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Free transverse vibration of shear deformable super-elliptical plates

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Abstract. Free transverse vibration of shear deformable super-elliptical plates with uniform thickness was studied based on Mindlin plate theory using finite element method. Quadrilateral isoparametric elements were used in the paper. Sensitivity analysis was made to determine the influence of the thickness, the aspect ratio, and the shape of the plate on the natural frequency. Accuracy of the results computed in the current study was validated by comparing them with the solutions available in the literature. The results reveal that the frequencies of clamped super-elliptical plates lie in the range bounded by elliptical and rectangular plates irrespective of the aspect ratio, and furthermore, the frequency decreases if the super-elliptical power increases. A similar trend was observed for simply supported plates with high aspect ratio. The free vibration response for the first and the second symmetric-antisymmetric (SA) modes were found to be different for high aspect ratio. The results reveal that using insufficient number of degrees of freedom results in finding a totally different relation between the super-elliptical power and the frequency.

Keywords: plate; vibration; frequency; Mindlin; finite element

1. Introduction

Plates are straight, plane, two-dimensional structural components of which thickness is much smaller than the other dimensions (Szilard 2004). Plates with different shapes are used as primary or secondary structural components in various disciplines such as civil, mechanical, aerospace, and naval engineering (Senjanovic *et al*. 2014). Like all structural members plates are subject to excitation, and therefore understanding the dynamic behavior is crucial for design purposes. Since the data of transient vibration are used to determine the resonance frequencies which are essential for engineers the extensive literature on the vibration of plates has basically focused on free vibration analysis which may be induced by wind, blast, earthquake or machines (Leissa 1973, Rao and Prasad 1975, Iyengaar and Raman 1978, Narita 1986, Liew *et al*. 1990, Lam *et al*. 1992, Singh and Chakraverty 1992, Liew *et al*. 1993a, Wang and Wang 1994, Wang 1994, Liew *et al*. 1995, Wang *et al*. 1995, Geannakakes 1995, Wang *et al*. 1995a, Liew 1996, Bert and Malik 1996, Lin and Tseng 1998, Han and Liew 1999, Liew and Theo 1999, Duran *et al*. 1999, Wu and Liu 2001, Zhou *et al*. 2002, Bayer *et al*. 2002, Kim 2003, Zhou *et al*. 2003, Hashemi and Arsanjani 2005, Zhou *et al*. 2006, Zhong and Yu 2007, Nallim and Grossi 2008, Civalek and Ersoy 2009, Civalek and Ozturk 2010, El-Sayad and Ghazy 2012, Farag *et al*. 2013, Eftekhari and Jafari 2013,

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Thai and Choi 2013, Senjanovic *et al*. 2013, Wang 2015a, 2015b, Abdelbari *et al*. 2016). Except a few exceptions 2-D equations have almost always been used in these investigations. This is partly because dealing with 2-D equations is relatively simple than 3-D equations, and partly because 3-D equations involve inevitably numerical errors of experimental nature as well as 2-D equations do (Altay and Dokmeci 2006). Apart from a limited number of studies, numerical methods have been used in the publications since closed form solutions are available only for a few cases depending on (i) the geometry, (ii) the material properties, (iii) the plate theory, and (iv) the boundary conditions. Most of these published papers deal with thin plates for which the effect of shear deformation is neglected (Liew *et al*. 1995). However, ignoring this effect results in overestimating the vibration frequencies (Wang 1994, Liew 1996). To overcome this defect, a significant rise in the number of studies taking into account the shear deformation has been observed especially in the last two decades, and most of these investigations have been reported for rectangular or circular plates.

Super-elliptical plates include a large variety of plate shapes ranging from an ellipse to a rectangle with rounded corners (Liew *et al*. 1998). Accordingly, an ellipse and a rectangle are the limiting shapes from geometrical aspect. Due to the diffusion of stress concentration at the corners this class of plates is superior to the rectangular plates in engineering applications (Lim *et al*. 1998). Such a plate shape is widely used in automative, and ship building industries, and also in aircrafts. However, despite their common use, the studies on super-elliptical plates have been relatively scanty (Wu and Liu 2005, Ceribasi 2012, Jazi and Farhatnia 2012, Tang *et al*. 2012, Zhang 2013, Zhang and Zhou 2014, Hasheminejad 2014), and these studies have mostly investigated the vibration analysis by the Ritz method (DeCapua and Sun 1972, Wang and Wang 1994, Lim and Liew 1995, Liew *et al*. 1998, Altekin 2008, 2009, 2010a, 2010b, 2014, Lim *et al*. 1998, Chen *et al*. 1999, Chen and Kitipornchai 2000, Liew and Feng 2001, Zhou *et al*. 2004, Altekin and Altay 2008, Ceribasi and Altay 2009, Wang 2015). Free vibration of perforated super-elliptical plates was analyzed based on the linear, small strain, three dimensional elasticity theory (Liew and Feng 2001). Chen *et al*. (1999) employed Reddy's third order plate theory to investigate the vibration of laminated thick plates. Liew *et al*. (1998) conducted a study on the free vibration of thick plates using a higher order shear deformation theory. Vibration of perforated plates was presented on the basis of Kirchhoff plate theory by Lim and Liew (1995). Transverse and inplane vibration of thin plates were studied numerically by DeCapua and Sun (1972), Wang and Wang (1994), Altekin (2008, 2009, 2010b), and Ceribasi and Altay (2009). Lim *et al*. (1998) reported the free vibration analysis of laminated composite plates by the Ritz method, and LUSAS (a commercial finite element package). Zhou *et al*. (2004) made three-dimensional free vibration analysis based on the exact, small-strain and linear elasticity theory. Chen and Kitipornchai (2000) sought the free vibration of symmetrically laminated thick-perforated plates using Reddy's higher order theory. Recently Wang (2015) examined completely free thin plates by the Ritz method.

The current work was motivated by the scarcity of the studies on the dynamic behavior of moderately thick super-elliptical plates. The solution was obtained by the finite element method (FEM) based on the first order shear deformation theory (FSDT). Since quadrilateral elements have been effectively used in plates with curved edges (Kutlu and Omurtag 2012, Kutlu *et al*. 2012), the plate domain was discretized by quadrilateral isoparametric elements. As far as the author knows, the relation between the super-elliptical power, and the frequency parameter has not been investigated in detail due to lack of numerical results for greater values of aspect ratio than those presented in the previous studies. The current work attempts to address this gap in the literature by investigating the influence of (i) the boundary conditions, (ii) the super-elliptical power, (iii) the aspect ratio, and (iv) the thickness of the plate on the frequency parameter of shear deformable super-elliptical plates by sensitivity analysis. Convergence studies were made for h-refinement (i.e., more of the same kind of elements (Reddy 1993)). Accuracy of the results obtained in the current paper was validated through comparison studies, and good agreement was obtained. Since not only the mode sequence number but also the type of vibration mode is essential for design purposes, the first two modes corresponding to doubly symmetric (SS), doubly antisymmetric (AA), and symmetric-antisymmetric (SA and AS) classes were presented in the current study. Some of the numerical results were tabulated for future reference.

2. Formulation

The perimeter of the plate is defined by

$$\left(\frac{x}{a}\right)^{2k} + \left(\frac{y}{b}\right)^{2k} = 1, \qquad k = 1, 2, \ldots, \infty \tag{1}$$

where k is an integer controlling the roundness of the corner (Fig. 1). Four-noded quadrilateral plate bending element with straight boundaries developed by Hughes *et al*. (1977) was used in discretizing the plate domain (Krishnamoorthy 1994). The geometry of the isoparametric element is identified by

$$x = \sum_{i=1}^{4} N_i x_i, \qquad y = \sum_{i=1}^{4} N_i y_i, \qquad N_i = \frac{1}{4}\left(1 + r r_i\right)\left(1 + s s_i\right) \tag{2}$$

Three field variables (i.e., degrees of freedom) per node were introduced for each element. These field variables are the deflection, and the rotations defined by

$$w = \sum_{i=1}^{4} N_i w_i, \qquad \theta_x = \sum_{i=1}^{4} N_i \theta_{xi}, \qquad \theta_y = \sum_{i=1}^{4} N_i \theta_{yi} \tag{3}$$

The element shape functions are bilinear for transverse displacement and rotations. C^0 continuity for the displacement model was ensured on the basis of Mindlin's plate theory (Krishnamoorthy 1994). The shear locking was prevented by separating the shear and bending energy terms and using selective integration procedure (Krishnamoorthy 1994). The nodal displacement vector and the strain vector are related by (Krishnamoorthy 1994)

$$\left\{d_i\right\}^T = \left\{w_i \quad \theta_{xi} \quad \theta_{yi}\right\}, \qquad \left\{\varepsilon\right\} = \sum_{i=1}^{4} \left[B_i\right]\left\{d_i\right\} \tag{4}$$

$$\left[B_i\right] = \begin{bmatrix} 0 & 0 & \dfrac{\partial N_i}{\partial x} \\ 0 & -\dfrac{\partial N_i}{\partial y} & 0 \\ 0 & -\dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial x} & 0 & N_i \\ \dfrac{\partial N_i}{\partial y} & -N_i & 0 \end{bmatrix}, \qquad \left\{\varepsilon\right\} = \begin{Bmatrix} k_x = \sum_{i=1}^{4} \theta_{yi} \dfrac{\partial N_i}{\partial x} \\ k_y = -\sum_{i=1}^{4} \theta_{xi} \dfrac{\partial N_i}{\partial y} \\ k_{xy} = \sum_{i=1}^{4} \theta_{yi} \dfrac{\partial N_i}{\partial y} - \sum_{i=1}^{4} \theta_{xi} \dfrac{\partial N_i}{\partial x} \\ \phi_x = \sum_{i=1}^{4} w_i \dfrac{\partial N_i}{\partial x} + \sum_{i=1}^{4} \theta_{yi} N_i \\ \phi_y = \sum_{i=1}^{4} w_i \dfrac{\partial N_i}{\partial y} - \sum_{i=1}^{4} \theta_{xi} N_i \end{Bmatrix} \tag{5}$$

The element stiffness matrix is given by (Krishnamoorthy 1994)

$$[k_e] = \iint_A [B]^T [C][B] dxdy, \qquad [k_e] = [k_B] + [k_S] \tag{6}$$

where

$$[B] = \big[[B_1] \quad [B_2] \quad [B_3] \quad [B_4]\big], \qquad [C] = \begin{bmatrix} [C_B] & [0] \\ [0] & [C_S] \end{bmatrix} \tag{7}$$

$$[C_B] = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix}, \qquad [C_S] = D_S \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{8}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}, \qquad G = \frac{E}{2(1+\nu)}, \qquad D_s = Gh\kappa \tag{9}$$

The element mass matrix is given by

$$[m_e] = \iint_{(A)} [\bar{M}] dx\, dy, \qquad [\bar{M}] = [N]^T [I][N] \tag{10}$$

where

$$[N] = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix} \tag{11}$$

$$[I] = \rho h \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{h^2}{12} \end{bmatrix}, \qquad m = \rho h \tag{12}$$

The circular frequencies are the eigenvalues of the equilibrium equation which governs the undamped free vibration response of a system of finite elements given by (Bathe 1996)

$$[K]\{U\} + [M]\{\ddot{U}\} = \{0\}, \qquad \{U\} = \{u\}\sin(\omega t), \qquad \{\ddot{U}\} = \frac{d^2}{dt^2}\{U\} \tag{13}$$

3. Analysis

The modes of transverse vibration which were analyzed separately, were categorized into four symmetry groups identified by SS, SA, AS, and AA where the first and the second letters indicate symmetry (S) or antisymmetry (A) about *y*-axis, and *x*-axis, respectively (Narita 1984). The quarter of the plate was considered in the solution. Simply supported (S) and clamped (C) super-elliptical plates with uniform thickness were examined. The geometrical boundary conditions along the plate perimeter, and the symmetric and/or antisymmetric boundary conditions along *x* and *y* axes were satisfied exactly (Table 1).

The mesh pattern used in the paper is composed of nonuniform four-noded quadrilateral elements (Fig. 2). An algorithm which was coded in Python by the author was used for automatic mesh generation. The material was assumed to be homogeneous and isotropic. The parameters used in the analysis are

$$c = \frac{a}{b}, \qquad \eta = \frac{h}{b}, \qquad \lambda = \omega a^2 \sqrt{\frac{m}{D}}, \qquad \kappa = \frac{5}{6}, \qquad \nu = 0.3 \tag{14}$$

4. Numerical results

Convergence studies were made for a large variety of plate shapes ranging from an ellipse to a rectangle with various parameters given by

$$\eta = \{0.002,\ 0.010,\ 0.020,\ 0.050,\ 0.100\}, \qquad c = \{1,\ 2,\ 3,\ 5,\ 10,\ 18,\ 20,\ 25\} \tag{15}$$

The accuracy of the results was validated by comparing the nondimensional frequencies with those obtained for elliptical (e), rectangular (r), and super-elliptical plates in Appendix A. Satisfactory convergence was achieved using fine mesh and close agreement was obtained in the comparison tests which were made for thin and moderately thick plates (Tables A1-A7).

The results presented in the study were obtained for p=23 which corresponds to 1657 nodes and 1587 meshes in the quarter of the plate domain. Some of the tabulated numerical values of the nondimensional frequency of super-elliptical plates were presented for future reference (Tables 2-11).

Table 1 Boundary and continuity conditions in the quarter of the plate $\left(x \in [0 \quad a],\quad y \in [0 \quad b]\right)$

	(C)	(C)	(C)	(C)	(S)	(S)	(S)	(S)
Natural mode	SA	SS	AS	AA	SA	SS	AS	AA
On the boundary defined by $\left(\frac{x}{a}\right)^{2k} + \left(\frac{y}{b}\right)^{2k} = 1$	$w=0$ $\theta_x=0$ $\theta_y=0$	$w=0$ $\theta_x=0$ $\theta_y=0$	$w=0$ $\theta_x=0$ $\theta_y=0$	$w=0$ $\theta_x=0$ $\theta_y=0$	$w=0$	$w=0$	$w=0$	$w=0$
Along *x*-axis	$w=0$	$\theta_x=0$	$\theta_x=0$	$w=0$	$w=0$	$\theta_x=0$	$\theta_x=0$	$w=0$
Along *y*-axis	$\theta_y=0$	$\theta_y=0$	$w=0$	$w=0$	$\theta_y=0$	$\theta_y=0$	$w=0$	$w=0$

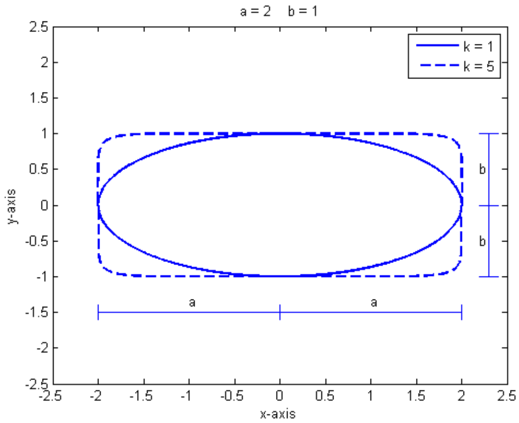

Fig. 1 Geometry of a super-elliptical plate (c=2)

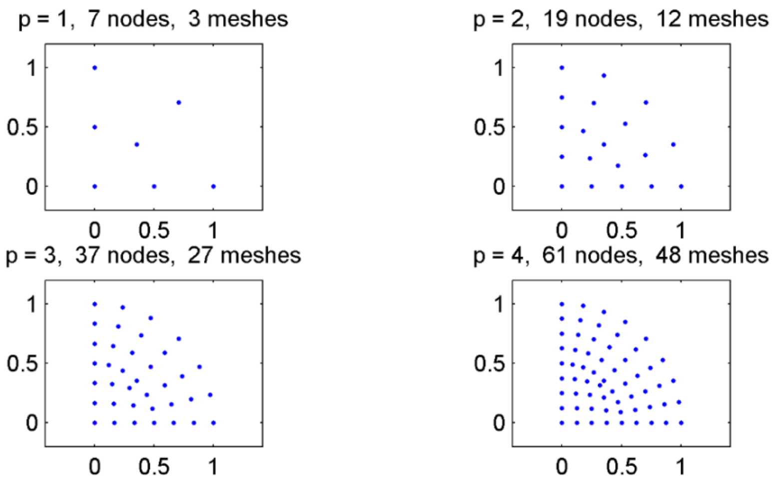

Fig. 2 Location of the nodes in a quarter of the plate domain (k=1, c=1)

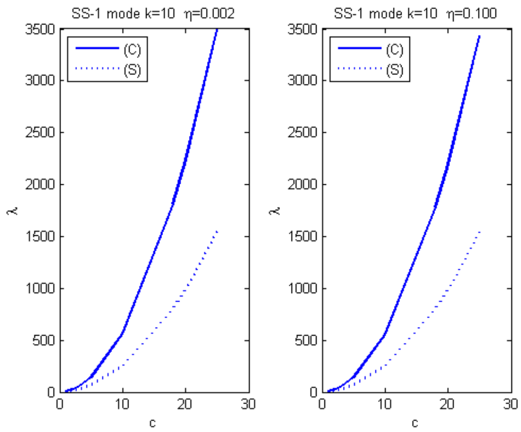

Fig. 3 Influence of thickness on λ

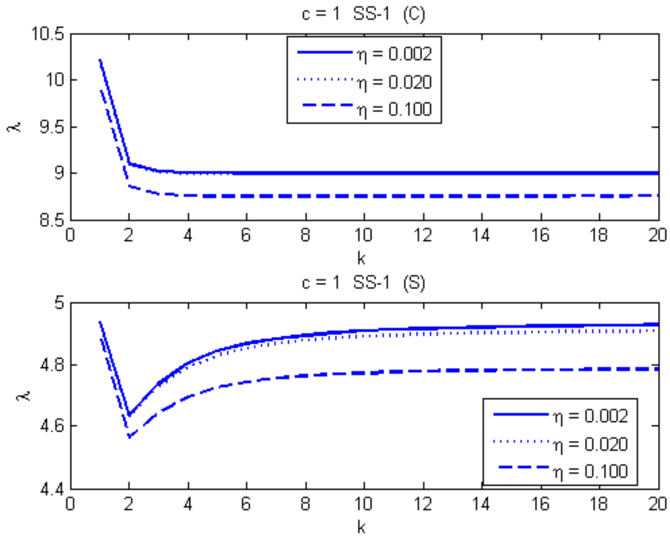

Fig. 4 Influence of thickness on λ for c=1

Table 2 Nondimensional circular frequency λ for SA-1 modes of (C) plate (p=23, η=0.050)

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
2	20.9307	69.0051	148.386	399.090	1560.57	5006.92	6173.84	9625.45
4	18.4105	64.1834	141.107	386.678	1534.53	4958.06	6119.18	9556.17
6	18.1496	63.5133	139.962	384.749	1530.98	4952.29	6112.90	9548.68
8	18.0977	63.3542	139.638	384.117	1529.84	4950.51	6110.99	9546.45
10	18.0835	63.3056	139.527	383.857	1529.32	4949.73	6110.14	9545.47
12	18.0786	63.2879	139.483	383.738	1529.04	4949.30	6109.69	9544.95
14	18.0767	63.2806	139.463	383.680	1528.88	4949.05	6109.42	9544.64
16	18.0758	63.2772	139.454	383.650	1528.79	4948.89	6109.24	9544.44
18	18.0754	63.2755	139.449	383.633	1528.73	4948.78	6109.13	9544.30
20	18.0752	63.2746	139.446	383.623	1528.69	4948.71	6109.05	9544.21
22	18.0750	63.2741	139.445	383.618	1528.67	4948.66	6108.99	9544.14
24	18.0749	63.2737	139.444	383.614	1528.65	4948.62	6108.95	9544.09
26	18.0749	63.2735	139.443	383.611	1528.64	4948.59	6108.92	9544.06
28	18.0749	63.2734	139.443	383.610	1528.63	4948.57	6108.90	9544.03
30	18.0748	63.2733	139.443	383.609	1528.63	4948.56	6108.88	9544.01
32	18.0748	63.2733	139.442	383.608	1528.62	4948.55	6108.87	9543.99
34	18.0748	63.2732	139.442	383.607	1528.62	4948.54	6108.86	9543.98
36	18.0748	63.2732	139.442	383.607	1528.62	4948.53	6108.85	9543.97
38	18.0748	63.2732	139.442	383.607	1528.61	4948.52	6108.84	9543.96
40	18.0748	63.2732	139.442	383.607	1528.61	4948.52	6108.84	9543.95
100	18.0747	63.2731	139.442	383.606	1528.61	4948.50	6108.81	9543.92
400	18.0747	63.2731	139.442	383.606	1528.61	4948.51	6108.82	9543.93

Table 3 Nondimensional circular frequency λ for SS-1 modes of (C) plate (p=23, η=0.050)

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
2	10.1508	27.2453	56.5368	148.972	575.376	1837.21	2264.11	3526.39
4	9.03668	24.7691	52.6282	142.342	561.650	1811.37	2235.14	3489.43
6	8.95247	24.5243	52.1197	141.303	559.641	1808.01	2231.46	3484.96
8	8.93859	24.4783	52.0074	141.006	558.973	1806.94	2230.29	3483.57
10	8.93522	24.4661	51.9747	140.904	558.677	1806.45	2229.76	3482.93
12	8.93418	24.4620	51.9631	140.863	558.531	1806.18	2229.46	3482.58
14	8.93379	24.4605	51.9584	140.845	558.455	1806.02	2229.29	3482.37
16	8.93362	24.4598	51.9562	140.837	558.413	1805.91	2229.17	3482.23
18	8.93354	24.4595	51.9551	140.832	558.388	1805.85	2229.10	3482.13
20	8.93350	24.4593	51.9545	140.829	558.374	1805.80	2229.05	3482.06
22	8.93347	24.4592	51.9542	140.828	558.364	1805.77	2229.01	3482.01
24	8.93345	24.4591	51.9540	140.827	558.359	1805.75	2228.99	3481.98
26	8.93344	24.4591	51.9538	140.826	558.354	1805.74	2228.97	3481.95
28	8.93343	24.4591	51.9538	140.826	558.352	1805.73	2228.96	3481.93
30	8.93342	24.4590	51.9537	140.826	558.350	1805.72	2228.95	3481.92
32	8.93341	24.4590	51.9536	140.825	558.348	1805.71	2228.94	3481.91
34	8.93341	24.4590	51.9536	140.825	558.347	1805.71	2228.93	3481.90
36	8.93341	24.4590	51.9536	140.825	558.346	1805.71	2228.93	3481.89
38	8.93340	24.4590	51.9535	140.825	558.346	1805.70	2228.92	3481.88
40	8.93340	24.4590	51.9535	140.825	558.345	1805.70	2228.92	3481.88
100	8.93337	24.4589	51.9534	140.824	558.342	1805.69	2228.90	3481.85
400	8.93336	24.4589	51.9533	140.824	558.341	1805.68	2228.90	3481.84

Table 4 Nondimensional circular frequency λ for AS-1 modes of (C) plate (p=23, η=0.050)

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
2	20.9307	39.1693	71.1090	170.076	613.911	1904.87	2339.25	3620.56
4	18.4105	32.5654	60.0128	150.892	574.386	1830.39	2255.68	3513.76
6	18.1496	31.8189	58.4320	147.486	567.630	1819.05	2243.23	3498.57
8	18.0977	31.6623	58.0591	146.483	565.255	1815.17	2239.02	3493.52
10	18.0835	31.6173	57.9437	146.127	564.183	1813.35	2237.03	3491.15
12	18.0786	31.6015	57.9008	145.982	563.649	1812.33	2235.92	3489.83
14	18.0767	31.5951	57.8827	145.916	563.367	1811.72	2235.25	3489.01
16	18.0758	31.5921	57.8742	145.884	563.210	1811.33	2234.81	3488.47
18	18.0754	31.5907	57.8699	145.867	563.120	1811.08	2234.52	3488.09
20	18.0752	31.5899	57.8676	145.857	563.065	1810.91	2234.33	3487.83
22	18.0750	31.5895	57.8662	145.851	563.030	1810.80	2234.19	3487.64
24	18.0749	31.5892	57.8654	145.848	563.008	1810.72	2234.10	3487.51
26	18.0749	31.5890	57.8648	145.845	562.993	1810.66	2234.03	3487.41
28	18.0749	31.5889	57.8645	145.844	562.982	1810.62	2233.98	3487.33

Table 4 Continued

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
30	18.0748	31.5888	57.8642	145.843	562.975	1810.59	2233.94	3487.27
32	18.0748	31.5887	57.8640	145.842	562.970	1810.57	2233.91	3487.23
34	18.0748	31.5887	57.8639	145.841	562.965	1810.55	2233.89	3487.19
36	18.0748	31.5886	57.8638	145.841	562.962	1810.54	2233.87	3487.17
38	18.0748	31.5886	57.8637	145.841	562.960	1810.53	2233.86	3487.14
40	18.0748	31.5886	57.8637	145.840	562.958	1810.52	2233.85	3487.13
100	18.0747	31.5884	57.8632	145.839	562.948	1810.47	2233.78	3487.02
400	18.0747	31.5884	57.8633	145.839	562.948	1810.47	2233.78	3487.01

Table 5 Nondimensional circular frequency λ for AA-1 modes of (C) plate (p=23, η=0.050)

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
2	33.8261	86.3594	171.658	435.062	1629.20	5128.01	6307.88	9791.05
4	27.4436	72.9097	151.068	399.863	1555.18	4989.13	6152.64	9595.23
6	26.6177	70.8212	147.403	393.511	1543.33	4969.62	6131.38	9569.88
8	26.4316	70.2936	146.326	391.325	1539.25	4963.24	6124.50	9561.84
10	26.3759	70.1239	145.943	390.405	1537.35	4960.32	6121.36	9558.20
12	26.3558	70.0594	145.786	389.979	1536.32	4958.72	6119.64	9556.22
14	26.3474	70.0316	145.715	389.767	1535.72	4957.75	6118.60	9555.02
16	26.3435	70.0184	145.680	389.654	1535.35	4957.12	6117.92	9554.23
18	26.3415	70.0117	145.662	389.591	1535.12	4956.69	6117.46	9553.69
20	26.3405	70.0079	145.651	389.554	1534.98	4956.39	6117.13	9553.31
22	26.3398	70.0058	145.645	389.531	1534.88	4956.18	6116.90	9553.03
24	26.3394	70.0045	145.641	389.517	1534.81	4956.02	6116.73	9552.83
26	26.3392	70.0036	145.639	389.508	1534.77	4955.91	6116.60	9552.67
28	26.3390	70.0030	145.637	389.501	1534.74	4955.82	6116.50	9552.55
30	26.3388	70.0026	145.636	389.497	1534.71	4955.76	6116.43	9552.46
32	26.3388	70.0024	145.635	389.493	1534.69	4955.71	6116.37	9552.38
34	26.3387	70.0022	145.635	389.491	1534.68	4955.67	6116.33	9552.32
36	26.3386	70.0020	145.634	389.489	1534.67	4955.64	6116.29	9552.28
38	26.3386	70.0019	145.634	389.488	1534.66	4955.62	6116.26	9552.24
40	26.3385	70.0018	145.634	389.487	1534.66	4955.60	6116.24	9552.21
100	26.3382	70.0012	145.632	389.483	1534.63	4955.50	6116.12	9552.03
400	26.3382	70.0013	145.633	389.484	1534.64	4955.51	6116.13	9552.03

Table 6 Nondimensional circular frequency λ for SA-1 modes of (S) plate (p=23, η=0.050)

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
2	13.7477	45.8135	97.9101	261.262	1014	3241.21	3994.70	6222.60
4	12.1183	42.2069	92.1458	250.911	991.336	3197.72	3945.86	6160.23
6	12.0096	41.6864	91.1608	249.144	987.924	3191.99	3939.59	6152.66

Table 6 Continued

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
8	12.0282	41.5874	90.8907	248.555	986.774	3190.14	3937.59	6150.28
10	12.0555	41.5755	90.8088	248.318	986.246	3189.30	3936.67	6149.20
12	12.0772	41.5825	90.7843	248.214	985.967	3188.84	3936.17	6148.61
14	12.0931	41.5926	90.7789	248.166	985.808	3188.55	3935.87	6148.25
16	12.1047	41.6018	90.7798	248.143	985.715	3188.37	3935.67	6148.01
18	12.1132	41.6095	90.7828	248.132	985.657	3188.25	3935.53	6147.84
20	12.1196	41.6156	90.7862	248.126	985.621	3188.17	3935.44	6147.73
22	12.1245	41.6206	90.7894	248.124	985.598	3188.11	3935.37	6147.64
24	12.1283	41.6246	90.7922	248.123	985.583	3188.07	3935.32	6147.58
26	12.1312	41.6278	90.7947	248.122	985.572	3188.03	3935.29	6147.53
28	12.1336	41.6304	90.7968	248.123	985.565	3188.01	3935.26	6147.49
30	12.1356	41.6326	90.7986	248.123	985.560	3187.99	3935.24	6147.46
32	12.1372	41.6344	90.8002	248.124	985.556	3187.98	3935.22	6147.44
34	12.1385	41.6360	90.8015	248.124	985.553	3187.97	3935.21	6147.43
36	12.1396	41.6372	90.8027	248.125	985.551	3187.96	3935.20	6147.41
38	12.1405	41.6383	90.8036	248.125	985.550	3187.96	3935.20	6147.40
40	12.1413	41.6393	90.8045	248.126	985.548	3187.95	3935.19	6147.39
100	12.1474	41.6467	90.8117	248.131	985.544	3187.92	3935.16	6147.34
400	12.1488	41.6485	90.8136	248.133	985.546	3187.92	3935.15	6147.34

Table 7 Nondimensional circular frequency λ for SS-1 modes of (S) plate (p=23, η=0.050)

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
2	4.92681	13.1811	27.0124	69.5410	262.218	827.184	1017.79	1580.71
4	4.60996	11.9856	24.6079	64.8965	251.750	806.579	994.547	1550.72
6	4.70177	12.0258	24.4260	64.1654	249.947	803.379	991.012	1546.35
8	4.76133	12.0952	24.4430	64.0094	249.349	802.283	989.807	1544.88
10	4.79620	12.1426	24.4747	63.9778	249.112	801.779	989.249	1544.20
12	4.81750	12.1734	24.5002	63.9765	249.010	801.518	988.953	1543.82
14	4.83120	12.1939	24.5188	63.9824	248.963	801.373	988.785	1543.60
16	4.84043	12.2081	24.5324	63.9892	248.941	801.290	988.686	1543.46
18	4.84689	12.2182	24.5423	63.9955	248.930	801.240	988.626	1543.38
20	4.85157	12.2256	24.5498	64.0007	248.925	801.210	988.589	1543.32
22	4.85504	12.2311	24.5555	64.0051	248.922	801.190	988.564	1543.28
24	4.85769	12.2354	24.5599	64.0086	248.921	801.177	988.548	1543.26
26	4.85974	12.2388	24.5635	64.0116	248.921	801.169	988.537	1543.24
28	4.86137	12.2414	24.5663	64.0140	248.921	801.163	988.529	1543.23
30	4.86267	12.2436	24.5686	64.0160	248.921	801.159	988.523	1543.22
32	4.86373	12.2453	24.5705	64.0177	248.922	801.156	988.519	1543.21
34	4.86460	12.2468	24.5720	64.0191	248.922	801.154	988.516	1543.21

Table 7 Continued

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
36	4.86533	12.2480	24.5734	64.0203	248.923	801.152	988.514	1543.20
38	4.86594	12.2490	24.5745	64.0214	248.923	801.151	988.512	1543.20
40	4.86646	12.2499	24.5754	64.0223	248.923	801.150	988.511	1543.20
100	4.87035	12.2565	24.5829	64.0296	248.928	801.147	988.505	1543.18
400	4.87123	12.2580	24.5846	64.0315	248.930	801.147	988.505	1543.18

Table 8 Nondimensional circular frequency λ for AS-1 modes of (S) plate ($p=23$, $\eta=0.050$)

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
2	13.7477	23.4569	39.8413	88.3392	296.502	887.128	1084.34	1664.13
4	12.1183	19.3375	32.2373	74.0987	265.760	827.553	1017.20	1577.55
6	12.0096	19.1170	31.5028	71.7551	259.800	816.886	1005.40	1562.92
8	12.0282	19.1783	31.4615	71.2258	257.759	813.020	1001.13	1557.69
10	12.0555	19.2514	31.5145	71.1040	256.941	811.204	999.108	1555.17
12	12.0772	19.3078	31.5708	71.0891	256.585	810.255	998.026	1553.79
14	12.0931	19.3488	31.6165	71.1032	256.421	809.732	997.415	1552.97
16	12.1047	19.3786	31.6516	71.1234	256.343	809.431	997.057	1552.47
18	12.1132	19.4006	31.6784	71.1430	256.305	809.254	996.840	1552.15
20	12.1196	19.4171	31.6991	71.1601	256.287	809.145	996.706	1551.95
22	12.1245	19.4297	31.7152	71.1746	256.279	809.077	996.620	1551.82
24	12.1283	19.4396	31.7279	71.1866	256.276	809.033	996.564	1551.73
26	12.1312	19.4474	31.7380	71.1967	256.276	809.005	996.526	1551.66
28	12.1336	19.4536	31.7462	71.2051	256.278	808.986	996.501	1551.62
30	12.1356	19.4587	31.7530	71.2122	256.280	808.974	996.484	1551.59
32	12.1372	19.4629	31.7586	71.2182	256.283	808.965	996.472	1551.57
34	12.1385	19.4664	31.7632	71.2233	256.285	808.960	996.464	1551.55
36	12.1396	19.4693	31.7672	71.2276	256.288	808.956	996.458	1551.54
38	12.1405	19.4718	31.7705	71.2314	256.290	808.953	996.454	1551.53
40	12.1413	19.4739	31.7734	71.2347	256.292	808.952	996.451	1551.53
100	12.1474	19.4902	31.7963	71.2621	256.317	808.962	996.456	1551.52
400	12.1488	19.4941	31.8019	71.2696	256.327	808.976	996.471	1551.53

Table 9 Nondimensional circular frequency λ for AA-1 modes of (S) plate ($p=23$, $\eta=0.050$)

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
2	24.9567	61.7652	119.539	294.805	1078.06	3354.79	4120.73	6379.76
4	19.7132	50.6222	102.208	264.515	1012.78	3230.09	3980.81	6201.40
6	19.0695	48.8733	98.9986	258.692	1001.43	3210.83	3959.68	6175.79
8	18.9811	48.4922	98.0878	256.676	997.385	3204.27	3952.55	6167.29
10	18.9929	48.4179	97.7978	255.849	995.478	3201.18	3949.21	6163.34
12	19.0207	48.4215	97.7049	255.481	994.456	3199.47	3947.35	6161.14

Table 9 Continued

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
14	19.0468	48.4438	97.6803	255.309	993.872	3198.41	3946.20	6159.77
16	19.0683	48.4680	97.6801	255.225	993.525	3197.73	3945.45	6158.87
18	19.0852	48.4896	97.6883	255.183	993.312	3197.27	3944.93	6158.24
20	19.0984	48.5077	97.6988	255.163	993.177	3196.95	3944.58	6157.80
22	19.1088	48.5226	97.7093	255.154	993.090	3196.72	3944.32	6157.47
24	19.1171	48.5349	97.7188	255.150	993.033	3196.56	3944.13	6157.23
26	19.1238	48.5449	97.7273	255.149	992.993	3196.45	3944	6157.04
28	19.1291	48.5532	97.7346	255.150	992.966	3196.36	3943.90	6156.91
30	19.1336	48.5601	97.7409	255.152	992.947	3196.29	3943.82	6156.80
32	19.1372	48.5659	97.7464	255.154	992.934	3196.25	3943.76	6156.72
34	19.1403	48.5708	97.7511	255.156	992.924	3196.21	3943.72	6156.65
36	19.1429	48.5750	97.7551	255.158	992.917	3196.18	3943.68	6156.60
38	19.1450	48.5785	97.7587	255.160	992.911	3196.16	3943.65	6156.56
40	19.1469	48.5816	97.7617	255.162	992.907	3196.14	3943.63	6156.53
100	19.1614	48.6061	97.7879	255.183	992.897	3196.04	3943.51	6156.34
400	19.1649	48.6121	97.7949	255.191	992.904	3196.05	3943.51	6156.34

Table 10 Nondimensional circular frequency λ for SA-1 modes of (C) plate ($p=23$, $\eta=0.100$)

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
2	20.0893	66.6245	143.459	386.250	1511.59	4683.76	5243.08	6644.14
4	17.7250	62.1014	136.650	374.669	1487.38	4806.28	5830.64	7442.40
6	17.4676	61.4671	135.578	372.876	1484.10	4800.98	5926.18	8223.73
8	17.4133	61.3136	135.274	372.289	1483.04	4799.36	5924.43	8878.68
10	17.3974	61.2657	135.167	372.046	1482.56	4798.64	5923.66	9254.22
12	17.3915	61.2478	135.125	371.934	1482.31	4798.26	5923.25	9253.75
14	17.3891	61.2401	135.106	371.880	1482.16	4798.03	5923.01	9253.47
16	17.3879	61.2365	135.097	371.851	1482.07	4797.88	5922.85	9253.29
18	17.3872	61.2346	135.092	371.835	1482.02	4797.78	5922.75	9253.17
20	17.3868	61.2335	135.089	371.825	1481.98	4797.71	5922.67	9253.08
22	17.3866	61.2329	135.087	371.820	1481.96	4797.67	5922.62	9253.02
24	17.3865	61.2325	135.086	371.816	1481.94	4797.63	5922.58	9252.98
26	17.3864	61.2323	135.086	371.814	1481.93	4797.61	5922.55	9252.94
28	17.3863	61.2321	135.085	371.812	1481.92	4797.59	5922.53	9252.92
30	17.3862	61.2320	135.085	371.811	1481.92	4797.57	5922.52	9252.90
32	17.3862	61.2319	135.085	371.810	1481.91	4797.56	5922.50	9252.88
34	17.3862	61.2318	135.085	371.810	1481.91	4797.56	5922.49	9252.87
36	17.3862	61.2318	135.084	371.809	1481.91	4797.55	5922.49	9252.86
38	17.3861	61.2318	135.084	371.809	1481.91	4797.54	5922.48	9252.86
40	17.3861	61.2317	135.084	371.809	1481.91	4797.54	5922.48	9252.85

Table 10 Continued

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
100	17.3860	61.2316	135.084	371.808	1481.90	4797.52	5922.45	9252.82
400	17.3860	61.2316	135.084	371.808	1481.90	4797.52	5922.46	9252.82

Table 11 Nondimensional circular frequency λ for SS-2 modes of (C) plate ($p=23$, $\eta=0.050$)

2k	c=1	c=2	c=3	c=5	c=10	c=18	c=20	c=25
2	34.1991	55.5668	89.7413	195.109	656.974	1979.09	2421.79	3724.89
4	32.3592	46.0576	72.3845	163.780	592.550	1857.65	2285.26	3549.29
6	32.2719	44.9039	69.8662	157.844	580.124	1836.49	2261.98	3520.66
8	32.2622	44.6338	69.2296	156.065	575.526	1828.76	2253.51	3510.40
10	32.2606	44.5487	69.0177	155.410	573.413	1824.98	2249.38	3505.40
12	32.2603	44.5166	68.9343	155.134	572.351	1822.84	2247.03	3502.54
14	32.2602	44.5029	68.8973	155.004	571.785	1821.54	2245.57	3500.73
16	32.2602	44.4964	68.8794	154.938	571.468	1820.71	2244.63	3499.53
18	32.2602	44.4931	68.8701	154.903	571.283	1820.17	2244	3498.70
20	32.2603	44.4912	68.8649	154.883	571.170	1819.80	2243.57	3498.11
22	32.2603	44.4902	68.8619	154.871	571.098	1819.56	2243.27	3497.69
24	32.2603	44.4896	68.8601	154.863	571.052	1819.38	2243.06	3497.38
26	32.2603	44.4891	68.8590	154.858	571.021	1819.26	2242.91	3497.15
28	32.2603	44.4889	68.8582	154.855	571	1819.17	2242.80	3496.98
30	32.2603	44.4887	68.8577	154.853	570.984	1819.11	2242.72	3496.85
32	32.2603	44.4885	68.8574	154.852	570.974	1819.06	2242.65	3496.75
34	32.2603	44.4885	68.8571	154.851	570.966	1819.02	2242.61	3496.67
36	32.2604	44.4884	68.8569	154.850	570.960	1819	2242.57	3496.61
38	32.2604	44.4883	68.8568	154.849	570.955	1818.97	2242.54	3496.56
40	32.2604	44.4883	68.8567	154.849	570.952	1818.96	2242.52	3496.52
100	32.2605	44.4880	68.8564	154.848	570.938	1818.87	2242.40	3496.31
400	32.2605	44.4881	68.8568	154.849	570.943	1818.88	2242.41	3496.31

The influence of the aspect ratio and the parameter of thickness on the fundamental frequency were investigated (Figs. 3-4). Numerical simulation shows that the difference between (C) and (S) plates becomes evident with increasing aspect ratio (Fig. 3), and λ decreases with increasing parameter of thickness (Fig. 4).

5. Conclusions

Among the numerical techniques presently available for solutions of various plate problems, finite difference method (FDM) is probably the most transparent and the most general due to its straightforward approach (Szilard 2004). Like differential quadrature (DQ) method which requires fewer grid points for satisfactory accuracy than FDM does (Han and Liew 1997, Civalek and Ulker 2004) and discrete singular convolution (DSC) which is one of the most recent and the most effective methods applied to plate problems (Civalek and Ozturk 2010), both FDM and FEM can be considered to be grid-based methods (http://people.sc.fsu.edu/~jpeterson/FEMbook.pdf). However, unlike FDM which is generally used for rectangular or axisymmetric geometries, uniform grid spacing is not required in FEM, and therefore FEM provides more flexibility and versatility in handling arbitrary shapes such as regions with curved boundaries.

From a mathematical standpoint, the displacement version of the FEM can be considered as a special case of the Ritz method. That is, both methods are essentially equal, since each uses a set of assumed displacement functions for obtaining approximate solutions. Furthermore, the principle of minimum potential energy is applied by both methods to make these functions stationary. The major difference between the two approaches is that the assumed displacement functions in the FEM are not defined over the whole domain of the plate (i.e., they are local rather than global), since they represent piecewise trial functions that must satisfy certain continuity and completeness conditions. Recently, FEM has rapidly become the most dominant numerical technique in almost all fields of engineering and applied science due to its robustness and simplicity in modelling (Szilard 2004).

Free transverse vibration of moderately thick super-elliptical plates was reported in this paper by the finite element method using (i) a relatively large interval for the value of the super-elliptical power, (ii) a large number of aspect ratio, and (iii) several values for the parameter of thickness.

The results of the numerical simulation reveal that the aspect ratio has a prominent effect on the free vibration response of super-elliptical plates as well as the super-elliptical power does.

It is observed that the k versus λ curve shows a behavior such that for all symmetry classes of (S) super-elliptical plates the minimum value of λ is obtained for higher super-elliptical power with increasing aspect ratio (Tables 2-9). λ decreases for all k for $c>10$ except AS mode (Tables 2-9). Hence, interpolation may be used to predict the SA, SS, and AA modes of (S) super-elliptical plates for $c\geq18$, and therefore, it can be stated that the aforementioned modes are bounded by elliptical and rectangular plates.

Irrespective of the aspect ratio the response of (C) super-elliptical plates is definitely more predictable than that of (S) super-elliptical plates. Apart from a few cases in which a negligible departure from the trend is detected, the variation of the super-elliptical power versus λ is found to be in descending order (Tables 2-5). The curve representing the variation of λ versus k exhibits an asymptotic behavior such that the influence of k declines with increasing k. However, the only unexpected anomaly is discovered for SA mode of clamped super-elliptical plates with $\eta=0.100$ and $c\geq18$ (Table 10). In this exceptional case variation of λ versus k increases first, and then decreases.

The fundamental frequencies (SS-1 mode) of clamped super-elliptical plates lie between those bounded by elliptical and rectangular plates (Table 3) such that as k is raised, λ decreases. Consequently, regardless of the aspect ratio, interpolation may be used to predict the frequencies of clamped super-elliptical plates (Tables 2-5). However, for simply supported super-elliptical plates the aspect ratio is decisive on the variation of λ versus k curve. That is, the aforementioned asymptotic trend is observed for $c\geq18$ (Table 7).

The main goal of the current study is to predict the free vibration characteristics of super-elliptical plates with a clamped or a simply supported boundary. The basic difficulty in analyzing super-elliptical plates is that unlike rectangular and elliptical plates, results with admissible accuracy may be obtained by considerable computational effort. Use of insufficient number of degrees of freedom in the plate domain results in finding a totally different trend between λ and k. For example, Ceribasi and Altay (2009) considered relatively fewer terms in the trial function, and therefore the frequency parameter λ was found to be increasing for k≥8 in the solution of clamped super-elliptical plates (Table A5). Nevertheless, λ was found to be decreasing with increasing k in the current study. The comparison studies show that the mesh pattern used in the study is capable of simulating the free transverse vibration of super-elliptical plates with admissible accuracy.

The highlights of the present study may be presented as follows:

• The aspect ratio has a prominent effect on the free vibration response of super-elliptical plates as well as the super-elliptical power does.

• The fundamental frequencies (SS-1 mode) of clamped super-elliptical plates may be predicted by interpolation. However, when the boundary is simply supported, interpolation may be used for high aspect ratio.

• The trend between λ and k is found to be asymptotic. Decreasing λ with increasing k is observed for the first and second modes of all symmetry classes of clamped super-elliptical plates except a few cases in which a slight discrepancy from the trend is detected (Table 11).

• Compared to (S) thin rectangular plates, super-elliptical Mindlin plates with η=0.010 agree better than those with η=0.002 for c≥5 (Table A4).

• The comparison of fundamental frequencies reveal that compared to thin plates, super-elliptical Mindlin plates with η=0.010 agree better than those with η=0.002 (Table A6-A7).

• Compared to SA, SS, and AA modes of (S) and (C) plates, AS mode of (S) plates is observed to be more sensitive to the aspect ratio such that a regular trend (i.e., decreasing λ with increasing k) is detected for c>20. However, the same trend can be seen for (S) plates for all mode categories for c>10 (Tables 6-9).

• The use of sufficient number of degrees of freedom in the plate is crucial to obtain convergence with admissible accuracy.

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Nomenclature

a, b, c	semi-major, and semi-minor axes of the plate, aspect ratio
h, k, m	thickness of the plate, super-elliptical power, mass of the plate per unit area
p, t, w	number of partitions in the quadrant of the plate domain, time, deflection
D, E, G	flexural rigidity, Young's modulus, shear modulus
ω, λ, ρ	natural frequency, nondimensional frequency parameter, mass density
k_x , k_y , k_{xy}	curvatures, twist
r_i, s_i	local coordinates of node i (i = 1, 2, 3, 4)
D_s, N_i	shear rigidity, shape function (i = 1, 2, 3, 4)
κ, η, ν	shear correction factor , parameter of thickness, Poisson's ratio
θ_x, θ_y	rotations
ϕ_x, ϕ_y	average shear deformations
$[k_e]$, [K]	element, and global stiffness matrices
$[m_e]$, [M]	element, and global mass matrices
[B], [C], [U]	strain-displacement matrix, constitutive matrix, global displacement vector
$[C_B]$, $[C_S]$	contribution of bending, and shear deformation on [C],
$[k_B]$, $[k_S]$	bending, and shear stiffness contribution of $[k_e]$

Appendix A: Convergence studies and comparison tests

Some of the results cited in Tables A1-A7 were scaled by the author. The results obtained from Liew *et al*. (1998a) were taken from Tables 4.16-4.17 using Eq. (4.14) on page 92 in Liew *et al*. (1998a). The results obtained from Chakraverty (2009) were taken from Table 7.1-7.2 and 7.11-7.12 on pages 203 and 209-210 in Chakraverty (2009). Some of the results obtained from Leissa (1993) were computed by the author using Eq. (4.20) on page 44, and some other results were taken from Table 4.30 on pages 66-67 in Leissa (1993).

Table A1 Convergence study and comparison of λ for thick super elliptical plates (c=1)

	η	2k	SS-1	SS-2	SA-1	SA-2	AA-1	AA-2	Reference
(S)	0.020	2	4.9335	25.5325	13.8758	39.7677	25.5325	56.4837	Liew *et al*. (1998b)
(S)	0.020	2	4.93552	25.5549	13.8646	39.7472	25.4303	56.4134	p=23
(S)	0.020	2	4.93571	25.5577	13.8656	39.7533	25.4328	56.4256	p=22
(S)	0.020	2	4.93592	25.5610	13.8668	39.7603	25.4357	56.4396	p=21
(S)	0.020	4	4.6275	24.5850	12.2334	33.3271	20.1404	49.5880	Liew *et al*. (1998b)
(S)	0.020	4	4.62750	24.6024	12.2217	33.2928	20.0585	49.5162	p=23
(S)	0.020	4	4.62763	24.6051	12.2226	33.2982	20.0607	49.5267	p=22
(S)	0.020	4	4.62778	24.6081	12.2236	33.3045	20.0632	49.5388	p=21
(S)	0.020	8	4.7926	24.4445	12.1582	31.7971	19.4109	49.0782	Liew *et al*. (1998b)
(S)	0.020	8	4.79011	24.4834	12.1428	31.7661	19.3298	48.9966	p=23
(S)	0.020	8	4.79019	24.4874	12.1436	31.7715	19.3317	49.0069	p=22
(S)	0.020	8	4.79029	24.4920	12.1445	31.7776	19.3340	49.0187	p=21
(S)	0.020	20	4.8934	24.5304	12.2656	31.8161	19.5662	48.8734	Liew *et al*. (1998b)
(S)	0.020	20	4.88916	24.5610	12.2471	31.7742	19.4759	48.8607	p=23
(S)	0.020	20	4.88922	24.5648	12.2478	31.7792	19.4775	48.8756	p=22
(S)	0.020	20	4.88930	24.5691	12.2486	31.7849	19.4795	48.8927	p=21
(C)	0.020	2	10.2276	34.8629	21.2647	50.9147	34.8299	69.3954	Zhou *et al*. (2004)
(C)	0.020	2	10.2053	34.7613	21.2168	50.7875	34.7572	69.2145	Liew *et al*. (1998b)
(C)	0.020	2	10.2098	34.8123	21.2092	50.8015	34.6494	69.2120	p=23
(C)	0.020	2	10.2103	34.8175	21.2113	50.8110	34.6538	69.2293	p=22
(C)	0.020	2	10.2108	34.8235	21.2137	50.8218	34.6588	69.2491	p=21
(C)	0.020	4	9.1040	32.9628	18.6789	43.0747	28.1464	61.0679	Zhou *et al*. (2004)
(C)	0.020	4	9.0883	32.8918	18.6508	42.9789	28.0977	60.9564	Liew *et al*. (1998b)
(C)	0.020	4	9.08935	32.9297	18.6381	42.9845	28.0170	60.9321	p=23
(C)	0.020	4	9.08971	32.9343	18.6398	42.9928	28.0207	60.9467	p=22
(C)	0.020	4	9.09013	32.9396	18.6418	43.0024	28.0249	60.9634	p=21
(C)	0.020	8	9.0131	32.8554	18.3650	41.3646	27.1138	60.3409	Zhou *et al*. (2004)
(C)	0.020	8	8.9908	32.7860	18.3336	41.2762	27.0692	60.2154	Liew *et al*. (1998b)
(C)	0.020	8	8.99325	32.8335	18.3257	41.2786	26.9868	60.2084	p=23

Table A1 Continued

	η	2k	SS-1	SS-2	SA-1	SA-2	AA-1	AA-2	Reference
(C)	0.020	8	8.99355	32.8381	18.3273	41.2867	26.9901	60.2225	p=22
(C)	0.020	8	8.99390	32.8433	18.3291	41.2960	26.9940	60.2386	p=21
(C)	0.020	20	9.0049	32.8472	18.1915	41.1581	27.0229	60.2914	Zhou et al. (2004)
(C)	0.020	20	8.9859	32.7835	18.3105	41.0796	26.9750	60.1823	Liew et al. (1998b)
(C)	0.020	20	8.98862	32.8317	18.3046	41.0879	26.8994	60.1874	p=23
(C)	0.020	20	8.98890	32.8363	18.3061	41.0957	26.9025	60.2017	p=22
(C)	0.020	20	8.98922	32.8415	18.3079	41.1046	26.9060	60.2182	p=21
(C)	0.050	2	10.147						Civalek and Ersoy (2009)
(C)	0.050	2	10.047						Lin and Tseng (1998)
(C)	0.050	2	9.9887						Iyengar and Raman (1978)
(C)	0.050	2	10.159						Zhou et al. (2006)
(C)	0.050	2	10.1508	34.1991	20.9307	49.4387	33.8261	66.7377	p=23
(C)	0.050	2	10.1513	34.2041	20.9327	49.4474	33.8302	66.7532	p=22
(C)	0.050	2	10.1518	34.2098	20.9350	49.4574	33.8349	66.7710	p=21

Table A2 Convergence study and comparison of λ for thick rectangular plates (SS-1 mode, 2k=400)

	(S)	(S)	(C)	(C)	
η	c=1	c=2	c=1	c=2	Reference
0.002	4.922959	12.338979	9.048207	24.608872	Civalek (2007)
0.002	4.935	12.34	8.995	24.58	Liew *et al.* (1998a)
0.002	4.93367	12.3373	8.99913	24.5856	p=23
0.002	4.93379	12.3375	8.99923	24.5864	p=22
0.002	4.93392	12.3379	8.99972	24.5872	p=21
0.002	4.93862	12.3501	9.01183	24.6204	p=10
0.002	4.99465	12.509	9.1734	25.0546	p=3
0.020		12.3342			Thai and Choi (2013)
0.020	4.9237				Liew *et al.* (2004)
0.020	4.9331				Zhou *et al.* (2002)
0.020	4.92305		8.86125		Farag *et al.* (2013)
0.020	4.94171		9.00996		Liew and Theo (1999)
0.020	4.93308	12.33404	9.00404	24.60788	Liew *et al.* (1993b)
0.020	4.91255	12.3113	8.98851	24.5653	p=23
0.020	4.91267	12.3116	8.98878	24.5661	p=22
0.020	4.91282	12.3119	8.98909	24.5669	p=21
0.020	4.91837	12.3257	9.00106	24.5999	p=10
0.020	4.9857	12.4985	9.16084	25.0319	p=3

Table A3 Convergence study and comparison of λ for thick elliptical plates (SS-1 mode, 2k=2)

	(C)	(S)	(C)	(S)	(C)	(S)	
η	c=1	c=1	c=2	c=2	c=3	c=3	Reference
0.002	10.21	4.935	27.38	13.20	56.79	27.09	Liew *et al*. (1998a)
0.002	10.2210	4.93717	27.3913	13.2181	56.8298	27.0909	p=23
0.002	10.2215	4.93736	27.3926	13.2185	56.8327	27.0922	p=22
0.002	10.2221	4.93757	27.3942	13.2192	56.8358	27.0932	p=21
0.100	9.931						Liew *et al*. (2004)
0.100	9.9425	4.895	26.81	13.10	55.665	26.8425	Liew *et al*. (1998a)
0.100	9.94887	4.89620	26.8224	13.1044	55.6863	26.8591	p=23
0.100	9.94934	4.89638	26.8237	13.1049	55.6889	26.8601	p=22
0.100	9.94987	4.89660	26.8251	13.1054	55.6920	26.8613	p=21

Table A4 Convergence study and comparison of λ for thin rectangular (S) plates (SS-1 mode, 2k=400)

η	c=1	c=2	c=3	c=5	c=10	Reference
	4.93480	12.33701	24.67401	64.15243	249.20751	Leissa (1993)
0.002	4.93367	12.3373	24.6775	64.1664	249.268	p=23
0.002	4.93379	12.3375	24.6781	64.1681	249.278	p=22
0.002	4.93392	12.3379	24.6791	64.1701	249.287	p=21
0.010	4.92467	12.3263	24.6661	64.1539	249.251	p=23
0.010	4.92479	12.3266	24.6667	64.1555	249.257	p=22
0.010	4.92493	12.3270	24.6674	64.1573	249.264	p=21

Table A5 Convergence study and comparison of λ for thin super-elliptical plates (SS-1 mode, η=0.002)

	(S)	(S)	(S)	(C)	(C)	(C)	
2k	c=1	c=2	c=3	c=1	c=2	c=3	Reference
(e)				10.2661			Liew et al. (2004)
(e)		13.213					Narita (1986)
(e)				10.211	27.273		Bert and Malik (1996)
(e)	4.9352	13.2135	27.0805	10.2160	27.4773	56.8995	Lam et al. (1992)
(e)				10.2160	27.377		Singh and Chakraverty (1992)
(e)	4.935			10.2160			Wu and Liu (2001)
(e)	4.977			10.2158	27.5	56.9	Leissa (1993)
(e)	4.9351	13.213		10.216	27.377		Nallim and Grossi (2008)
2		13.1636			27.3733		Chen et al. (1999)
2	4.9348	13.2129	27.0822	10.2175	27.3882	56.8292	Wang and Wang (1994)
2	4.9351	13.2136	27.0810	10.2158	27.3776	56.8071	Ceribasi and Altay (2009)
2	4.93717	13.2181	27.0909	10.2210	27.3913	56.8298	p=23
2	4.93736	13.2185	27.0922	10.2215	27.3926	56.8327	p=22
2	4.93757	13.2192	27.0932	10.2221	27.3942	56.8358	p=21

Table A5 Continued

	(S)	(S)	(S)	(C)	(C)	(C)	
2k	c=1	c=2	c=3	c=1	c=2	c=3	Reference
4	4.6338	12.0212	24.6617	9.0998	24.9059	52.9134	Wang and Wang (1994)
4	4.7359	12.4336	25.5222	8.6017	25.0144	52.2540	Ceribasi and Altay (2009)
4	4.63405	12.0243	24.6691	9.09946	24.8953	52.8874	p=23
4	4.63415	12.0246	24.6698	9.09984	24.8963	52.8893	p=22
4	4.63434	12.0252	24.6709	9.10026	24.8974	52.8918	p=21
8	4.8040	12.1520	24.5112	9.0011	24.6025	52.2571	Wang and Wang (1994)
8	4.80262	12.1512	24.5163	9.00379	24.6045	52.2625	p=23
8	4.80266	12.1514	24.5166	9.00410	24.6054	52.2642	p=22
8	4.80281	12.1518	24.5177	9.00446	24.6063	52.2663	p=21
16	5.0655		25.0488	9.3005	25.1064	52.8444	Ceribasi and Altay (2009)
16	4.89333	12.2765	24.6161	8.99936	24.5865	52.2113	p=23
16	4.89334	12.2766	24.6164	8.99964	24.5873	52.2129	p=22
16	4.89346	12.2770	24.6172	8.99997	24.5882	52.2149	p=21
20		12.3000			24.5803		Chen et al. (1999)
20	4.9101	12.3025	24.6419	8.9863	24.5531	52.1411	Wang and Wang (1994)
20	5.1810	12.8632	25.4331	9.3763	25.2272	52.9434	Ceribasi and Altay (2009)
20	4.90690	12.2966	24.6357	8.99926	24.5860	52.2096	p=23
20	4.90693	12.2966	24.6363	8.99955	24.5868	52.2114	p=22
20	4.90704	12.2972	24.6370	8.99987	24.5877	52.2132	p=21
400	4.93367	12.3373	24.6775	8.99913	24.5856	52.2086	p=23
400	4.93379	12.3375	24.6781	8.99923	24.5864	52.2101	p=22
400	4.93392	12.3379	24.6791	8.99972	24.5872	52.2120	p=21
(r)	4.9348			8.99688			Liew et al. (1993a)
(r)	4.9348			8.998			Leissa (1973)
(r)	4.9348			8.9963			Geannakakes (1995)
(r)	4.93475			8.997	24.579		El-Sayad and Ghazy (2012)
(r)	4.9357			9.0253			Wu and Liu (2005)
(r)	4.935						Liew et al. (1990)
(r)	4.9348						Szilard (1974)
(r)				8.9997	24.56	52.1775	Leissa (1993)

Table A6 Convergence study and comparison of λ for thin elliptical plates (k=1)

	η	c	SS-1	SS-2	SA-1	SA-2	AS-1	AS-2	AA-1	AA-2	Reference
(S)		1	4.9352	25.613	13.898				25.593		Narita (1986)
(S)		1	4.9351	25.613			13.898	39.981			Chakraverty (2009)
(S)	0.010	1	4.93676	25.6115	13.8904	39.9143	13.8904	39.9143	25.5469	56.7524	p=23
(S)	0.010	1	4.93696	25.6144	13.8914	39.9204	13.8914	39.9204	25.5494	56.7649	p=22
(S)	0.010	1	4.93717	25.6177	13.8926	39.9275	13.8926	39.9275	25.5523	56.7791	p=21

Table A6 Continued

η	c	SS-1	SS-2	SA-1	SA-2	AS-1	AS-2	AA-1	AA-2	Reference
(S)0.002	1	4.93717	25.6398	13.9061	40.0064	13.9061	40.0064	25.6235	56.9413	p=23
(S)0.002	1	4.93736	25.6427	13.9070	40.0127	13.9075	40.0127	25.6259	56.9547	p=22
(S)0.002	1	4.93757	25.6460	13.9083	40.0195	13.9083	40.0195	25.6285	56.9683	p=21
(S)	2	13.213	38.354			23.641	57.625			Chakraverty (2009)
(S)0.010	2	13.2145	38.3477	46.1524	82.9752	23.6289	57.5434	62.6874	107.059	p=23
(S)0.010	2	13.2150	38.3528	46.1560	82.9887	23.6306	57.5558	62.6938	107.086	p=22
(S)0.010	2	13.2156	38.3587	46.1602	83.0041	23.6326	57.5700	62.7011	107.117	p=21
(S)0.002	2	13.2181	38.3752	46.1834	83.1264	23.6540	57.6030	62.8053	107.328	p=23
(S)0.002	2	13.2185	38.3800	46.1869	83.1401	23.6555	57.6154	62.8120	107.361	p=22
(S)0.002	2	13.2192	38.3862	46.1911	83.1548	23.6578	57.6296	62.8180	107.385	p=21
(C)	1	10.216	34.877	21.260	51.030			34.877	69.666	Kim (2003)
(C)	1	10.216	34.878			21.260	51.033			Chakraverty (2009)
(C)0.010	1	10.2183	34.9032	21.2577	51.0303	21.2577	51.0303	34.8169	69.6470	p=23
(C)0.010	1	10.2188	34.9084	21.2597	51.0399	21.2597	51.0399	34.8214	69.6646	p=22
(C)0.010	1	10.2194	34.9145	21.2621	51.0509	21.2621	51.0509	34.8265	69.6848	p =21
(C)0.002	1	10.2210	34.9325	21.2797	51.1235	21.2797	51.1235	34.9082	69.8337	p=23
(C)0.002	1	10.2215	34.9377	21.2818	51.1331	21.2818	51.1331	34.9130	69.8518	p=22
(C)0.002	1	10.2221	34.9438	21.2842	51.1442	21.2842	51.1442	34.9176	69.8720	p=21
(C)	2	27.377	55.976	69.858	109.94	39.497	76.995	88.047	135.71	Kim (2003)
(C)	2	27.377	55.985			39.497	77.037			Chakraverty (2009)
(C)0.010	2	27.3856	56.0604	69.8790	109.985	39.5086	77.1672	88.0041	135.815	p=23
(C)0.010	2	27.3870	56.0703	69.8861	110.006	39.5123	77.1886	88.0154	135.854	p=22
(C)0.010	2	27.3885	56.0815	69.8942	110.031	39.5166	77.2131	88.0285	135.900	p=21
(C)0.002	2	27.3913	56.0807	69.9286	110.146	39.5338	77.2191	88.1441	136.093	p=23
(C)0.002	2	27.3926	56.0906	69.9357	110.167	39.5376	77.2409	88.1579	136.136	p=22
(C)0.002	2	27.3942	56.1019	69.9439	110.192	39.5419	77.2650	88.1678	136.178	p=21

Table A7 Convergence study and comparison of λ for thin rectangular (C) plates (2k=400)

η	c	SS-1	SS-2	SA-1	SA-2	AS-1	AS-2	AA-1	AA-2	Reference
	1	9								Gutierrez *et al*. (2000)
	1	8.99975								Bhat *et al*. (1993)
	1	8.9963	32.8952	18.34845	41.2501	18.34845	41.2501	27.05413	60.53848	Leissa (1993)
0.002	1	8.99913	32.9441	18.3625	41.3257	18.3624	41.3257	27.0757	60.6848	p=23
0.002	1	8.99923	32.9488	18.3640	41.3336	18.3640	41.3336	27.0789	60.6995	p=22

Table A7 Continued

η	c	SS-1	SS-2	SA-1	SA-2	AS-1	AS-2	AA-1	AA-2	Reference
0.002	1	8.99972	32.9542	18.3657	41.3426	18.3657	41.3426	27.0822	60.7181	p=21
0.010	1	8.99654	32.9168	18.3442	41.2531	18.3442	41.2531	27.0125	60.5359	p=23
0.010	1	8.99682	32.9214	18.3458	41.2609	18.3458	41.2609	27.0155	60.5510	p=22
0.010	1	8.99713	32.9267	18.3475	41.2699	18.3475	41.2699	27.0191	60.5684	p=21
	2	24.6305								Gutierrez *et al.* (2000)
	2	24.5809								Bhat *et al.* (1993)
	2	24.5777	44.7696	63.9831	83.2727	31.8260	63.3308	71.0763	100.7921	Leissa (1993)
0.002	2	24.5856	44.8428	64.0426	83.4173	31.8473	63.5084	71.1553	101.057	p=23
0.002	2	24.5864	44.8497	64.0485	83.4324	31.8497	63.5254	71.1645	101.087	p=22
0.002	2	24.5872	44.8576	64.0552	83.4496	31.8521	63.5449	71.1731	101.117	p=21
0.010	2	24.5807	44.8286	64.0048	83.3077	31.8296	63.4667	71.0724	100.864	p=23
0.010	2	24.5814	44.8355	64.0107	83.3227	31.8318	63.4836	71.0811	100.892	p=22
0.010	2	24.5823	44.8433	64.0174	83.3399	31.8344	63.5031	71.0909	100.923	p=21
	5	141.4531	155.4256	387.4638	403.7456	146.4925	168.8488	393.5256	418.2775	Leissa (1993)
0.002	5	141.498	155.664	387.847	404.544	146.592	169.315	394.055	419.474	p=23
0.002	5	141.503	155.686	387.883	404.620	146.601	169.363	394.106	419.598	p=22
0.002	5	141.508	155.711	387.925	404.707	146.612	169.409	394.163	419.720	p=21
0.010	5	141.472	155.631	387.673	404.283	146.560	169.266	393.843	419.124	p=23
0.010	5	141.476	155.654	387.710	404.359	146.569	169.310	393.893	419.240	p=22
0.010	5	141.481	155.679	387.751	404.446	146.580	169.360	393.951	419.369	p=21
	10	560.8075	572.8875	1543.743	1559.17	565.28	583.835	1549.515	1572.753	Leissa (1993)
0.002	10	560.985	573.724	1545.27	1562.29	565.655	585.401	1551.63	1577.27	p=23
0.002	10	561.002	573.801	1545.42	1562.58	565.689	585.548	1551.83	1577.70	p=22
0.002	10	561.021	573.887	1545.58	1562.90	565.728	585.703	1552.05	1578.15	p=21
0.010	10	560.883	573.616	1544.62	1561.54	565.549	585.282	1550.93	1576.44	p=23
0.010	10	560.900	573.692	1544.76	1561.82	565.583	585.425	1551.13	1576.85	p=22
0.010	10	560.919	573.779	1544.92	1562.15	565.622	585.583	1551.35	1577.31	p=21