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Dynamic analysis of a transversely isotropic non-classical thin plate

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Abstract. This study investigates the dynamic analysis of a transversely isotropic thin plate. The plate is made of hyperelastic John's material and its constitutive law is obtained by taken the Frechect derivative of the highlighted energy function with respect to the geometry of deformation. The three-dimensional equation governing the motion of the plate is expressed in terms of first Piola-Kirchhoff's stress tensor. In the reduction to an equivalent two-dimensional plate equation, the obtained model generalizes the classical plate equation of motion. It is obtained that the plate under consideration exhibits harmonic force within its planes whereas this force varnishes in the classical plate model. The presence of harmonic forces within the planes of the considered plate increases the natural and resonance frequencies of the plate in free and forced vibrations respectively. Further, the parameter characterizing the transversely isotropic structure of the plate is observed to increase the plate flexural rigidity which in turn increases both the natural and resonance frequencies. Finally, this study reinforces the view that non-classical models of problems in elasticity provide ample opportunity to reveal important phenomena which classical models often fail to apprehend.

Keywords: dynamic analysis; thin plate; transversely isotropic

1. Introduction

In the last few decades, renewed efforts of researchers in elasticity have been vigorously channeled toward investigating theoretical and industrial problems through the prism of finite deformation. One reason behind this recent attention is that it has been increasingly acknowledged that classical models of elasticity, whose mathematical theory though now firmly established, have a limited range of applicability; and consequently, must be replaced by genuine non-classical models which they originally approximate (Ciarlet 1988). Another reason, similar in its underlying principle, is that finite deformation considerations provide avenue for revealing important phenomena which the classical infinitesimal theory of elasticity often fails to apprehend (Akinola 2001, Fadodun 2014, Fadodun and Akinola 2017). Efforts directed at apprehending these phenomena have utility in many industrial settings such as aerospace, automobile, and rubber industries. The solution of elasticity problems for the general case of three-dimensional bodies

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involves great mathematical difficulties; we are compelled by this circumstance to turn to the solution of more or less wide classes of special problems, one of which are thin plate problems (Amenzade 1979). Thin plates are common structural elements utilized in buildings, bridges, aircraft, and machinery, to name but a few. They are fabricated from steel, aluminum, concrete, and composite material. The phenomenon of vibration involves an alternating interchange of potential and kinetic energies. Vibrations serve some useful purposes in engineering applications, such as in vibratory testing of materials, dentist drills, electric massaging units, and material processing operations. However, vibration is undesirable in many other cases. The failure of most mechanical and structural elements can be associated with vibrations. For instance, wind-induced vibration in machine leads to rapid wear of parts such as gears and bearings, loosening of fasteners, poor surface finish during metal cutting, and excessive noise (Rao 2007). Furthermore, supersonic aircraft creates sonic booms that shatter doors and windows; and several spectacular failures of bridges, buildings, and dams are associated with the wind-induced vibration. In engineering and related fields, the dynamic (vibration) analysis is an important aspect in any complete structural investigation; and an understanding of dynamic analysis of thin plates is of much importance in ensuring safety in designs, constructions, and operations of a variety of machines and structures. In fact, the study of dynamics and statics analyses of thin plates is a subject of considerable scientific and practical interests that has been examined extensively, and is still receiving attention in literature. Imrak and Fetvaci (2009) solved the problem of deflection of a clamped rectangular thin plate carrying uniform loads. The approach of double series was employed and the exact solution was given in terms of a set of orthogonal functions. Shooshtari and Razavi (2015) investigated nonlinear vibration analysis of rectangular magneto-electro-elastic thin plates. Both the cases of free and force vibrations were considered. The governing plate equation was transformed to an equivalent ordinary differential equation. The method of perturbation was used to construct an approximate analytical solution of the considered problem. Furthermore, the presented numerical results showed the effects of several parameters on the nonlinear behavior of the plate. Lychev *et al.* (2011) studied the problem of unsteady vibration of a growing circular plate. They investigated the case of forced vibration within the framework of small deformation theory. The material of the plate was assumed to be elastic and isotropic; and that the plate thickness was continually increasing due to influx of the material from outside. The analysis of the obtained solution indicated the characteristics features of the dynamic growth of the plates. Pan (2001) gave exact solution for simply supported and multilayered magneto-electro-elastic thick plates under statics loading. He expressed the homogeneous solution in terms of a new and simple formalism that resembled the Stroh formalism. In addition, the solutions for multilayered plates were expressed in terms of the propagator matrix. It must be mentioned that his solutions generalized all the previous solutions, such as piezoelectric, piezomagnetic, and purely elastic solutions, as special cases. An *et al.* (2015) presented exact solution of bending problem of clamped orthotropic rectangular thin plates. They transformed the governing plate equation to a coupled system of fourth order ordinary differential equations using the generalized integral transform technique. The resulting ordinary differential equations were then solved numerically. Liu and Chang (2010) derived a closed form expression for the transverse vibration of a magneto-electro-elastic (MEE) thin plate, and obtained exact solution for the free vibration of a two-layered composite. Based on the Kirchhoff's plate theory, they investigated the bending of a transversely isotropic MEE rectangular plate and presented the governing equation in terms of transverse displacement. The natural frequencies of the MEE plate were evaluated analytically, and the effects of different volume fractions were presented. Altekin (2017) used Mindlin plate theory and studied free transverse vibration of shear deformable super-elliptical plates with uniform thickness. Sensitivity analysis was carried out to determine the influence of the thickness, the aspect ratio, and the shape of the plate on the natural frequencies. Wu *et al*. (2007) proposed Bessel function approach for obtaining exact solutions for free-vibration analysis of rectangular plates with three edge conditions. The cases of fully simply supported, fully clamped, and two opposite edges simply supported and the other two edges clamped were considered. It was shown that the proposed method provided simple, direct, and highly accurate solutions for the considered family of problems. Abdelbari *et al*. (2016) presented a simple hyperbolic shear deformation theory for analysis of functionally graded plates resting on elastic foundation, and the obtained Navier-type analytical solutions for simply-supported plates were compared with the existing solutions to demonstrate the accuracy of the proposed theory. Zhong *et al*. (2013) used the finite cosine integral transform method to obtain analytical solution for the natural frequencies and mode shapes of a vibrating rectangular thin plate on foundation with four edges free. In the analysis procedure, Kirchhoff's plate was considered and the foundation was modelled as the Winkler elastic foundation. The presented numerical result was in agreement with known results in literature. In view of recent design in structural engineering, Fadodun and Akinola (2017) proposed flexural model for the design of an isotropic non-classical thin plate. The obtained plate equation generalized the famous Kirchhoff's plate model. It was observed that the plate made of hyperelastic John material exhibits in-plane forces which classical model fails to apprehend. Further, it was shown that the considered plate could be used as a substitute for Kirchhoff's plate on elastic foundation. The present study is motivated by the increasing use of transversely isotropic material in the design of modern structures. Furthermore, most structural elements such as thin plates, rods, beams, columns, pipes, and fibers use for the design of modern aircrafts, ships, bridges, missiles, and railway tracks are manufactured by processes that induce transversely isotropic elastic properties in them. The dynamic analysis of these bodies due to wind-induced load is of great importance for safety of designs. Therefore, this work investigates the free and forced vibrations of transversely isotropic thin plate. It is assumed that the plate is made of non-classical hyperelastic John's material. The three-dimensional motion equation is presented in terms of first Piola-Kirchhoff's stress tensor. In the reduction to an equivalent two-dimensional plate equation, the obtained model generalizes the classical motion equation of plate. It is obtained that the plate under consideration exhibits harmonic forces within its planes. In the case of free vibration, the presence of harmonic forces within the planes of the plate increases its natural frequencies. Similarly, in the case of forced vibration, the harmonic forces increase the resonance frequency of the plate. Further, the parameter characterizing the transversely isotropic structure of the plate is observed to increase the plate flexural rigidity which in-turn increases both the natural and resonance frequencies. This paper is organized as follows: section two presents the three-dimensional equation of motion, section three details a reduced two-dimensional equation of a vibrating plate, section four highlights the effect of transversely isotropic on the flexural rigidity of the plate, sections five and six present the natural frequencies and exact solution of forced vibration of thin plate respectively, while section seven concludes the study.

2. Three-dimensional equation of motion

2.1 Statement of the problem

Consider a rectangular plate in the reference configuration $\Omega_0 \in \Re^3$, with arbitrary supports. Assume the plate deforms onto current configuration $\Omega \in \Re^3$ due to certain transverse surface loads distributed on its surface such that the particles in the middle surface attain the deflections and velocities directed perpendicularly to the reference middle surface. At a certain time, which is assumed to be the initial, the plate is suddenly released from all external loads. The unloaded plate, which has initial deflection and velocity, begin to execute natural or free vibration. The particles located in the middle surface move in the direction perpendicular to the plate; and the deformation function $\vec{\varphi} = (\varphi_i)$, $i = 1,2,3$, is assumed to take the form

$$\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$$
$$\begin{cases} \varphi_\alpha = x_\alpha - x_3 \dfrac{\partial}{\partial x_\alpha} w(x_1, x_2, t) & \alpha = 1,2 \\ \varphi_3 = cx_3 + w(x_1, x_2, t) \end{cases} \tag{1}$$

where (x_1, x_2, x_3) is the material coordinate in the reference configuration Ω_0, t is the time, $w(x_1, x_2, t)$ is the transverse displacement (deflection) of the plate, and $c \in \Re$.

2.2 Energy function and constitutive relation

The energy function for an isotropic hyperelastic non-classical John's material is (John 1960, Fadodun and Akinola 2017)

$$W = \mu S_1(\tilde{U} - \tilde{E})^2 + \frac{1}{2}\lambda S_1^2(\tilde{U} - \tilde{E}) \tag{2}$$

where μ, λ are the Lame's constants, $S_1(\tilde{U} - \tilde{E})$ is the trace of second rank tensor $(\tilde{U} - \tilde{E})$, $\tilde{E}$ is the unit second rank tensor, $\tilde{U}$ is the left stretch symmetric second rank tensor such that $\nabla\vec{\varphi} = \tilde{U} \cdot \tilde{O}^D$ and $\tilde{U}^2 = \nabla\vec{\varphi} \cdot \nabla\vec{\varphi}^T$, $\nabla\vec{\varphi}$ is the gradient of deformation function $\vec{\varphi}$, $\nabla\vec{\varphi}^T$ is the transpose of $\nabla\vec{\varphi}$, $\tilde{O}^D$ is the orthogonal rotation tensor, and $\cdot$ is the usual scalar (dot) product.

On the basis of Eq. (2), Akinola (1999) employed asymptotic averaging method to construct energy function for the corresponding transversely isotropic material

$$W = \lambda_2 S_1(\tilde{U} - \tilde{E})^2 + \frac{1}{2}\lambda_1 S_1^2(\tilde{U} - \tilde{E}) + \lambda_0 \vec{\tau} \cdot \tilde{U}^2 \cdot \vec{\tau} \tag{3}$$

where $\vec{\tau}$ is the unit vector characterizing the direction of anisotropy of the medium, and $\lambda_0, \lambda_1, \lambda_2$ are the effective material constants defined by (Akinola 1999)

$$\lambda_2 = <\mu>, \quad \lambda_0 = 2\left(\frac{1}{<1/\mu>} - \lambda_2\right), \quad \lambda_1 = <\lambda> + \frac{<\lambda/(\lambda+2\mu)>}{1/(\lambda+2\mu)} - <\frac{\lambda^2}{\lambda+2\mu}>$$

and for any function $\eta \in \Omega \times [0,T)$, $<\eta>$ denotes its geometric average over Ω given by

$$< \eta >= \frac{1}{|\Omega|}\int_{\Omega} \eta d\Omega, \quad (|\Omega| \text{ being volume of } \Omega).$$

In the case of degeneracy to isotropic, the energy function in Eq. (3) naturally reduces to energy function in Eq. (2) when

$$\lambda_2 = \mu, \quad \lambda_1 = \lambda, \quad \text{and} \quad \lambda_0 = 0$$

Let $\tilde{P}$ denote the first Piola-Kirchhoff's stress tensor which is energy conjugate to the geometry of deformation $\nabla\vec{\varphi}$ (deformation gradient), then the Frechet derivative of the energy function in Eq. (3) with respect to the geometry of deformation $\nabla\vec{\varphi}$ gives the constitutive law $\tilde{P}$.

$$\tilde{P} = \frac{\partial W}{\partial \nabla\vec{\varphi}} \tag{4}$$

The Frechet derivatives of terms in Eq. (3) are

$$\frac{\partial S_1(\tilde{U} - \tilde{E})^2}{\partial \nabla\vec{\varphi}} = 2(\nabla\vec{\varphi} - \tilde{O}^D) \tag{5}$$

$$\frac{\partial S_1^2(\tilde{U} - \tilde{E})}{\partial \nabla\vec{\varphi}} = 2S_1(\tilde{U} - \tilde{E})\tilde{O}^D \tag{6}$$

and

$$\frac{\partial(\vec{\tau} \cdot \tilde{U}^2 \cdot \vec{\tau})}{\partial \nabla\vec{\varphi}} = 2\vec{\tau}\vec{\tau} \cdot \nabla\vec{\varphi} \tag{7}$$

respectively.

Substituting Eqs. (3) and (5)-(7) in Eq. (4) gives the constitutive law for the material of the plate under consideration

$$\tilde{P} = \lambda_2 \nabla\vec{\varphi} + (\lambda_1 S_1(\tilde{U} - \tilde{E}) - 2\lambda_2)\tilde{O}^D + 2\lambda_0 \vec{\tau}\vec{\tau} \cdot \nabla\vec{\varphi} \tag{8}$$

2.3 Three-dimensional equation of state

Using the constitutive law in Eq. (8), the associated three-dimensional motion equation for the considered problem is

$$\nabla \cdot \tilde{P} + \vec{f} = \rho \frac{\partial^2 \vec{\varphi}}{\partial t^2} \tag{9}$$

where $\vec{f}$ is the body force and ρ is the mass density of the material of the plate.

3. An equivalent two-dimensional equation of a vibrating thin plate

3.1 Three-dimensional equation of state

Let the gradient of deformation $\nabla\vec{\varphi}$ be the geometry of deformation of the plate from the reference configuration Ω_0 onto current configuration Ω. Then, by definition (Fadodun and Akinola 2017)

$$\nabla\vec{\varphi} = \begin{pmatrix} \frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_1}{\partial x_2} & \frac{\partial\varphi_1}{\partial x_3} \\ \frac{\partial\varphi_2}{\partial x_1} & \frac{\partial\varphi_2}{\partial x_2} & \frac{\partial\varphi_2}{\partial x_3} \\ \frac{\partial\varphi_3}{\partial x_1} & \frac{\partial\varphi_3}{\partial x_2} & \frac{\partial\varphi_3}{\partial x_3} \end{pmatrix} \tag{10}$$

Substituting Eq. (1) in Eq. (10) gives

$$\nabla\vec{\varphi} = \begin{pmatrix} 1 - x_3\frac{\partial^2 w}{\partial x_1^2} & -x_3\frac{\partial^2 w}{\partial x_1\partial x_2} & -\frac{\partial w}{\partial x_1} \\ -x_3\frac{\partial^2 w}{\partial x_1\partial x_2} & 1 - x_3\frac{\partial^2 w}{\partial x_2^2} & -\frac{\partial w}{\partial x_2} \\ \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial x_2} & c \end{pmatrix} \tag{11}$$

where $w = w(x_1, x_2, t)$.

The polar decomposition of the gradient of deformation in Eq. (11) into product of left symmetric stretch tensor $\tilde{U}$ and orthogonal rotation tensor $\tilde{O}^D$

$$\nabla\vec{\varphi} = \tilde{U}\cdot\tilde{O}^D \tag{12}$$

such that $\tilde{U}^2 = \nabla\vec{\varphi}\cdot\nabla\vec{\varphi}^T$ gives

$$\tilde{U} = \begin{pmatrix} x_3\frac{\partial^2 w}{\partial x_1^2} - 1 & x_3\frac{\partial^2 w}{\partial x_1\partial x_2} & -\frac{\partial w}{\partial x_1} \\ x_3\frac{\partial^2 w}{\partial x_1\partial x_2} & x_3\frac{\partial^2 w}{\partial x_2^2} - 1 & -\frac{\partial w}{\partial x_2} \\ -\frac{\partial w}{\partial x_1} & -\frac{\partial w}{\partial x_2} & c \end{pmatrix} \tag{13}$$

and

$$\tilde{O}^D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{14}$$

respectively.

3.2 Two-dimensional motion equation of plate

Let P_{mn}, $m,n=1,2,3$ denote the components of the first Piola-Kirchhoff's stress tensor $\tilde{P}$.

Substituting Eqs. (11), (13)-(14) in Eq. (8) and invoking the constrain of zero stress ($P_{33}=0$) along the perpendicular axis to the plane of the plate give

$$P_{11}=-2x_3\lambda_2\frac{\partial^2 w}{\partial x_1^2}-x_3\left(\frac{2\lambda_1(\lambda_2+\lambda_0)}{2\lambda_2+\lambda_1+2\lambda_0}\right)\nabla^2 w+K \tag{15a}$$

$$P_{12}=-2x_3\lambda_2\frac{\partial^2 w}{\partial x_1\partial x_2},\qquad P_{13}=-2\lambda_2\frac{\partial w}{\partial x_1} \tag{15b}$$

$$P_{22}=-2x_3\lambda_2\frac{\partial^2 w}{\partial x_2^2}-x_3\left(\frac{2\lambda_1(\lambda_2+\lambda_0)}{2\lambda_2+\lambda_1+2\lambda_0}\right)\nabla^2 w+K \tag{15c}$$

$$P_{21}=-2x_3\lambda_2\frac{\partial^2 w}{\partial x_1\partial x_2},\qquad P_{23}=-2\lambda_2\frac{\partial w}{\partial x_2} \tag{15d}$$

$$P_{31}=2(\lambda_2+\lambda_0)\frac{\partial w}{\partial x_1},\ \ P_{32}=2(\lambda_2+\lambda_0)\frac{\partial w}{\partial x_2} \tag{15e}$$

where $\nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}$ is the Laplacian operator and $K=\frac{10\lambda_0\lambda_1+12\lambda_1\lambda_2+8\lambda_0\lambda_2+8\lambda_2^2}{2\lambda_0+\lambda_1+2\lambda_2}$ is a constant.

In view of Eq. (1), and neglecting the body force ($\vec{f}=\vec{0}$), the components form of the governing equation in Eq. (9) are

$$\frac{\partial}{\partial x_1}P_{11}+\frac{\partial}{\partial x_2}P_{21}+\frac{\partial}{\partial x_3}P_{31}=0 \tag{16a}$$

$$\frac{\partial}{\partial x_1}P_{12}+\frac{\partial}{\partial x_2}P_{22}+\frac{\partial}{\partial x_3}P_{32}=0 \tag{16b}$$

$$\frac{\partial}{\partial x_1}P_{13}+\frac{\partial}{\partial x_2}P_{23}+\frac{\partial}{\partial x_3}P_{33}=\rho\frac{\partial^2}{\partial t^2}w \tag{16c}$$

Substituting Eqs. (15(a))-(15(e)) in Eqs. (16(a))-(16(c)) give the two-dimensional equation governing the free-vibration of thin plate under consideration

$$D\nabla^4 w-(4\lambda_2+\lambda_0)h\nabla^2 w+\rho\frac{\partial^2 w}{\partial t^2}=0 \tag{17}$$

where the coefficient D given by

$$D = \frac{h^3}{12}\left(\frac{2\lambda_1(\lambda_0+\lambda_2)}{2\lambda_2+\lambda_1+2\lambda_0}+2\lambda_2\right) \tag{18}$$

is the flexural rigidity of the plate, h is the uniform thickness of the thin plate, ∇^2 is the Laplacian operator, $\nabla^4 = \nabla^2\nabla^2$ is the Biharmonic operator, and $w = w(x_1, x_2, t)$ is the transverse displacement.

In the degeneracy to isotropic thin plate case

$$\lambda_0 = 0, \qquad \lambda_2 = \mu = \frac{E}{(1+\upsilon)}, \qquad \lambda_1 = \lambda = \frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)},$$

E, υ being the Young Modulus and Poisson's ratio of the material of thin plate respectively, then, the two-dimensional Eq. (17) reduces to

$$D^*\nabla^4 w - 4h\frac{E}{(1+\nu)}\nabla^2 w + \rho\frac{\partial^2 w}{\partial t^2} = 0 \tag{19}$$

where the coefficient D^* given by

$$D^* = \frac{h^3}{12}\left(\frac{2\lambda_1\lambda_2}{2\lambda_2+\lambda_1}+2\lambda_2\right) \tag{20a}$$

$$D^* = \frac{h^3}{12}\left(\frac{4\mu(\mu+\lambda)}{2\mu+\lambda}\right) = \frac{Eh^3}{12(1-\upsilon^2)} \tag{20b}$$

is the corresponding flexural rigidity of the resulting isotropic thin plate.

Meanwhile, the governing equation for the free vibration of an isotropic classical Kirchhoff's plate is (Ventsel and Krauthammer 2001, Eq. (9.3))

$$D^*\nabla^4 w + \rho\frac{\partial^2 w}{\partial t^2} = 0 \tag{21}$$

Remark 1. *The obtained Eq. (17) describes the free-vibration of a transversely isotropic non-classical thin plate made of hyperelastic John's material.*

Remark 2. *It is observed that Eq. (19) which governs the free-vibration of the corresponding isotropic plate is a special case of Eq. (17); and is a generalization of the bending model obtained by Fadodun and Akinola (2017).*

Remark 3. *The existence of the middle term in Eq. (17)-(19) shows that the non-classical thin plate under consideration exhibits time-dependent in-plane harmonic force which classical Kirchhoff's plate model does not apprehend.*

4. Effect of transversely isotropic on the flexural rigidity of thin plate

Theorem: Given the effective material constants, $\lambda_j \in \Re^+$, $j = 0,1,2$, the flexural rigidities D and D^* defined in Eqs. (18) and (20(a)) for transversely isotropic and isotropic thin plates respectively, satisfy the inequality $D > D^*$.

Proof:

From the definitions of D and D^*, it is sufficient to show that $F^{TI} > F^I$, where

$$F^{TI} = \frac{2\lambda_1(\lambda_2 + \lambda_0)}{(2\lambda_2 + \lambda_1 + 2\lambda_0)} \tag{22}$$

and

$$F^I = \frac{2\lambda_1\lambda_2}{(2\lambda_2 + \lambda_1)} \tag{23}$$

respectively.
Now, using Eqs. (22) and (23)

$$\frac{F^{TI}}{F^I} = \frac{\left(1 + \dfrac{\lambda_0}{\lambda_2}\right)}{\left(1 + \dfrac{2\lambda_0}{2\lambda_2 + \lambda_1}\right)} \tag{24}$$

Since $\lambda_0 \in \Re^+$, then, Eq. (24) implies that

$$\frac{F^{TI}}{F^I} > 1 \tag{25}$$

Finally, in view of Eqs. (18), (20(a)), (22)-(25), we conclude that

$$D > D^* \tag{26}$$

Remark 4: *The implication of the above theorem is that a transversely isotropic non-classical thin plate under consideration is stiffer than the corresponding isotropic thin plate. That is, the transversely isotropic structure of the plate increases its flexural rigidity.*

5. Natural frequencies

It is well-known that the problem of a freely vibrating plate is often reduced to an eigenvalue problem. The most important part of the problem of free vibration of plate is to determine the natural frequencies and the mode shapes of the vibration associated with each natural frequency. The natural frequencies are the eigenvalues and associated shape functions are the eigenfunctions.

Values of these parameters are necessary for establishing the dynamic stresses caused by variable load. In the present case, we investigate the natural frequencies of a simply supported transversely isotropic non-classical thin plate. The plate occupies the region

$$0 \le x_1 \le a, \quad 0 \le x_2 \le b \;, \; -\frac{h}{2} \le x_3 \le \frac{h}{2}$$

Then, we solve the equation

$$D\nabla^4 w - (4\lambda_2 + \lambda_0)h\nabla^2 w + \rho \frac{\partial^2 w}{\partial t^2} = 0 \tag{27}$$

subject to the boundary and initial conditions

$$w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x_1^2} = 0 \quad \text{at} \quad x_1 = 0, a \tag{28}$$

$$w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x_2^2} = 0 \quad \text{at} \quad x_2 = 0, b \tag{29}$$

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial t} = v_0(x_1, x_2) \quad \text{at} \quad t = 0 \tag{30}$$

where v_0 is the initial velocity at point $P(x_1, x_2)$ and $a, b, h \in \Re^+$.

Let the natural frequencies of the plate problem under consideration be denoted as ϖ_{mn}, $m, n = 1,2,3,\ldots$.

In order to determine ϖ_{mn}, we assume a solution of the form

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin \alpha_m x_1 \sin \beta_n x_2 \sin \varpi_{mn} t \tag{31}$$

where $\alpha_m = \frac{m\pi}{a}$, $\beta_n = \frac{n\pi}{b}$.

It is obvious that Eq. (31) satisfies the boundary and initial conditions in Eqs. (28)-(30). Substituting Eq. (31) in Eq. (27) yields

$$\varpi_{mn} = \sqrt{\frac{D(\alpha_m^2 + \beta_n^2)^2 + h(4\lambda_2 + \lambda_0)(\alpha_m^2 + \beta_n^2)}{\rho h}} \tag{32}$$

In the reduction to an isotropic non-classical thin plate ($D = D^*$, $\varpi_{mn} = \varpi_{mn}^*$, $\lambda_2 = \mu$, $\lambda_0 = 0$), we have

$$\varpi_{mn}^* = \sqrt{\frac{D^*(\alpha_m^2 + \beta_n^2)^2 + 4h\mu(\alpha_m^2 + \beta_n^2)}{\rho h}} \tag{33}$$

Meanwhile, the natural frequencies ϖ_{mn}^K for a rectangular, simply supported, isotropic classical Kirchhoff's plate are (Ventsel and Krauthammer 2001, Eq. (9.11))

$$\varpi_{mn}^{K} = \sqrt{\frac{D^*}{\rho h}}(\alpha_m^2 + \beta_n^2) \tag{34}$$

Comparing Eqs. (32) and (33), and in view of Eq. (26)

$$\varpi_{mn} > \varpi_{mn}^* \tag{35}$$

Further, comparing Eqs. (33) and (34) yields the inequality

$$\varpi_{mn}^* > \varpi_{mn}^K \tag{36}$$

Remark 5: *The inequality in Eq. (35) shows that a transversely isotropic non-classical thin plate has higher natural frequencies than the corresponding isotropic thin plate. That is, the transversely isotropic structure of the plate increases its natural frequencies. Further, it is observed in Eq. (36) that an isotropic non-classical thin plate under consideration has higher natural frequencies than an isotropic classical Kirchhoff's plate.*

6. Exact solution of forced vibration of thin plate

Consider the forced vibration of a transversely isotropic non-classical thin rectangular plate of sides a and b. The plate is assumed to be fully simply supported and subjected to a surface transverse load $p = p_0(x_1, x_2)\cos\theta t$, where θ is the frequency of forced vibration, which is equal to the frequency of a disturbing loading. In this case, the governing equation of motion of plate is

$$D\nabla^4 w - (4\lambda_2 + \lambda_0)h\nabla^2 w + \rho\frac{\partial^2 w}{\partial t^2} = p \tag{37}$$

where $p = p_0(x_1, x_2)\cos\theta t$ is a variable, time-dependent, transverse load.

The boundary and initial conditions are

$$w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x_1^2} = 0 \quad \text{at} \quad x_1 = 0, a \tag{38}$$

$$w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x_2^2} = 0 \quad \text{at} \quad x_2 = 0, b \tag{39}$$

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial t} = 0 \quad \text{at} \quad t = 0 \tag{40}$$

In the solution of the above problem Eqs. (37)-(40), the deflection $w = w(x_1, x_2, t)$ and applied load $p = p(x_1, x_2, t)$ are expressed in the form of infinite Fourier series

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\varphi_{mn}(t)\sin\alpha_m x_1 \sin\beta_n x_2 \tag{41}$$

$$p = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}g_{mn}(t)\sin\alpha_m x_1 \sin\beta_n x_2 \tag{42}$$

where $\varphi_{mn}(t)$ and $g_{mn}(t)$ are functions to be determined. It can be easily verify that Eq. (41) automatically satisfies the prescribed boundary conditions in Eqs. (38) and (39).
Substituting Eqs. (41) and (42) in Eq. (37) gives

$$\frac{d^2}{dt^2}\varphi_{mn} + \varpi_{mn}^2\varphi_{mn}(t) = \frac{g_{mn}(t)}{\rho h} \tag{43}$$

where

$$\varpi_{mn} = \sqrt{\frac{D(\alpha_m^2+\beta_n^2)^2 + h(4\lambda_2+\lambda_0)(\alpha_m^2+\beta_n^2)}{\rho h}}$$

Expanding the applied load $p = p_0 \cos\theta t$ in the form

$$p = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}p_{mn}\cos\theta t \sin\alpha_m x_1 \sin\beta_n x_2 \tag{44}$$

implies that

$$g_{mn}(t) = p_{mn}\cos\theta t \tag{45}$$

$$p_{mn} = \int_0^a\int_0^b p_0 \sin\alpha_m x_1 \sin\beta_n x_2 dx_1 dx_2 \tag{46}$$

The solution of Eq. (43) in view of Eqs. (44) and (45) gives

$$\varphi_{mn}(t) = X_m \cos\varpi_{mn}t + Y_{mn}\sin\varpi_{mn}t + \frac{p_{mn}\cos\theta t}{\rho h(\varpi_{mn}^2-\theta^2)} \tag{47}$$

Using the initial conditions in Eq. (40) yields

$$X_{mn} = \frac{p_{mn}\cos\theta t}{\rho h(\varpi_{mn}^2-\theta^2)} \quad \text{and} \quad Y_{mn} = 0 \tag{48}$$

The constants p_{mn} are determined from Eq. (46)

$$p_{mn} = \frac{16p_0}{\pi^2 mn}, \qquad m,n = 1,3,5,\ldots \tag{49}$$

Substituting Eqs. (47)-(49) in Eq. (41) gives

$$w = \frac{16 p_0}{\rho h \pi^2} \sum_m^{\infty} \sum_n^{\infty} \left(\frac{\cos \theta t - \cos \varpi_{mn} t}{\varpi_{mn}^2 - \theta^2} \right) \sin \alpha_m x_1 \sin \beta_n x_2 \tag{50}$$

$m, n = 1,3,5,7,...$

In the case when the load frequency θ coincides with any natural frequencies ϖ_{mn} of the plate, that is

$$\theta = \varpi_{mn} \tag{51}$$

the plate vibrates in the resonance state.

The corresponding resonance term of series in Eq. (51) takes the form

$$w_{mn} = \left(\frac{8t \sin \varpi_{mn} t}{\rho h \pi^2 \varpi_{mn}} \right) p_0 \sin \alpha_m x_1 \sin \beta_n x_2 \tag{52}$$

Eq. (52) represents a vibration with amplitude that indefinitely increases with time.

Remark 6: *Eq. (51) and the inequalities in Eqs. (35) and (36) indicate that a transversely isotropic non-classical thin plate resonates at higher frequency than the corresponding isotropic thin plate. Further, an isotropic non-classical thin plate also resonates at higher frequency than an isotropic classical Kirchhoff plate.*

7. Conclusions

The study presents a two-dimensional equation for investigating and analyzing the vibration of a transversely isotropic thin plate structures in various areas of engineering. In the case of free and force vibrations of a fully simply supported rectangular plate, it is obtained that the non-classical plate under consideration has higher natural and resonance frequencies than the classical Kirchhoff' plate. Further, the parameter characterizing the transversely isotropic structure of the plate increases its flexural rigidity which in turn increases both the natural and renounce frequencies. This work finds applications in the analysis and modern design of building foundations and bridge decks.

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