt}E[Y_1Y_2] &= E[Y_2^2] - 2\xi_\alpha\omega_\alpha E[Y_1Y_2] - \omega_\alpha^2 E[Y_1^2] \\
&\quad + \frac{\rho B_d^2}{I}\left\{\begin{array}{l} kA_2^* B_d U E[Y_1Y_2] + \left(k^2 A_3^* U^2 - \dfrac{c_{2M}U^2}{2}\right)E[Y_1^2] + \\ + \dfrac{c_{1M}U^2}{2}E[Y_1] + c_{1M}UE[Y_1Y_3] \end{array}\right\} \\
\frac{d}{dt}E[Y_1Y_3] &= E[Y_2Y_3] - \theta_1 E[Y_1Y_3] \\
\frac{d}{dt}E[Y_2Y_3] &= -\theta_1 E[Y_2Y_3] - 2\xi_\alpha\omega_\alpha E[Y_2Y_3] - \omega_\alpha^2 E[Y_1Y_3] \\
&\quad + \frac{\rho B_d^2}{I}\left\{\begin{array}{l} kA_2^* B_d U E[Y_2Y_3] + \\ + \left(k^2 A_3^* U^2 - \dfrac{c_{2M}U^2}{2}\right)E[Y_1Y_3] + c_{1M}UE[Y_3^2] \end{array}\right\}
\end{aligned}
\tag{9}
$$

In the following the solution of this system is obtained with a mathematical closed form.

4. Mathematical solution of the dynamic response

To simplify the analytical study, the following positions are assumed:

$$
\begin{aligned}
& A = -2\xi_\alpha\omega_\alpha \;; \qquad B = -\omega_\alpha^2 \;; \\
& C = \rho\frac{B_d^3}{I}kA_2^* U \;; \qquad D = \rho\frac{B_d^3}{I}kA_2^* \\
& E = \rho\frac{B_d^3 U^2}{I}\left(k^2 A_3^* - \frac{c_{2M}}{2}\right) \;; \quad F = \rho\frac{B_d^2 U}{I}(2k^2 A_3^* - c_{2M}) \;; \\
& G = \rho\frac{B_d^2 U^2}{2I}c_{1M} \;; \qquad L = \rho\frac{B_d^2 U}{I}c_{1M}
\end{aligned}
\tag{10}
$$

Substituting Eq. (10) in Eq. (9) and considering that $E[Y_3] = 0$:

$$\begin{cases} E[Y_2] = \dfrac{d}{dt}E[Y_1] \\ E[Y_2Y_3] = \dfrac{d}{dt}E[Y_1Y_3] + \theta_1 E[Y_1Y_3] \\ \dfrac{d}{dt}E[Y_1^2] = 2E[Y_1Y_2] \\ \dfrac{d}{dt}E[Y_2^2] = 2GE[Y_2] + 2(A+C)E[Y_2^2] + 2(B+E)E[Y_1Y_2] + 2L\cdot E[Y_2Y_3] \\ \dfrac{d}{dt}E[Y_3^2] = -2\vartheta_1 E[Y_3^2] + \vartheta_2^2 \\ \dfrac{d}{dt}E[Y_1Y_2] = G\cdot E[Y_1] + (A+C)E[Y_1Y_2] + (B+E)E[Y_1^2] + L\cdot E[Y_1Y_3] + E[Y_2^2] \\ \dfrac{d^2}{dt^2}E[Y_1] = (A+C)\dfrac{d}{dt}E[Y_1] + (B+E)\cdot E[Y_1] + D\cdot E[Y_2Y_3] + F\cdot E[Y_1Y_3] + G \\ \dfrac{d^2}{dt^2}E[Y_1Y_3] + (2\vartheta_1 - A - C)\dfrac{d}{dt}E[Y_1Y_3] + (\vartheta_1^2 - \vartheta_1 A - B - C\vartheta_1 - E)\cdot E[Y_1Y_3] - L\cdot E[Y_3^2] = 0 \end{cases} \tag{11}$$

The solution of the fifth equation of (11) is:

$$E[Y_3^2] = \frac{\vartheta_2^2}{2\vartheta_1} + a_1\cdot\exp(-2\vartheta_1 t) \tag{12}$$

where a_1 is a constant.

Since in the eighth equation results:

$$\Delta = (2\vartheta_1 - A - C)^2 - 4(\vartheta_1^2 - \vartheta_1 A - B - C\vartheta_1 - E) \le 0,$$

its general integral is:

$$E[Y_1Y_3] = 2\exp(\alpha_1 t)[c_1\cos(\beta_1 t) + c_2\sin(\beta_1 t)] + \frac{L\vartheta_2^2}{2\vartheta_1[\vartheta_1^2 - \vartheta_1(A+C) - B - E]} + \frac{L\cdot a_1}{[\vartheta_1^2 + \vartheta_1(A+C) - B - E]}\exp(-2\vartheta_1 t) \tag{13}$$

where:

$\alpha_1 = -\dfrac{2\vartheta_1 - A - C}{2}$, $\beta_1 = \dfrac{\sqrt{-\Delta}}{2}$, c_1 and c_2 are constants.

Substituting Eq. (13) in the second equation of (11):

$$E[Y_2Y_3] = 2\exp(\alpha_1 t)[(\alpha_1 c_1 + \vartheta_1 c_1 + \beta_1 c_2)\cos(\beta_1 t) + (\alpha_1 c_2 - \beta_1 c_1 + \vartheta_1 c_2)\sin(\beta_1 t)] + \frac{L\vartheta_2^2}{2[\vartheta_1^2 - \vartheta_1(A+C) - B - E]} - \frac{L\cdot a_1\cdot\vartheta_1}{[\vartheta_1^2 + \vartheta_1(A+C) - B - E]}\exp(-2\vartheta_1 t) \tag{14}$$

The seventh equation is of the kind:

$$\frac{d^2}{dt^2}E[Y_1]-(A+C)\frac{d}{dt}E[Y_1]-(B+E)\cdot E[Y_1]=F\cdot E[Y_1Y_3]+D\cdot E[Y_2Y_3]+G \tag{15}$$

Since $\Delta=(-A-C)^2-4(-B-D)<0$, the integral of the associated homogenea is:

$$E[Y_1]=c_3\exp(\alpha_2 t)\cos(\beta_2 t)+c_4\exp(\alpha_2 t)\sin(\beta_2 t)$$

where:

$\alpha_2=\dfrac{A+C}{2},\;\; \beta_2=\dfrac{\sqrt{-\Delta}}{2},\;\; c_3$ and c_4 are constants.

Since the second member results:

$$\sum_1^3 f_i=2\exp(\alpha_1 t)\{[Fc_1+(\alpha_1c_1+\vartheta_1c_1+\beta_1c_2)D]\cos(\beta_1 t)+[Fc_2+(\alpha_1c_2-\beta_1c_1+\vartheta_1c_2)D]\sin(\beta_1 t)\}$$
$$+\left\{\frac{L\cdot a_1\cdot(F-D\cdot\vartheta_1)}{[\vartheta_1^2+\vartheta_1(A+C)-B-E]}\exp(-2\vartheta_1 t)\right\}+\left\{\frac{L\vartheta_2^2}{2[\vartheta_1^2-\vartheta_1(A+C)-B-E]}\left(\frac{F}{\vartheta_1}+D\right)+G\right\} \tag{16}$$

the solution is obtained by the addition of three integrals.

Assuming :

$$M=2(Fc_1+\alpha_1c_1D+\vartheta_1c_1D+\beta_1c_2D);\qquad N=2(Fc_2+\alpha_1c_2D-\beta_1c_1D+\vartheta_1c_2D);$$
$$P=\frac{L\vartheta_2^2}{2[\vartheta_1^2-\vartheta_1(A+C)-B-E]}\left(\frac{F}{\vartheta_1}+D\right)+G;\quad Q=\frac{L\cdot a_1\cdot(F-D\cdot\vartheta_1)}{[\vartheta_1^2+\vartheta_1(A+C)-B-E]}, \tag{17}$$

the first integral is:

$$\{E[Y_1]\}_1=\exp(\alpha_2 t)[c_3\cos(\beta_2 t)+c_4\sin(\beta_2 t)]+\exp(\alpha_1 t)[a_2\cos(\beta_1 t)+a_3\sin(\beta_2 t)] \tag{18}$$

where:

$$a_2=\frac{M[\alpha_1^2-\beta_1^2-\alpha_1(A+C)-(B+E)]-N[2\alpha_1\beta_1-\beta_1(A+C)]}{[\alpha_1^2-\beta_1^2-\alpha_1(A+C)-(B+E)]^2+[2\alpha_1\beta_1-\beta_1(A+C)]^2}$$
$$a_3=\frac{N[\alpha_1^2-\beta_1^2-\alpha_1(A+C)-(B+E)]+M[2\alpha_1\beta_1-\beta_1(A+C)]}{[\alpha_1^2-\beta_1^2-\alpha_1(A+C)-(B+E)]^2+[2\alpha_1\beta_1-\beta_1(A+C)]^2}$$

The second integral is:

$$\{E[Y_1]\}_2=\exp(\alpha_2 t)[c_3\cos(\beta_2 t)+c_4\sin(\beta_2 t)]+a_4\exp(-2\vartheta_1 t) \tag{19}$$

where: $a_4=\dfrac{Q}{4\vartheta_1^2+2(A+C)-(B+E)}$

The third integral is:

$$\{E[Y_1]\}_3 = \exp(\alpha_2 t)[c_3\cos(\beta_2 t) + c_4\sin(\beta_2 t)] - \frac{P}{B+E} \tag{20}$$

Finally the solution of Eq. (15) is:

$$E[Y_1] = 3\exp(\alpha_2 t)[c_3\ \cos(\beta_2 t) + c_4\sin(\beta_2 t)] + \exp(\alpha_1 t)[a_2\cos(\beta_1 t) + a_3\sin(\beta_1 t)]$$
$$+ a_4\ \exp(-2\vartheta_1 t) - \frac{P}{B+E} \tag{21}$$

Moreover the solution of the first equation of (11) is :

$$E[Y_2] = 3\exp(\alpha_2 t)[(c_3\alpha_2 + c_4\beta_2)\cos(\beta_2 t) + (c_4\alpha_2 - c_3\beta_2)\sin(\beta_2 t)]$$
$$+ \exp(\alpha_1 t)[(a_2\alpha_1 + a_3\beta_1)\cos(\beta_1 t) + (a_3\alpha_1 - a_2\beta_1)\sin(\beta_1 t)] - 2a_4\vartheta_1\exp(-2\vartheta_1 t) \tag{22}$$

The last two relations give, in a closed form, the temporal evolution of the first order torsional rotation moment and the angular rotation: they are the most important moments for the probabilistic description of the output process. In the same way it is possible to obtain the remaining moments.

5. Application to Tacoma narrows bridge

As example, the dynamic response under turbulent wind of the one d.o.f. model-section of Tacoma Narrows Bridge will be determined. The time dependent plots associated to the mean and to the mean square value of the response process will be determined, as they very well describe the dynamic response.

The Tacoma bridge deck represents a very significant case from a structural point of view; moreover it is one of the most sensational example of collapse due to the wind action. The principal characteristics of the bridge are summarised in Table 1. The dynamic parameters necessary to define the model are listed in Table 2. The expression of the aeroelastic coefficient A_2^* has been obtained interpolating, with a second order curve, the experimental values of the nodes proposed in Simiu and Scanlan (1986). In this procedure the minimum square principle has been utilised to calculate the coefficients of the polynomial. The dynamic response has been evaluated for three rough length:

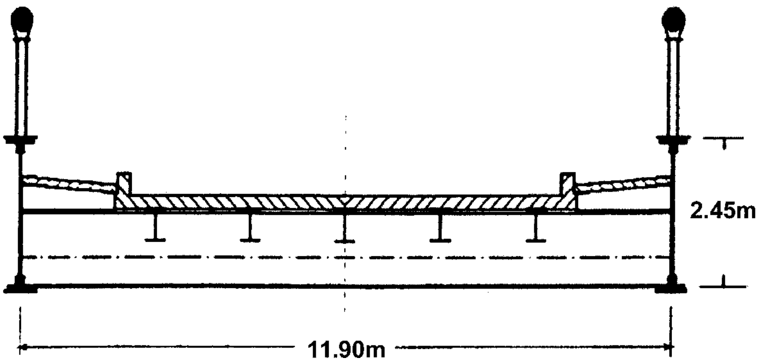

Fig. 2 Tacoma bridge deck

Table 1 Geometric and mechanical characteristics of Tacoma bridge

Span	L	854 m
Cable sag	f	70 m
Deck width	B_d	11.90 m
Total weight of the bridge (per length unity)	g	8483 Kg/m
Deck mass (per length unity)	m	865 Kgs^2/m
Moment of polar inertia of the deck (per length unity)	I	9490 Kgs^2/rad

Table 2 Dynamic characteristics of Tacoma bridge and the wind input

Torsional frequency	$\omega_\alpha / 2\pi$	0.167 Hz
Damping coefficient	ξ_α	0.01
Buffeting coefficient	c_{1M}	-0.005
Buffeting coefficient	c_{2M}	-0.559

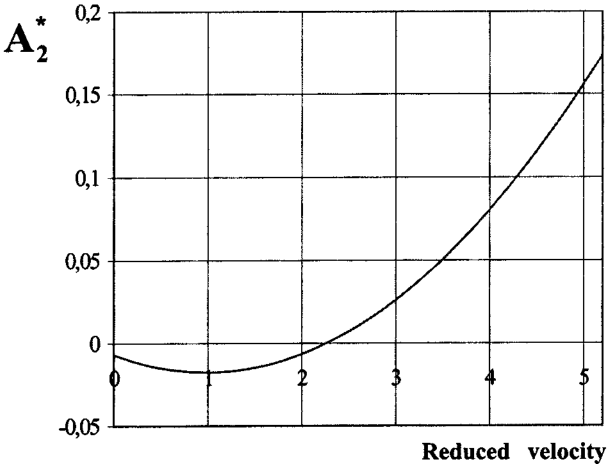

Fig. 3 Aeroelastic coefficient

$z_0 = 0.01$, $z_0 = 0.10$ and $z_0 = 0.80$.

Figs. 4, 5, 6 show the time dependent expected value of the torsional rotation for $U_{10} = 6.0$ m/s and $U_{10} = 4.5$ m/s respectively for less and high turbulence intensities; in Fig. 7 the same diagram for $U_{10} = 7$ m/s and high turbulence intensity is plotted. The Tacoma bridges supposed height from the ground was calculated at 50 m. In the first three diagrams the time evolution moment is decreasing, which means a stable condition of the motion. In the fourth plot the time dependent moment is increasing; which characterises an unstable condition. Figs. 8 and 9, respectively, show the expected values of the standard deviation of the rotation and the angular velocity for high turbulence intensity.

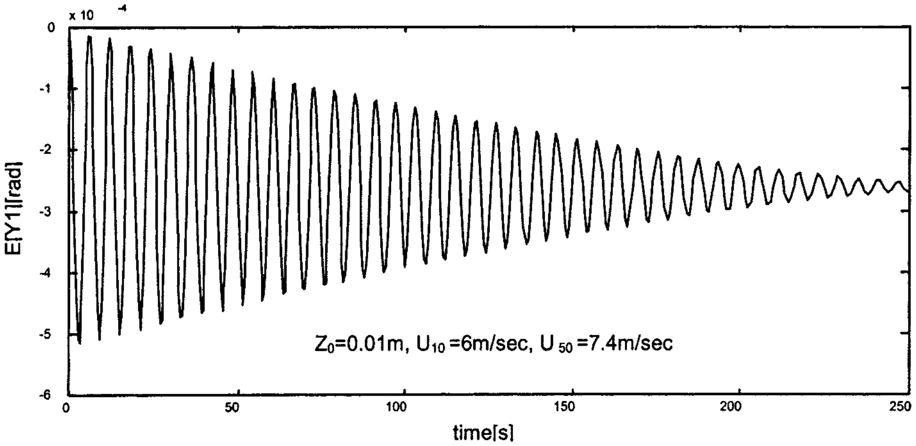

Fig. 4 Expected value of the angular rotation for U_{10} = 6.0 m/s

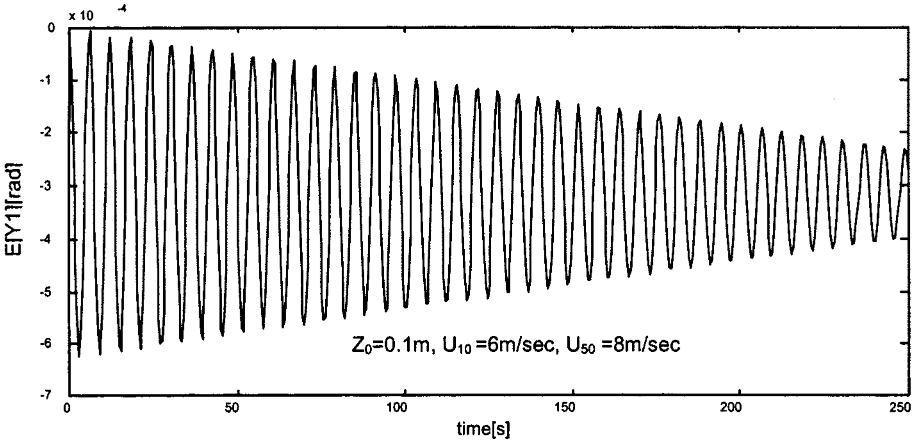

Fig. 5 Expected value of the angular rotation for U_{10} = 6.0 m/s

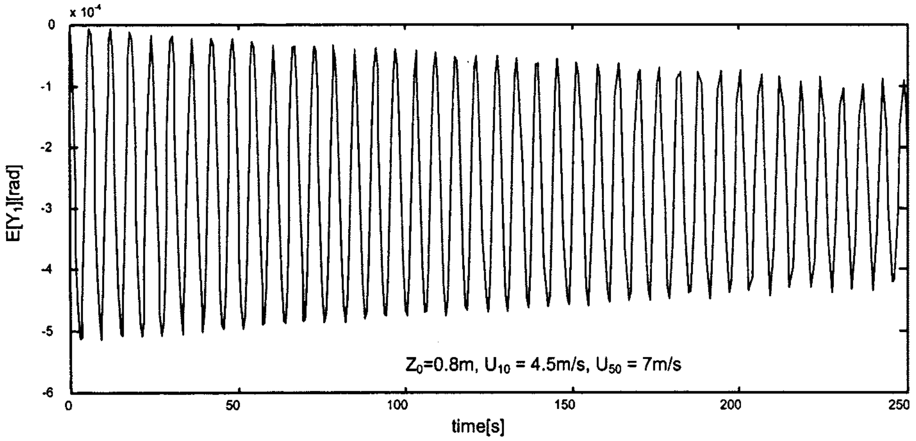

Fig. 6 Expected value of the angular rotation for U_{10} = 4.5 m/s

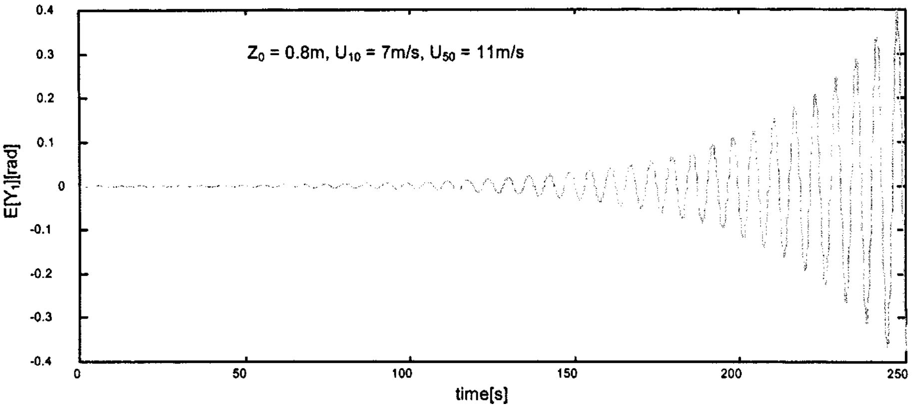

Fig. 7 Expected value of the angular rotation for U_{10} = 7 m/s

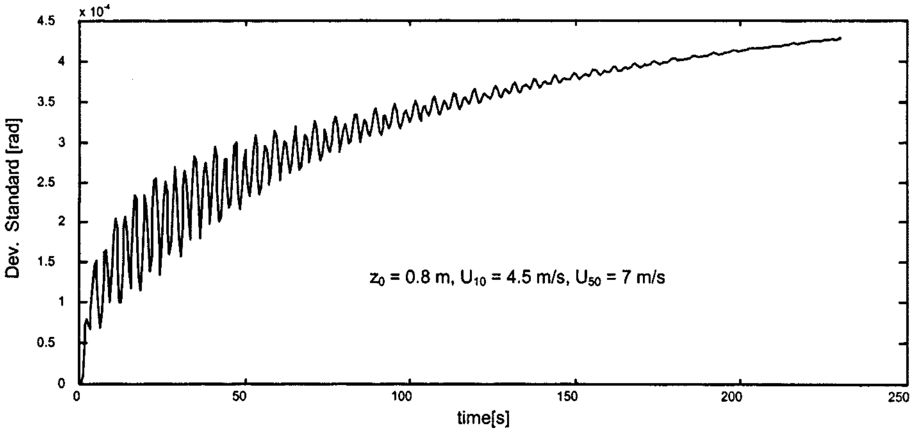

Fig. 8 Expected value of the standard deviation of the angular rotation

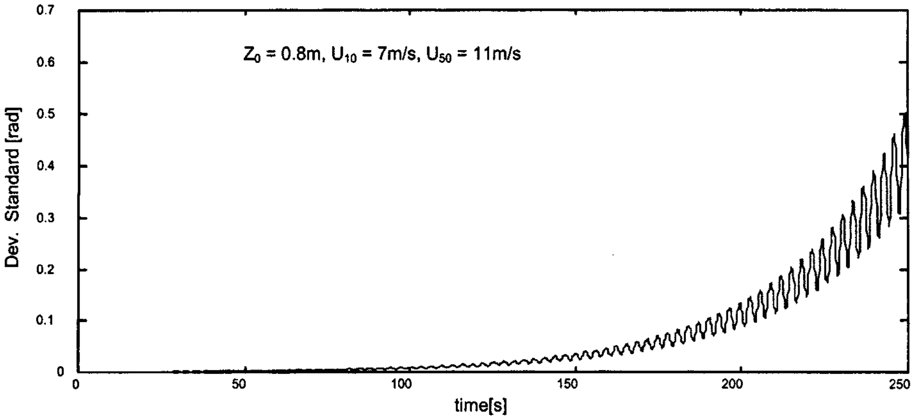

Fig. 9 Expected value of the standard deviation of the angular velocity

6. Conclusions

In this paper the closed form of the dynamic response of a system with one torsional degree of freedom has been determined. The system is represented by the section-model of a suspended or stayed bridge deck under turbulent wind. The analysis is referred to the instantaneous velocity of the wind (turbulent component).

Due to the stochastic term the torsional motion equation of the section-deck is a stochastic differential equation. The response process, represented by the time dependent moments, is obtained in a probabilistic way. The Gaussian processes has been utilised; therefore the proposed solution is good only for small displacements. As a consequence the dynamic response under high wind velocity will be more approximate.

The expressions of the expected values of the deck rotation and angular velocity have been determined in a closed form. These expressions have been utilised in an example to obtain the dynamic response of Tacoma bridge. The results practically coincide with those determined in the numerical analysis.

References

Bartoli, G. and Borri, C. (1997), “On the influence of wind turbulence on bridge decks flutter. Aspect in modern computational structural analysis”, Meskouris & Wittek. Balkema, Rotterdam.

Bontempi, F. and Malerba, P.G. (1994), “On the aeroelastic behaviour of cable stayed bridges”, *Proc. of the International Symposium on Cable Stayed Bridges*, Shanghai, May 10-13, 290-301.

Citrini, C. (1992). *Analisi Matematica*, **1, 2.** Ed. Bollati Boringhieri. Torino.

Iannuzzi, A. and Spinelli, P. (1986), “Effetti del vento sugli impalcati da ponte: comportamento aerodinamico e simulazione numerica. Parte I e II”, *Costruzioni Metalliche*, nn. 1, 2.

Ibrahim, R.A., Soundararajan, A. and Heo, H. (1985) “Stochastic response of non linear dynamic systems based on a non-Gaussian closure”, *Journal of Applied Mechanics*, **4**, 51-89.

Lin, Y.K. (1967). *Probabilistic Theory of Structural Dynamics*, McGraw-Hill Book Company.

Scanlan, R. H. and Tomko, J.J. (1971), “Airfoil and bridge deck flutter derivatives”, *Journal of the Engineering Mechanics Division* , December, 1717-1737.

Simiu, E. and Scanlan, R.H. (1986). *Wind Effects on Structure*, Second Edition, John Wiley & Sons, New York.

Solari G. (1987), “Turbulence modelling for gust loading”, *Journal Struct. Eng., ASCE*, **113**, 1550-1569.

Appendix - A

$$\frac{d}{dt}E[Y_1]=E[Y_2]$$

$$\frac{d}{dt}E[Y_2]=-2\xi_\alpha\omega_\alpha E[Y_2]-\omega_\alpha^2 E[Y_1]$$
$$+\frac{\rho B_d^2}{I}\left\{\begin{array}{l} kA_2^*B_dUE[Y_2]+\left(k^2A_3^*U^2-\dfrac{c_{2M}}{2}U^2\right)E[Y_1]+kA_2^*B_dE[Y_2Y_3]+(2k^2A_3^*U-c_{2M}U)E[Y_1Y_3] \\ +\dfrac{c_{1M}}{2}U^2+\dfrac{c_{3M}}{2}U^2E[Y_1^2]+c_{3M}UE[Y_1^2Y_3]+\dfrac{c_{4M}}{2}U^2E[Y_1^3]+c_{4M}UE[Y_1^3Y_3] \end{array}\right\}$$

$$\frac{d}{dt}E[Y_3]=-\theta_1 E[Y_3]=0$$

$$\frac{d}{dt}E[Y_1^2]=2E[Y_1Y_2]$$

$$\frac{d}{dt}E[Y_2^2]=-4\xi_\alpha\omega_\alpha E[Y_2^2]-2\omega_\alpha^2E[Y_1Y_2]$$

$$+2\frac{\rho B_d^2}{I}\left\{\begin{array}{l} kA_2^*B_dUE[Y_2^2]+\left(k^2A_3^*U^2-\frac{c_{2M}}{2}U^2\right)E[Y_1Y_2]+kA_2^*BE[Y_2^2Y_3] \\ +(2k^2A_3^*U-c_{2M}U)E[Y_1Y_2Y_3]+\frac{c_{1M}}{2}U^2E[Y_2]+c_{1M}UE[Y_2Y_3] \\ +\frac{c_{3M}}{2}U^2E[Y_1^2Y_2]+c_{3M}UE[Y_1^2Y_2Y_3]+\frac{c_{4M}}{2}U^2E[Y_1^3Y_2]+c_{4M}UE[Y_1^3Y_2Y_3] \end{array}\right\}$$

$$\frac{d}{dt}E[Y_3^2]=-2\theta_1E[Y_2^3]+\theta_2^2$$

$$\frac{d}{dt}E[Y_1Y_2]=E[Y_2^2]-2\xi_\alpha\omega_\alpha E[Y_1Y_2]-\omega_\alpha^2E[Y_1^2]$$

$$+\frac{\rho B_d^2}{I}\left\{\begin{array}{l} kA_2^*B_dUE[Y_1Y_2]+\left(k^2A_3^*U^2-\frac{c_{2M}}{2}U^2\right)E[Y_1^2]+kA_2^*B_dE[Y_1Y_2Y_3] \\ +(2k^2A_3^*U-c_{2M}U)E[Y_1^2Y_3]+\frac{c_{1M}}{2}U^2E[Y_1]+\frac{c_{3M}}{2}U^2E[Y_1^3]+ \\ +c_{3M}UE[Y_1^3Y_3]+\frac{c_{4M}}{2}U^2E[Y_1^4]+c_{4M}UE[Y_1^4Y_3] \end{array}\right\}$$

$$\frac{d}{dt}E[Y_1Y_3]=E[Y_2Y_3]-\theta_1E[Y_1Y_3]$$

$$\frac{d}{dt}E[Y_2Y_3]=-\theta_1E[Y_2Y_3]-2\xi_\alpha\omega_\alpha E[Y_2Y_3]-\omega_\alpha^2E[Y_1Y_3]$$

$$+\frac{\rho B_d^2}{I}\left\{\begin{array}{l} kA_2^*B_dUE[Y_2Y_3]+\left(k^2A_3^*U^2-\frac{c_{2M}}{2}U^2\right)E[Y_1Y_3]+kA_2^*B_dE[Y_2Y_3^2] \\ +(2k^2A_3^*U-c_{2M}U)E[Y_1Y_3^2]+c_{1M}UE[Y_3^2]+\frac{c_{3M}}{2}U^2E[Y_1^2Y_3] \\ +c_{3M}UE[Y_1^2Y_3^2]+\frac{c_{4M}}{2}U^2E[Y_1^3Y_3]+c_{4M}UE[Y_1^3Y_3^2] \end{array}\right\}$$

Appendix - B

Expressions of the third order of moments obtained in function of the moments of lower order by mean of a Gaussian closing on the cumulants:

$$E[Y_1Y_3^2]=E[Y_1]E[Y_3^2] \qquad E[Y_2Y_3^2]=E[Y_2]E[Y_3^2] \qquad E[Y_1^2Y_3]=2E[Y_1]E[Y_1Y_3]$$
$$E[Y_2^2Y_3]=2E[Y_2]E[Y_2Y_3] \qquad E[Y_1Y_2Y_3]=E[Y_1]E[Y_2Y_3]+E[Y_2]E[Y_1Y_3]$$

System with nine ordinary differential non-linear equations obtained:

$$\frac{d}{dt}E[Y_1]=E[Y_2]$$

$$
\begin{aligned}
\frac{d}{dt}E[Y_2] &= -2\xi_\alpha\omega_\alpha E[Y_2] - \omega_\alpha^2 E[Y_1] \\
&\quad + \frac{\rho B_d^2}{I}\left\{
\begin{array}{l}
kA_2^* B_d U E[Y_2] + \left(k^2 A_3^* U^2 - \dfrac{c_{2M}U^2}{2}\right)E[Y_1] + (2k^2 A_3^* U - c_{2M}U)E[Y_1Y_3] \\
+\dfrac{c_{1M}U^2}{2} + \dfrac{c_{3M}}{2}U^2 E[Y_1^2] + 2c_{3M}UE[Y_1]E[Y_1Y_3] + kA_2^* B_d E[Y_2Y_3] \\
+\dfrac{c_{4M}}{2}U^2(3E[Y_1^2]E[Y_1] - 2E([Y_1])^3) + 3c_{4M}UE[Y_1^2]E[Y_1Y_3]
\end{array}
\right\}
\end{aligned}
$$

$$
\frac{d}{dt}E[Y_3] = -\theta_1 E[Y_3] = 0
$$

$$
\frac{d}{dt}E[Y_1^2] = 2E[Y_1Y_2]
$$

$$
\begin{aligned}
\frac{d}{dt}E[Y_2^2] &= -4\xi_\alpha\omega_\alpha E[Y_2^2] - 2\omega_\alpha^2 E[Y_1Y_2] \\
&\quad + 2\frac{\rho B_d^2}{I}\left\{
\begin{array}{l}
kA_2^* B_d U E[Y_2^2] + \left(k^2 A_3^* U^2 - \dfrac{c_{2M}U^2}{2}\right)E[Y_1Y_2] + 2kA_2^* B_d E[Y_2]E[Y_2Y_3] \\
+(2k^2 A_3^* U - c_{2M}U)(E[Y_1]E[Y_2Y_3] + E[Y_2]E[Y_1Y_3]) + \dfrac{c_{1M}U^2}{2}E[Y_2] \\
+c_{1M}UE[Y_2Y_3] + \dfrac{c_{3M}}{2}U^2(E[Y_1^2]E[Y_2] + 2E[Y_1Y_2]E[Y_1] - 2(E[Y_1])^2E[Y_2]) \\
+c_{3M}U(E[Y_1^2]E[Y_2Y_3] + 2E[Y_1Y_2]E[Y_1Y_3]) + c_{4M}U^2\left(\begin{array}{l} \dfrac{3}{2}E[Y_1^2]E[Y_1Y_2] \\ -(E[Y_1]^3)E[Y_2] \end{array}\right) \\
+c_{4M}U\left(\begin{array}{l} 3E[Y_1^2]E[Y_2Y_3]E[Y_1] + 3E[Y_1^2]E[Y_1Y_3]E[Y_2] + 6E[Y_1Y_2]E[Y_1Y_3]E[Y_1] \\ -6E[Y_1Y_3](E[Y_1])^2E[Y_2] - 2E[Y_2Y_3](E[Y_1])^3 \end{array}\right)
\end{array}
\right\}
\end{aligned}
$$

$$
\frac{d}{dt}E[Y_3^2] = -2\theta_1 E[Y_3^2] + \theta_2^2
$$

$$
\begin{aligned}
\frac{d}{dt}E[Y_1Y_2] &= E[Y_2^2] - 2\xi_\alpha\omega_\alpha E[Y_1Y_2] - \omega_\alpha^2 E[Y_1^2] \\
&\quad + \frac{\rho B_d^2}{I}\left\{
\begin{array}{l}
kA_2^* B_d U E[Y_1Y_2] + \left(k^2 A_3^* U^2 - \dfrac{c_{2M}U^2}{2}\right)E[Y_1^2] \\
+kA_2^* B_d(E[Y_1]E[Y_2Y_3] + E[Y_2]E[Y_1Y_3]) \\
+(2k^2 A_3^* U - c_{2M}U)2E[Y_1]E[Y_1Y_3] + \dfrac{c_{1M}U^2}{2}E[Y_1] + c_{1M}UE[Y_1Y_3] \\
+c_{3M}U^2\left(\dfrac{3}{2}E[Y_1^2]E[Y_1] - (E[Y_1])^3\right) + c_{3M}U(3E[Y_1^2]E[Y_1Y_3]) \\
+c_{4M}U^2\left(\dfrac{3}{2}E[Y_1^2]E[Y_1^2] - (E[Y_1])^4\right) \\
+c_{4M}U(12E[Y_1^2]E[Y_1Y_3]E[Y_1] - 8E[Y_1Y_3](E[Y_1])^3)
\end{array}
\right\}
\end{aligned}
$$

$$\frac{d}{dt}E[Y_1Y_3] = E[Y_2Y_3] - \theta_1 E[Y_1Y_3]$$

$$\frac{d}{dt}E[Y_2Y_3] = -\theta_1 E[Y_2Y_3] - 2\xi_\alpha\omega_\alpha E[Y_2Y_3] - \omega_\alpha^2 E[Y_1Y_3]$$

$$+\frac{\rho B_d^2}{I}\left\{\begin{array}{l} kA_2^*B_dUE[Y_2Y_3] + kA_2^*B_dE[Y_2]E[Y_3^2] + \left(k^2A_3^*U^2 - \dfrac{c_{2M}U^2}{2}\right)E[Y_1Y_3] \\ +(2k^2A_3^*U - c_{2M}U)E[Y_1]E[Y_3^2] + c_{1M}UE[Y_3^2] + c_{3M}U^2E[Y_1]E[Y_1Y_3] \\ +c_{3M}U(E[Y_1^2]E[Y_3^2] + 2E[Y_1Y_3]E[Y_1Y_3]) + c_{4M}U^2\left(\dfrac{3}{2}E[Y_1^2]E[Y_1Y_3]\right) \\ +c_{4M}U\left(\begin{array}{l} 3E[Y_1^2]E[Y_3^2]E[Y_1] - 2E[Y_3^2](E[Y_1])^3 + \\ +6E[Y_1Y_3]E[Y_1Y_3]E[Y_1] \end{array}\right) \end{array}\right\}$$

(*Communicated by Giovanni Solari*)