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Conformable solution of fractional vibration problem of plate subjected to in-plane loads

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Abstract. This study provides an approximate analytical solution to the fractional vibration problem of thin plate governing anomalous motion of plate subjected to in-plane loads. The method of variable separable is employed to transform the fractional partial differential equations under consideration into a fractional ordinary differential equation in temporal variable and a bi-harmonic plate equation in spatial variable. The technique of conformable fractional derivative is utilized to solve the resulting fractional differential equation and the approach of finite sine integral transform method is used to solve the accompanying bi-harmonic plate equation. The deflection field which measures the transverse displacement of the plate is expressed in terms of product of Bessel and trigonometric functions via the temporal and spatial variables respectively. The obtained solution reduces to the solution of the free vibration problem of thin plate in literature. This work shows that conformable fractional derivative is an efficient mathematical tool for tracking analytical solution of fractional partial differential equation governing anomalous vibration of thin plates.

Keywords: conformable solution; fractional vibration; in-plane loads; thin plate

1. Introduction

Thin plates subjected to in-plane loads are common structural elements with widespread applications in many areas of technology such as civil, mechanical, aeronautical, marine, and chemical to mention a few (Fadodun and Akinola 2017, Ventsel and Krauthammer 2001, Shooshtari and Razavi 2015, Altekin 2017, An *et al*. 2015). The vibrations of thin plates with various boundary conditions are of much importance in all fields of engineering; and an accurate stability analysis of vibration of thin plate is necessary for the control of resonance effect thus ensuring safety of thin-walled structures (Zhong *et al.* 2014, Fadodun *et al*. 2017a, Lychev 2011, Hadji *et al.* 2017). In fact, the studies of vibration of plates have been investigated extensively for years (Hadji *et al.* 2016, Bennoun *et al.* 2016, Berferhat *et al*. 2016, Bourada *et al.* 2016, Rao 2007); and a literature survey shows that most considered problems involve cases of free and forced vibrations.

For instance, Abdelaziz *et al.* (2017) developed and applied a simple shear deformation theory to buckling, bending and free vibration of functionally graded material (FGM) sandwich plate with various boundary conditions. They demonstrated the accuracy of the obtained numerical results for the natural frequencies, deflections and critical buckling of sandwich plates in comparison with the existing results in literature. Bakhadda *et al*. (2018) examined the vibration and bending response of carbon nanotube-reinforced composite plates resting on the Pasternak elastic foundation using hyperbolic shear deformation plate theory. They presented the effects of parameters associated with nanotube volume fraction, spring constant, plate thickness and aspect ratio on the plate vibration and bending behavior and in addition, validated the accuracy of the obtained results in comparison with some available solutions in the literature. Abualnour *et al.* (2018) proposed stretching effect shear deformation theory for free vibration of simply supported functionally graded plates. Both the validation of the theory and the accuracy of numerical solutions for natural frequencies of functionally graded plates considered were presented. Younsi *et al.* (2018) developed two-dimensional and quasi three-dimensional higher shear deformation theories (HSDT) for bending and free vibration of functionally graded plate using hyperbolic shape functions. It was shown that the theories rely on undetermined integral terms and fewer number of the unknowns. The accuracy of the obtained results was illustrated in comparison with the existing results in literature. Bourada *et al.* (2019) examined the free vibration analysis of porous functionally graded beam using sinusoidal shear deformation theory. The accuracy of the obtained results in comparison with the existing solutions was demonstrated. Bouhadra *et al.* (2018) incorporated the influence of thickness stretching and improved the higher shear deformation theory for the investigation of advanced composite plates. The obtained analytical solution for the case of simply supported plates is accurate in comparison with the known three-dimensional solution and those previously generated by the other higher shear deformation theories. Belabed *et al.* (2018) developed a simple and accurate three-unknown hyperbolic shear deformation theory for vibration of functionally graded sandwich plates. The obtained results for the vibration analysis of the

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sandwich plate are in agreement with the solutions obtained from the classical plate theory, first order shear deformation theory, and the existing higher order shear deformation theories. Bouafia *et al*. (2017) investigated size-dependent bending and free flexural vibration of behaviors of functionally graded nanobeams using nonlocal quasi-three-dimensional theory involving both shear deformation and thickness stretching effects. The presented numerical solutions showed the effects of material gradient index, the nonlocal parameter, and the beam aspect ratio on the global response of the functionally graded (FG) nanobeams. Bounouara (2016) used the zeroth-order shear deformation theory for free vibration analysis of functionally graded (FG) nanoscale plates resting on elastic foundation. The study showed the effects of shear deformation, gradient index, Winkler modulus parameter, and Pasternak shear modulus parameter on the vibration responses of the functionally graded (FG) nanoscale plates considered. Hebali *et al.* (2014) developed a new quasi-three-dimensional (3D) hyperbolic shear deformation theory for the bending and free vibration analysis of functionally graded plates. Unlike other theories in literature, the theory uses fewer number of the unknown functions for the displacement field in comparison with the other shear and normal deformation theories. Amine *et al.* (2015) used a nonlocal shear deformation beam theory for bending, buckling, and vibration of functionally graded (FG) nanobeams based on nonlocal differential constitutive relations of Eringen. The obtained analytical solutions for the simply supported FG nanobeams are in agreement with the results predicted by nonlocal Timoshenko beam theory.

Furthermore, Lindsay *et al.* (2015) studied the out-of-plane modes of vibration of thin plates perforated by collection of small clamped patches. As the radius of each patch shrinks to zero, they derived a point constraint eigenvalue problem so that each patch is replaced by a homogeneous Dirichlet condition at its center. The outcome of their work showed that the vibrational frequencies are dependent very sensitively on the number and center of the clamped patches. Park *et al.* (2015) developed a frequency-domain spectral element model using the boundary splitting and the Kantorovich method-based super spectral element for the transverse vibration of thin plates. In comparison with the standard finite element solutions, both the accuracy of the solutions and the efficiency of the method employed were presented. Jaroszewicz (2017) investigated natural frequencies of homogeneous and isotropic circular thin plates with nonlinear thickness variation and clamped edge conditions. Using the method of successive approximation, he expressed the frequency equations in terms of power series**.** Lal and Saini (2017) presented analysis and numerical results for free vibrations of isotropic rectangular plates having arbitrarily varying non-homogeneity with the in-plane coordinates along the two concurrent edges using Kirchhoff's plate theory. With the aid of MATLAB, they presented the effects of various plate parameters on the vibration characteristics of the plate. Bao and Wang (2017) developed a generalized solution procedure for in-plane free vibration of rectangular and annular sectorial plates with general boundary conditions. For the case of annular sectorial plate, they introduced a logarithmic radial function which simplifies the plate theory and the expression for the total potential energy. Their method was shown to be computationally effective in comparison with some existing techniques in literature. Senjanovic *et al*. (2017) worked out an approximate procedure for the vibration analysis of circular thin plates with multiple openings using assumed mode approach and illustrated the effect of the opening on the plate vibration through sample problems. Fadodun *et al.* (2017a) analyzed free and forced vibrations of a transversely isotropic non-classical thin plate made of semi-linear hyperelastic John's material**.** In the study, they showed that a non-classical plate made of John semilinear material exhibits in-plane loads which the classical Kirchhoff's plate model fails to apprehend. Tahouneh (2017) presented the free vibration characteristics of sandwich sectorial plates with multiwalled carbon nanotube-(MWCNT)-reinforced composite core using three dimensional theory of elasticity. Using a two-dimensional differential quadrature approach together with a semi-analytical technique, he obtained series solution of the plate equation of motion.

Recently, fractional derivatives have been employed to generalize differential equations governing numerous physical processes in media (Fadodun *et al.* 2017b, Fu *et al.* 2013, Li 2014) with a view to examine some effects associated with anomalous processes in materials. For instance, Treeby and Cox (2010) used fractional Laplacian to model power law absorption and dispersion of acoustic wave propagation in media. A framework for encoding the developed wave equation using three coupled first-order constitutive equations was discussed. Chen *et al.* (2010) considered solution of fractional diffusion equation by Kansa method. Applying the MultiQuadrics and thin plate spline serve as radial basis functions, the numerical solutions for one- and two-dimensional cases which agree with the corresponding analytical exact solutions were presented and discussed. Li (2014) constructed analytic solution of a fractional generalized two phase Lame-Clapeyron Stefan problem and showed the performance of several parameters on the obtained solutions. Fu et al. (2013) used boundary particle approach for the Laplace transform time-fractional diffusion equation. They demonstrated both the high accuracy and computational efficiency of the method via error analysis and numerical experiment. Fadodun *et al.* (2017b) presented exact solution of fractional vibration problem of radially vibrating non-classical cylinder and obtained the existence of non-smooth waves which characterizes the anomalous vibration of the cylinder**.** Du *et al.* (2010) considered a compact difference scheme for fractional diffusion wave equation and presented both convergence and stability of the scheme.

In all of the above mentioned studies, the problems concerned with the anomalous vibration of thin plate under the influence of in-plane loads in the sense of time-fractional derivative approach have not been investigated. Therefore, this study considers the time-fractional plate equation governing the anomalous vibration of thin plate subjected to in-plane loads. The objective of the study is to obtain an approximate analytical solution for the transverse displacement of the plate. The form of the time-fractional equation governing the anomalous vibration of the plate under consideration permits us to employ the method of variable separable which in turn transforms the plate fractional equation into a fractional ordinary differential equation in time and a bi-harmonic plate equation in space. In this work, the conformable fractional derivative approach developed by Khalil *et al.* (2014) is used to solve the resulting fractional differential equation due to its amenability for initial-boundary value problems and the technique of finite sine integral transform approach is utilized for the solution of the accompanying bi-harmonic equation. The obtained solution reduces to the solution of the free vibration problem of thin plates in literature. The rest of the paper is organized as follows: section two presents time-fractional generalization of plate equation, section three gives the initial-boundary value problem, section four highlights the solution procedure, section five details the analytical solution of the problem under consideration, while section six concludes the study.

2. Fractional generalization of equation of thin plate subjected to in-plane loads

The partial differential equation governing the free vibration of an isotropic thin plate subjected to in-plane loads reads (Rao, 2007; pp. 527, Eq. (14.455))

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\rho h\frac{\partial^2 w}{\partial t^2}= N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_y\frac{\partial^2 w}{\partial y^2} \tag{1}$$

where $w=w(x,y,t)$ is the transverse displacement (deflection) of the plate, x,y are the independent spatial variables, t is the temporal variable, N_x, N_y, N_{xy} are the in-plane loads per unit length, h is the plate thickness, and ρ is the plate material density.

Furthermore, the coefficient D in Eq. (1) defined by

$$D=\frac{Eh^3}{12(1-\upsilon^2)} \tag{2}$$

is the flexural rigidity of the plate, E is the Young's modulus, and υ is the Poisson's ratio.

In view of Fadodun and Akinola (2017), we consider in-plane loads of the form

$$N_x=\frac{Eh}{(1+\upsilon)},\quad N_y=\frac{Eh}{(1+\upsilon)},\ \text{and}\ \ N_{xy}=0 \tag{3}$$

Substituting Eq. (3) into Eq. (1) yields

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)= \tag{4}$$

$$\frac{Eh}{(1+\upsilon)}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-\rho h\frac{\partial^2 w}{\partial t^2}$$

Using Eq. (2), the coefficient $\frac{Eh}{(1+\upsilon)}$ on the R.H.S of Eq. (4) takes the form

$$\frac{Eh}{(1+\upsilon)}=\frac{12D(1-\upsilon)}{h^2} \tag{5}$$

Substituting Eq. (5) into Eq. (4) gives

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)= \frac{12D(1-\upsilon)}{h^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-\rho h\frac{\partial^2 w}{\partial t^2} \tag{6}$$

In view of Eq. (5), the governing Eq. (6) coincides completely with the free vibration equation of a non-classical thin plate made of John's material (Fadodun *et al*. 2017a).

The time-fractional generalization of the free vibration Eq. (6) is

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)= \frac{12D(1-\upsilon)}{h^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-\rho h\frac{1}{\tau^{2-\alpha}}\frac{\partial^\alpha w}{\partial t^\alpha} \tag{7}$$

where α , $1<\alpha\le 2$ and the parameter τ with dimension of time is introduced to ensure all terms in Eq. (7) are dimensionally consistent.

3. Initial-boundary value problem

Consider anomalous vibration of rectangular thin plate subjected to in-plane loads. The plate under consideration is simply supported and occupies the region

$$0\le x\le a,\quad 0\le y\le b,\quad -\frac{h}{2}\le z\le\frac{h}{2}$$

where a, b, and h are the length, width, and thickness of the plate respectively.

The motion equation and the initial-boundary conditions governing the fractional vibration of the thin plate under consideration are

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)= \frac{12D(1-\upsilon)}{h^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-\rho h\frac{1}{\tau^{2-\alpha}}\frac{\partial^\alpha w}{\partial t^\alpha} \tag{8a}$$

$$w=0 \text{ and } \frac{\partial^2 w}{\partial x^2}=0, \quad \text{at} \quad x=0 \tag{8b}$$

$$w=0 \text{ and } \frac{\partial^2 w}{\partial x^2}=0, \quad \text{at} \quad x=a \tag{8c}$$

$$w=0 \text{ and } \frac{\partial^2 w}{\partial y^2}=0, \quad \text{at} \quad y=0 \tag{8d}$$

$$w=0 \text{ and } \frac{\partial^2 w}{\partial y^2}=0, \quad \text{at} \quad y=b \tag{8e}$$

$$w=0 \text{ and } \lim_{t\to 0^+}\frac{\partial w}{\partial t}=g_0, \quad \text{at} \quad t=0, \tag{8f}$$

where $1<\alpha\le 2$ and $g_0\in\Re$, $g_0\neq 0$.

4. Solution approach

This section highlights the conformable fractional derivative and finite sine integral transform techniques which we shall employ in the solution of the problem.

4.1 Conformable fractional derivative

Definition: Let $\alpha\in(n,n+1]$, and f be an $n-$differentiable function at t, where $t>0$. Then, the conformable fractional derivative of function f of order α is defined by (Khalil *et al*. 2014)

$$\frac{d^\alpha}{dt^\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f^{(\lceil\alpha\rceil-1)}(t+\varepsilon t^{(\lceil\alpha\rceil-\alpha)})-f^{(\lceil\alpha\rceil-1)}}{\varepsilon} \tag{9}$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

As a consequence of the definition in Eq. (9), one notes that

$$\frac{d^\alpha}{dt^\alpha}f(t)=t^{(\lceil\alpha\rceil-\alpha)}\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}f(t) \tag{10}$$

where $\alpha\in(n,n+1)$, and f is $(n+1)-$ differentiable at $t>0$.

Also, if f is $\alpha-$differentiable in $(0,T)$, $T>0$, and $\lim_{t\to 0^+}\frac{d^\alpha}{dt^\alpha}f(t)$ exists, then (Khalil *et al*. 2014)

$$\frac{d^\alpha}{dt^\alpha}f(0)=\lim_{t\to 0^+}\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}f(t) \tag{11}$$

4.2 Finite sine integral transform technique

The finite sine integral transform of a function $W=W(x,y)$, defined on a rectangular region $0\le x\le a$, $0\le y\le b$, is (Li *et al*. 2011)

$$\Omega_{mn}=\int_0^a\int_0^b W(x,y)\sin\beta_m x\sin\chi_n y\,dxdy \tag{12}$$

where $\beta_m=\frac{\pi m}{a}$, $\chi_n=\frac{\pi n}{b}$, $m,n=1,2,3,\ldots$

The inversion formula of Eq. (12) is

$$W(x,y)=\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Omega_{mn}\sin\beta_m x\sin\chi_n y \tag{13}$$

5. Approximate analytical solution of the fractional vibration problem of thin plate

Using the method of variable separable, we introduce solution of the form

$$w(x,y,t)=W(x,y)\Phi(t) \tag{14}$$

where $\Phi(t)$ is a function of temporal variable t only and $W(x,y)$ is a function of spatial variables x, y.

Substituting Eq. (14) in Eq. (8(a)), the function $W=W(x,y)$ satisfies the partial differential equation

$$D\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)W-\frac{12D(1-\upsilon)}{h^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)W=\gamma^2 \tag{15}$$

and the function $\Phi(t)$ satisfies the fractional ordinary differential equation

$$\frac{1}{\tau^{2-\alpha}}\frac{d^\alpha}{dt^\alpha}\Phi(t)+\gamma^2\Phi(t)=0 \tag{16}$$

respectively, where $\gamma\in\Re$, $\gamma\neq 0$.

$$W=0 \text{ and } \frac{\partial^2 W}{\partial x^2}=0, \quad \text{at} \quad x=0 \tag{17a}$$

$$W=0 \text{ and } \frac{\partial^2 W}{\partial x^2}=0, \quad \text{at} \quad x=a \tag{17b}$$

$$W=0 \text{ and } \frac{\partial^2 W}{\partial y^2}=0, \quad \text{at} \quad y=0 \tag{17c}$$

$$W=0 \text{ and } \frac{\partial^2 W}{\partial y^2}=0, \quad \text{at} \quad y=b \tag{17d}$$

In view of Eq. (12) and using the method of integration by part, the fine sine integral transforms of terms in Eq. (15)

give

$$\int_0^a\int_0^b \frac{\partial^4 W}{\partial x^4} \sin\beta_m x \sin\chi_n y dxdy =$$

$$-\beta_m \int_0^b \left[(-1)^m \frac{\partial^2 W}{\partial x^2}\bigg|_{x=a} - \frac{\partial^2 W}{\partial x^2}\bigg|_{x=0} \right] \sin\chi_n y dy \tag{18a}$$

$$+\beta_m^3 \int_0^b \left[(-1)^m W\big|_{x=a} - W\big|_{x=0} \right] \sin\chi_n y dy + \beta_m^4 \Omega_{mn}$$

$$\int_0^a\int_0^b \frac{\partial^4 W}{\partial y^4} \sin\beta_m x \sin\chi_n y dxdy =$$

$$-\chi_n \int_0^b \left[(-1)^n \frac{\partial^2 W}{\partial y^2}\bigg|_{y=b} - \frac{\partial^2 W}{\partial y^2}\bigg|_{y=0} \right] \sin\beta_m x dx \tag{18b}$$

$$+\chi_n^3 \int_0^b \left[(-1)^n W\big|_{y=b} - W\big|_{y=0} \right] \sin\beta_m x dx + \chi_n^4 \Omega_{mn}$$

$$\int_0^a\int_0^b \frac{\partial^4 W}{\partial x^2 \partial y^2} \sin\beta_m x \sin\chi_n x dxdy = \beta_m^2 \chi_n^2 \Omega_{mn}$$

$$-\beta_m \int_0^b \left[(-1)^m \frac{\partial^2 W}{\partial y^2}\bigg|_{x=a} - \frac{\partial^2 W}{\partial y^2}\bigg|_{x=0} \right] \sin\chi_n y dy \tag{18c}$$

$$+\beta_m^2 \chi_n \int_0^b \left[(-1)^n W\big|_{y=b} - W\big|_{y=0} \right] \sin\beta_m x dx$$

$$\int_0^a\int_0^b \frac{\partial^2 W}{\partial x^2} \sin\beta_m x \sin\chi_n y dxdy = -\beta_m^2 \Omega_{mn}$$

$$-\beta_m \int_0^b \left[(-1)^m W\big|_{x=a} - W\big|_{x=0} \right] \sin\chi_n y dy \tag{18d}$$

$$\int_0^a\int_0^b \frac{\partial^2 W}{\partial y^2} \sin\beta_m x \sin\chi_n y dxdy = -\chi_n^2 \Omega_{mn}$$

$$-\chi_n \int_0^b \left[(-1)^n W\big|_{y=b} - W\big|_{y=0} \right] \sin\beta_m x dx \tag{18e}$$

It is well known that the conditions $W(0,y)=0=W(a,y)$ imply that the derivatives of $W(x,y)$ with respect to spatial variable y at $x=0$ and $x=a$ vanish (Ventsel and Krauthammer 2001). That is

$$\frac{\partial^2 W}{\partial y^2} = 0, \quad \text{at} \quad x = 0,\ a \tag{19}$$

Substituting Eqs. (17(a))-(17(d)) and Eq. (19) into Eqs. (18(a))-(18(e)) gives

$$\int_0^a\int_0^b \frac{\partial^4 W}{\partial x^4} \sin\beta_m x \sin\chi_n y dxdy = \beta_m^4 \Omega_{mn} \tag{20a}$$

$$\int_0^a\int_0^b \frac{\partial^4 W}{\partial y^4} \sin\beta_m x \sin\chi_n y dxdy = \chi_n^4 \Omega_{mn} \tag{20b}$$

$$\int_0^a\int_0^b \frac{\partial^4 W}{\partial x^2 \partial y^2} \sin\beta_m x \sin\chi_n x dxdy = \beta_m^2 \chi_n^2 \Omega_{mn} \tag{20c}$$

$$\int_0^a\int_0^b \frac{\partial^2 W}{\partial x^2} \sin\beta_m x \sin\chi_n y dxdy = -\beta_m^2 \Omega_{mn} \tag{20d}$$

$$\int_0^a\int_0^b \frac{\partial^2 W}{\partial y^2} \sin\beta_m x \sin\chi_n y dxdy = -\chi_n^2 \Omega_{mn} \tag{20e}$$

In view of Eqs. (20(a))-(20(e)), the finite integral transform of Eq. (15) leads to the eigenvalue problem

$$\left(\frac{D}{\rho h} \left((\beta_m^2 + \chi_n^2)^2 + \frac{12(1-\upsilon)}{h^2} (\beta_m^2 + \chi_n^2) \right) - \gamma^2 \right) \Omega_{mn} = 0, \quad m,n = 1,2,3,\ldots \tag{21}$$

where $\gamma = \gamma_{mn}$ are the eigenvalues and Ω_{mn} are the corresponding eigenfunctions.

For non-trivial solutions of eigenfunctions Ω_{mn}, we set

$$\frac{D}{\rho h} \left((\beta_m^2 + \chi_n^2)^2 + \frac{12(1-\upsilon)}{h^2} (\beta_m^2 + \chi_n^2) \right) - \gamma^2 = 0 \tag{22}$$

The solution of Eq. (22) gives the eigenvalue $\gamma = \gamma_{mn}$

$$\gamma_{mn} = \sqrt{\frac{D}{\rho h} \left((\beta_m^2 + \chi_n^2)^2 + \frac{12(1-\upsilon)}{h^2} (\beta_m^2 + \chi_n^2) \right)} \quad m,n = 1,2,3,\ldots \tag{23}$$

It is clear that for each eigenvalue γ_{mn} in Eq. (23), there is a corresponding and associated non-trivial eigenfunction Ω_{mn}; and consequently one obtains the solution $W(x,y)$ of Eq. (15)

$$W(x,y) = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Omega_{mn} \sin\beta_m x \sin\chi_n y \tag{24}$$

Now, using Eq. (10) and setting $\gamma = \gamma_{mn}$, the fractional order ordinary differential equation in Eq. (16) reduces to the variable coefficient ordinary differential equation

$$\frac{d^2}{dt^2}\Phi(t)+\left(\frac{t}{\tau}\right)^{\alpha-2}\gamma_{mn}^2\Phi(t)=0 \qquad (25)$$

where $1<\alpha\leq 2$.

The solution of Eq. (25) for special case of parameter $\tau=1$ and for each eigenvalue γ_{mn} is

$$\Phi(t)=\Phi_{mn}(t)=$$

$$\sqrt{t}\left(C_{mn}J_{\frac{1}{\alpha}}\left(\frac{2\gamma_{mn}}{\alpha}t^{\frac{\alpha}{2}}\right)+B_{mn}Y_{\frac{1}{\alpha}}\left(\frac{2\gamma_{mn}}{\alpha}t^{\frac{\alpha}{2}}\right)\right) \qquad (26)$$

where $J_{\frac{1}{\alpha}}\left(\frac{2\gamma_{mn}}{\alpha}t^{\frac{\alpha}{2}}\right)$ and $Y_{\frac{1}{\alpha}}\left(\frac{2\gamma_{mn}}{\alpha}t^{\frac{\alpha}{2}}\right)$ are the Bessel functions of order $\frac{1}{\alpha}$ of first and second kind respectively, and C_{mn}, B_{mn} are constants.

Substituting Eqs. (24) and (26) into Eq. (14) gives the deflection solution $w=w(x,y,t)$

$$w=\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sqrt{t}\left(C_{mn}^{*}J_{\frac{1}{\alpha}}\left(\frac{2\gamma_{mn}}{\alpha}t^{\frac{\alpha}{2}}\right)+B_{mn}^{*}Y_{\frac{1}{\alpha}}\left(\frac{2\gamma_{mn}}{\alpha}t^{\frac{\alpha}{2}}\right)\right)\sin\beta_m x\sin\chi_n y \qquad (27)$$

where $B_{mn}^{*}=\Omega_{mn}B_{mn}$ and $C_{mn}^{*}=\Omega_{mn}C_{mn}$.

Substituting Eq. (27) into first condition in Eq. (8f) gives

$$B_{mn}^{*}=0 \qquad (28)$$

Using Eq. (28) in Eq. (27) gives

$$w=\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sqrt{t}\left(C_{mn}^{*}J_{\frac{1}{\alpha}}\left(\frac{2\gamma_{mn}}{\alpha}t^{\frac{\alpha}{2}}\right)\right)\sin\beta_m x\sin\chi_n y \qquad (29)$$

Using the second initial condition in Eq. (8(f)) yields

$$g_0=\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}^{*}\delta(\alpha,\gamma_{mn})\sin\beta_m x\sin\chi_n y \qquad (30)$$

where the parameter $\delta(\alpha,\gamma_{mn})\in\Re$ is defined by

$$\delta(\alpha,\gamma_{mn})=\lim_{t\to 0^{+}}\frac{d}{dt}\left(\sqrt{t}\left(J_{\frac{1}{\alpha}}\left(\frac{2\gamma_{mn}}{\alpha}t^{\frac{\alpha}{2}}\right)\right)\right)\neq 0 \qquad (31)$$

In view of Eqs, (12) and (13), the constant C_{mn}^{*} in Eq. (30) is

$$C_{mn}^{*}=\frac{1}{\delta(\alpha,\gamma_{mn})}\int_0^a\int_0^b g_0\sin\beta_m x\sin\chi_n y\,dxdy \qquad (32)$$

$$m,n=1,2,3,\ldots$$

$$C_{mn}^{*}=\frac{4abg_0}{\delta(\alpha,\gamma_{mn})\pi^2 mn} \qquad (33)$$

$$m.n=1,3,5,7,\ldots$$

Substituting Eq. (33) into Eq. (29) gives the solution of the fractional vibration problem of thin plate subjected to in-plane loads

$$w=\frac{16g_0}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sqrt{t}\left(\frac{\sin\beta_m x\sin\chi_n y}{\delta(\alpha,\gamma_{mn})mn}J_{\frac{1}{\alpha}}\left(\frac{2\gamma_{mn}}{\alpha}t^{\frac{\alpha}{2}}\right)\right) \qquad (34)$$

where $m,n=1,3,5,\ldots$

In the special case of free vibration of thin plate subjected to in-plane loads, the parameter $\alpha=2$.

Using $\alpha=2$ in Eq. (34) yields

$$w=\frac{16g_0}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sqrt{t}\left(\frac{\sin\beta_m x\sin\chi_n y}{\delta(2,\gamma_{mn})mn}J_{\frac{1}{2}}(\gamma_{mn}t)\right) \qquad (35)$$

In view of Eq. (31)

$$\delta(2,\gamma_{mn})=\sqrt{\frac{2}{\pi\gamma_{mn}}}\gamma_{mn} \qquad (36)$$

Also, one knows that the Bessel function $J_{\frac{1}{2}}(\gamma_{mn}t)$ satisfies the relation

$$\sqrt{t}J_{\frac{1}{2}}(\gamma_{mn}t)=\sqrt{\frac{2}{\pi\gamma_{mn}}}\sin(\gamma_{mn}t) \qquad (37)$$

Using Eqs. (36) and (37) in Eq. (35) gives

$$w(x,y,t)=$$

$$\frac{16g_0}{\pi^2}\sum_{m=1,3,5,\ldots}^{\infty}\sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{\sin\gamma_{mn}t}{\gamma_{mn}mn}\right)\sin\beta_m x\sin\chi_n y \qquad (38)$$

Eq. (38) gives the transverse displacement solution of the free vibration problem of thin plates with simply supported boundary conditions in literature.

8. Conclusions

The work uses conformable fractional derivative approach and the method of variable separable to construct approximate analytical solution of fractional partial differential equation governing anomalous vibration of thin plate. The governing time-fractional partial differential equation is transformed into a bi-harmonic plate equation and a fractional ordinary differential equation. The obtained solution reduces to solution of free vibration problem of thin plate when the order of the fractional derivative becomes 2. The results in this work find applications in the analysis and design of modern foundations of bridge decks as well as rigid pavements of highway and airports.

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