Wind and Structures, *Vol. 23, No. 4 (2016) 351-366*
DOI: http://dx.doi.org/10.12989/was.2016.23.4.351

Mathematical modeling of wind power estimation using multiple parameter Weibull distribution

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(Received December 4, 2015, Revised August 9, 2016, Accepted August 13, 2016)

Abstract. Nowadays, wind energy is the most rapidly developing technology and energy source and it is reusable. Due to its cleanliness and reusability, there have been rapid developments made on transferring the wind energy systems to electric energy systems. Converting the wind energy to electrical energy can be done only with the wind turbines. So installing a wind turbine depends on the wind speed at that location. The expected wind power can be estimated using a perfect probability distribution. In this paper Weibull and Weibull distribution with multiple parameters has been used in deriving the mathematical expression for estimating the wind power. Statistically the parameters of Weibull and Weibull distribution are estimated using the maximum likelihood techniques. We derive a probability distribution for the power output of a wind turbine with given rated wind speeds for the regions where the wind speed histograms present a bimodal pdf and compute the first order moment of this distribution.

Keywords: Weibull & Weibull distribution; maximum likelihood method; capacity factor; wind power; V_{rated}

1. Introduction

Wind is the possible indicator of global climate change and its importance to effective wind power generation, it is fundamentally important to install high quality wind sensors and have best exposure conditions at observation sites. Substantial theoretical and empirical research is directed at wind power expansion and generation Park (1981). The focus of such research includes wind resource quantification, wind speed modeling Sen (2003), wind power production modeling, system reliability etc. The wind speed probability distribution for a certain location is crucial in determining the performance of energy conversion systems Seshaiah and Sukkiramathi (2016). When the wind speed distribution is determined, the wind power density distribution can easily obtained accordingly. For this reason, the proper specification of the wind speed distribution is of special importance in the assessment of wind energy potential .Garcia A and Torres (1998).

This paper is intended as an introduction to the use of the multiple-parameter Weibull and Weibull distribution to model single-site hourly average wind speeds He *et al*. (2013). Our goal is to write down an expression for the probability distribution of the power produced by a wind turbine at a fixed location. The Weibull distribution is widely used in life testing and reliability

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http://www.techno-press.org/?journal=was&subpage=8 ISSN: 1226-6116 (Print), 1598-6225 (Online)

studies. Weibull distribution that is the most widely used and accepted in the specialized literature on wind energy and other renewable energy sources. Several authors have indicated that Weibull pdf should not be used in a generalized way, as it is unable to represent some wind regimes, such as those which describe wind speed frequency histograms which present bimodality. Jaramillo and Borja (2004) have used a two component mixture Weibull distribution to overcome such situations. Let us here concentrate on only the mathematical modeling to estimate the wind power resources. In section 2, we define the Weibull and Weibull probability distribution, compute its mean, variance and prove some of its characteristics. In chapter 3 we estimate the parameters. In chapter 4 we derive a probability distribution for the electric power output of a wind turbine.

2. Weibull and Weibull distribution

In this section, we will see about the Weibull and Weibull distribution, denoted by WW $(v;\alpha_1,\beta_1,\alpha_2,\beta_2,\omega)$ and prove some of its basic properties. Here $v[m/s]$ is the wind speed, $\alpha_1$and α_2 are the scale parameters , $\beta_1 and\ \beta_2$ are dimensionless shape parameters.

2.1 Definition and basic properties

A random variable V that is distributed as V_i with mixing parameters ω_i (such that $\omega_1+\omega_2=1$) is said to have a two-component mixture Weibull and Weibull Distribution Carta and Ramirez (2007). The density function of V, which depends on the parameters$(\alpha_1,\beta_1,\alpha_2,\beta_2)$ is given by

$$ff(v;\alpha_1,\beta_1,\alpha_2,\beta_2,\omega) = \omega f(v;\alpha_1,\beta_1)+(1-\omega)f(v;\alpha_2,\beta_2)$$

$$= \omega\left\{\frac{\alpha_1}{\beta_1}\left(\frac{v}{\beta_1}\right)^{\alpha_1-1}\exp\left[-\left(\frac{v}{\beta_1}\right)^{\alpha_1}\right]\right\}+(1-\omega)\left\{\frac{\alpha_2}{\beta_2}\left(\frac{v}{\beta_2}\right)^{\alpha_2-1}\exp\left[-\left(\frac{v}{\beta_2}\right)^{\alpha_2}\right]\right\} \qquad (1)$$

Proposition 2.1 The Weibull and Weibull distribution with parameters $\alpha_1,\beta_1,\alpha_2,\beta_2>0$ has cumulative distribution function (cdf) given by

$FF(v;\alpha_1,\beta_1,\alpha_2,\beta_2,\omega) = P(V\le v) = \omega\, F(v;\alpha_1,\beta_1)+(1-\omega\,)F(v;\alpha_2,\beta_2)$, where F is the CDF of Weibull two parameter

$$= \begin{cases}\omega\left\{1-\exp\left[-\left(\frac{v}{\beta_1}\right)^{\alpha_1}\right]\right\}+(1-\omega\,)\left\{1-\exp\left[-\left(\frac{v}{\beta_2}\right)^{\alpha_2}\right]\right\}, & v\ge 0\\ 0, & v<0\end{cases} \qquad (2)$$

Proof:

By the fundamental theorem of calculus

$$P(V\le v)=\int_{-\infty}^{v} ff(v;\alpha_1,\beta_1,\alpha_2,\beta_2,\omega)\ dv$$

$$= \int_0^v \omega\left\{\frac{\alpha_1}{\beta_1}\left(\frac{v}{\beta_1}\right)^{\alpha_1-1}\exp\left[-\left(\frac{v}{\beta_1}\right)^{\alpha_1}\right]\right\}dv$$

$$+ \int_0^v (1-\omega\,)\left\{\frac{\alpha_2}{\beta_2}\left(\frac{v}{\beta_2}\right)^{\alpha_2-1}\exp\left[-\left(\frac{v}{\beta_2}\right)^{\alpha_2}\right]\right\}dv$$

$$= -\ \omega \int_0^v \frac{d}{dv}\left(e^{-\left(\frac{v}{\beta_1}\right)^{\alpha_1}}\right) dv -\ (1-\omega)\int_0^v \frac{d}{dv}\left(e^{-\left(\frac{v}{\beta_2}\right)^{\alpha_2}}\right) dv$$

$$= \omega\left\{1 - \exp\left[-\left(\frac{v}{\beta_1}\right)^{\alpha_1}\right]\right\} + (1-\omega)\left\{1 - \exp\left[-\left(\frac{v}{\beta_2}\right)^{\alpha_2}\right]\right\}$$

Proposition 2.2 Let $\boldsymbol{V} \sim$ WW$(v; \alpha_1, \beta_1, \alpha_2, \beta_2, \omega)$ then V has r^{th} moment given by

$$E[V^r] = \frac{\sum_{i=1}^{n} v_i{}^r}{n} = \omega\beta_1{}^r\boldsymbol{\Gamma}\left(\boldsymbol{1}+\frac{\boldsymbol{r}}{\alpha_1}\right) + (\boldsymbol{1}-\omega)\beta_2{}^r\boldsymbol{\Gamma}\left(\boldsymbol{1}+\frac{\boldsymbol{r}}{\alpha_2}\right) \quad [\boldsymbol{m}^{\boldsymbol{r}}/\boldsymbol{s}^{-\boldsymbol{r}}] \tag{3}$$

In particular V has mean, variance respectively as Hong and Li (2014)

$$E[V] = \omega\beta_1\boldsymbol{\Gamma}\left(\boldsymbol{1}+\frac{\boldsymbol{1}}{\alpha_1}\right) + (\boldsymbol{1}-\omega)\beta_2\boldsymbol{\Gamma}\left(\boldsymbol{1}+\frac{\boldsymbol{1}}{\alpha_2}\right) \tag{4}$$

$$Var\,(V) = \beta_1{}^2\left[\boldsymbol{\Gamma}\left(\boldsymbol{1}+\frac{\boldsymbol{2}}{\alpha_1}\right) - \boldsymbol{\Gamma^2}\left(\boldsymbol{1}+\frac{\boldsymbol{1}}{\alpha_1}\right)\right] + \beta_2{}^2\left[\boldsymbol{\Gamma}\left(\boldsymbol{1}+\frac{\boldsymbol{2}}{\alpha_2}\right) - \boldsymbol{\Gamma^2}\left(\boldsymbol{1}+\frac{\boldsymbol{1}}{\alpha_2}\right)\right] \tag{5}$$

Proof:

Recall Euler's integral for the gamma function , $\boldsymbol{\Gamma(z)} = \int_{\boldsymbol{0}}^{\infty} \boldsymbol{e^{-x}x^{n-1}dx}$ for $\boldsymbol{n} > \boldsymbol{0}$.

Hence ,for $r \in Z \geq 1$,

$$E[V^r] = \int_0^v v^r ff(v; \alpha_1, \beta_1, \alpha_2, \beta_2, \omega)\, dv$$

$$= \int_0^v v^r\left\{\omega\left\{\frac{\alpha_1}{\beta_1}\left(\frac{v}{\beta_1}\right)^{\alpha_1-1}\exp\left[-\left(\frac{v}{\beta_1}\right)^{\alpha_1}\right]\right\} + \int_0^v v^r(1-\omega)\left\{\frac{\alpha_2}{\beta_2}\left(\frac{v}{\beta_2}\right)^{\alpha_2-1}\exp\left[-\left(\frac{v}{\beta_2}\right)^{\alpha_2}\right]\right\}\right\}dv$$

$$= \omega\beta_1{}^r\ \boldsymbol{\Gamma}\left(\boldsymbol{1}+\frac{r}{\alpha_1}\right) + (\boldsymbol{1}-\omega)\beta_2{}^r\ \boldsymbol{\Gamma}\left(\boldsymbol{1}+\frac{r}{\alpha_2}\right)$$

Set $r = 1$ gives $E[V] = \omega\beta_1\ \boldsymbol{\Gamma}\left(\boldsymbol{1}+\frac{\boldsymbol{1}}{\alpha_1}\right) + (\boldsymbol{1}-\omega)\beta_2\ \boldsymbol{\Gamma}\left(\boldsymbol{1}+\frac{\boldsymbol{1}}{\alpha_2}\right)$

$$Var\,(V) = E[V^2] - (E[V])^2$$

$$= \beta_1{}^2\left[\boldsymbol{\Gamma}\left(\boldsymbol{1}+\frac{\boldsymbol{2}}{\alpha_1}\right) - \boldsymbol{\Gamma^2}\left(\boldsymbol{1}+\frac{\boldsymbol{1}}{\alpha_1}\right)\right] + \beta_2{}^2\left[\boldsymbol{\Gamma}\left(\boldsymbol{1}+\frac{\boldsymbol{2}}{\alpha_2}\right) - \boldsymbol{\Gamma^2}\left(\boldsymbol{1}+\frac{\boldsymbol{1}}{\alpha_2}\right)\right]$$

Proposition 2.3

Let $k > 0$ and $V \sim WW(v; \alpha_1, \beta_1, \alpha_2, \beta_2, \omega)$. Then $kV = Y \sim$WW$(v; \alpha_1, \beta_1, \alpha_2, \beta_2, \omega)$

Proof : Using proposition 2.1 we see that

$$F_Y(v) = P(kV \leq v) =$$

$$P(V \leq \frac{v}{k}) == \omega\left\{1 - \exp\left[-\left(\frac{v}{k\beta_1}\right)^{\alpha_1}\right]\right\} + (1-\omega)\left\{1 - \exp\left[-\left(\frac{v}{k\beta_2}\right)^{\alpha_2}\right]\right\} \tag{6}$$

which is the cdf of WW$(v; \alpha_1, \beta_1, \alpha_2, \beta_2, \omega)$ random variable. Since the distribution function uniquely characterizes the law of the random variable, the conclusion follows immediately. The above proposition confirms that the wind speed follows a Weibull and Weibull distribution regardless of the choice of units.

Proposition 2.4

Let $V_1, V_2, \ldots\ldots\ldots.., V_n$ be independent random variables with $V_i \sim WW(\alpha_i, \beta_i, \omega)$, for $1 \le i \le n$. Then

$$P(\min(V_1, V_2, \ldots\ldots\ldots.., V_n) > v) = \exp\left(-\sum_{i=1}^{n}\left(\frac{v}{\beta_i}\right)^{\alpha_i}\right) \quad (7)$$

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ then $(V_1, V_2, \ldots\ldots\ldots.., V_n) \sim Weib(\beta_{min}, \alpha)$, where $\beta_{min} = \sum_{i=1}^{n}({\beta_i}^{-\alpha})^{-\frac{1}{\alpha}}$

Proof: Clearly, min $(V_1, V_2, \ldots\ldots..V_n) > v \iff V_i > v \;\; \forall\, i = 1,2, \ldots...n$. Since v_i are independent.

$P(V_1 > v, V_2 > v, \ldots\ldots\ldots\ldots..V_n > v) = \prod_{i=1}^{n} P(V_i > v)$

$$= \prod_{i=1}^{n}\left\{1 - \left\{\omega\left\{1 - \exp\left[-\left(\frac{v}{\beta_i}\right)^{\alpha_i}\right]\right\} + (1-\omega)\left\{1 - \exp\left[-\left(\frac{v}{\beta_i}\right)^{\alpha_i}\right]\right\}\right\}\right\}$$

$$= \prod_{i=1}^{n}\exp\left[-\left(\frac{v}{\beta_i}\right)^{\alpha_i}\right] \quad = \exp\left[-\sum_{i=1}^{n}\left(\frac{v}{\beta_i}\right)^{\alpha_i}\right]$$

This completes the proof of the first assertation . Now suppose that $\alpha_i = \alpha$, $1 \le i \le n$. Then we may write the above as

$$= \exp\left[-\sum_{i=1}^{n}\left(\frac{v}{\beta_i}\right)^{\alpha}\right] \quad = \exp\left[-\frac{v^{\alpha}}{\sum_{i=1}^{n}({\beta_i}^{-\alpha})^{-1}}\right]$$

$$= \exp\left[-\frac{v^{\alpha}}{{\beta_{min}}^{\alpha}}\right], \;\; \beta_{min} = \sum_{i=1}^{n}({\beta_i}^{-\alpha})^{-1}$$

Which shows that min $(V_1, V_2, \ldots\ldots..V_n) \sim WW(\beta_{min}, \alpha)$ by the uniqueness of distribution functions and proposition 2.1.

2.2 Parameter estimation

Having established some basic properties of the Weibull and Weibull distribution and found a probability distribution for the power output, we now turn to the problem of estimating the parameters $\alpha_1, \beta_1, \alpha_2, \beta_2$. Estimation of parameters can be done using several methods, but here we use MLE technique Kolhe *et al*. (2003), Celik (2004).

2.2.1 Likelihood function

Let $(V_1, V_2, \ldots\ldots..V_n) : \varphi \to R^n$ be a random variable with probability density function $f(v, \theta)$ for a k tuple of parameters $\theta \in \emptyset$ C R^k. Recall that for a sample $V(\tau) = v \in R$, the likelihood function ofθ, denoted by $l_v(\theta) = l(\theta)$, is defined by $l : \emptyset \to R$, $l(\theta) = f_\theta(v)$ .Let $\overline{\emptyset}$ denote the topological closure of $\emptyset\, in\, R$. If $\hat{\theta} \in \overline{\emptyset}$ satisfies $l(\hat{\theta}) = sup_{\theta \in \overline{\emptyset}}\, l(\theta)$, we say that

$\hat{\theta}$ is a maximum likelihood estimate (MLE) of θ. If $\hat{\theta} : \varphi \to R , \tau \in \hat{\tau}$, $where\ l_{V(\tau)}(\hat{\tau}) = sup_{\theta \in \bar{\emptyset}}\, l(\theta)$, then we say that $\widehat{\theta}$ is a maximum likelihood estimator (MLE) of θ. We may define the log-likelihood function to be log $l\ (\theta)$.We first note that this definition makes sense since we may assume that $l\ (\theta) > 0\ for\ all\ \theta$. the following lemma often simplifies computations of MLEs since the logarithm converts products to sum.

Lemma 2.5 $\widehat{\theta}$ is a MLE if and only if log $l(\hat{\theta}) = sup_{\theta \in \bar{\emptyset}}\, l(\log \theta)$.

2.2.2 Maximum Likelihood

The maximum Likelihood technique, with many required features is the most widely used technique among parameter estimation techniques Cook (2001). The MLE method has many large sample properties that make it attractive for use. It is asymptotically consistent, which means that as the sample size gets larger, the estimates converge to the true values. the goal of the MLE method is to find the parameter values such that the theoretical probability of the sample data is maximized.

Let $V_1, V_2, \dots \dots . . V_n$ be independent identically distributed (i.i.d) W $W(v; \alpha_1, \beta_1, \alpha_2, \beta_2, \omega)$ samples. Since the joint density function of independent random variables factors, we have that

$$l(\alpha_1, \beta_1, \alpha_2, \beta_2) = \prod_{i=1}^{n} f(v_i;\ \alpha_1, \beta_1, \alpha_2, \beta_2, \omega)$$

$$= \prod_{i=1}^{n} \left\{ \omega \left\{ \frac{\alpha_1}{\beta_1} \left(\frac{v_i}{\beta_1}\right)^{\alpha_1 - 1} \exp\left[-\left(\frac{v_i}{\beta_1}\right)^{\alpha_1}\right]\right\} + (1-\omega)\left\{ \frac{\alpha_2}{\beta_2} \left(\frac{v_i}{\beta_2}\right)^{\alpha_2 - 1} \exp\left[-\left(\frac{v_i}{\beta_2}\right)^{\alpha_2}\right]\right\}\right\}$$

$$= \omega^n \left(\frac{\alpha_1}{\beta_1}\right)^n \exp\left[-\frac{1}{{\beta_1}^{\alpha_1}} \sum_{i=1}^{n} {v_i}^{\alpha_1}\right] \frac{\prod_{i=1}^{n} {v_i}^{\alpha_1 - 1}}{{\beta_1}^{n(\alpha_1 - 1)}}$$
$$+ (1-\omega)^n \left(\frac{\alpha_2}{\beta_2}\right)^n \exp\left[-\frac{1}{{\beta_2}^{\alpha_2}} \sum_{i=1}^{n} {v_i}^{\alpha_2}\right] \frac{\prod_{i=1}^{n} {v_i}^{\alpha_2 - 1}}{{\beta_2}^{n(\alpha_2 - 1)}} \quad (8)$$

Taking the natural log on both the sides, we obtain that the log-likelihood function is

$$\text{Log } l(\alpha_1, \beta_1, \alpha_2, \beta_2) = n \log \alpha_1 - n \log \beta_1 - \frac{1}{{\beta_1}^{\alpha_1}} \sum_{i=1}^{n} {v_i}^{\alpha_1} + \sum_{i=1}^{n} (\alpha_1 - 1) \log v_i - \text{n}(\alpha_1 -$$
$$1) \log \beta_1 + n \log \alpha_2 - n \log \beta_2 - \frac{1}{{\beta_2}^{\alpha_2}} \sum_{i=1}^{n} {v_i}^{\alpha_2} + \sum_{i=1}^{n} (\alpha_2 -$$
$$1) \log v_i - n\ (\alpha_2 - 1) \log \beta_2 + n \log \omega + n \log(1-\omega)$$
$$= n \log \alpha_1 - n\alpha_1 \log \beta_1 - \frac{1}{{\beta_1}^{\alpha_1}} \sum_{i=1}^{n} {v_i}^{\alpha_1} + \sum_{i=1}^{n} (\alpha_1 - 1) \log v_i +$$
$$n \log \alpha_2 - n \log \beta_2 - \frac{1}{{\beta_2}^{\alpha_2}} \sum_{i=1}^{n} {v_i}^{\alpha_2} + \sum_{i=1}^{n} (\alpha_2 - 1) \log v_i - n\ \alpha_2 \log \beta_2 \quad (9)$$

Taking partial derivatives w.r.t α_1, β_1, α_2, β_2 we obtain (Sedghi *et al.* 2015 a, b)

$$\frac{\partial}{\partial \beta_1} [\log l(\alpha_1, \beta_1, \alpha_2, \beta_2)] = \frac{-n\alpha_1}{\beta_1} + \frac{\alpha_1}{{\beta_1}^{\alpha_1 + 1}} \sum_{i=1}^{n} {v_i}^{\alpha_1} \quad (10)$$

$$\frac{\partial}{\partial\beta_2}\,[\log l(\alpha_1,\beta_1,\alpha_2,\beta_2)] = \frac{-n\alpha_2}{\beta_2} + \frac{\alpha_2}{\beta_2^{\alpha_2+1}}\sum_{i=1}^{n} v_i^{\alpha_2} \tag{11}$$

$$\frac{\partial}{\partial\alpha_1}\,[\log l(\alpha_1,\beta_1,\alpha_2,\beta_2)] = \frac{n}{\alpha_1} - n\log\beta_1 + \frac{\log\beta_1}{\beta_1^{\alpha_1}}\sum_{i=1}^{n} v_i^{\alpha_1} - \frac{1}{\beta_1^{\alpha_1}}\left(\sum_{i=1}^{n}\log(v_i)\, v_i^{\alpha_1}\right) + 2\sum_{i=1}^{n}\log(v_i) \tag{12}$$

$$\frac{\partial}{\partial\alpha_2}\,[\log l(\alpha_1,\beta_1,\alpha_2,\beta_2)] = \frac{n}{\alpha_2} - n\log\beta_2 + \frac{\log\beta_2}{\beta_2^{\alpha_2}}\sum_{i=1}^{n} v_i^{\alpha_2} - \frac{1}{\beta_2^{\alpha_2}}\left(\sum_{i=1}^{n}\log(v_i)\, v_i^{\alpha_2}\right) + 2\sum_{i=1}^{n}\log(v_i) \tag{13}$$

Proposition 2.6 The MLE $(\hat\alpha_1,\hat\beta_1,\hat\alpha_2,\hat\beta_2)$ exists and is unique if $v_1, v_2, v_{3,\dots\dots\dots,} v_n$ satisfying $\min(v_1, v_2, v_{3,\dots\dots\dots,} v_n) < max(v_1, v_2, v_{3,\dots\dots\dots,} v_n)\ and\ v_i > 0\ \forall i = 1,2,\dots\dots,n.$

Proof: The log-likelihood function is in $C^{1,1}(0,\infty)$. we set the proceeding equations to zero, we have

$$0 = -\frac{n}{\widehat{\alpha_1}} - n\log\hat\beta_1 + \frac{\log\hat\beta_1}{\hat\beta_1^{\hat\alpha_1}}\sum_{i=1}^{n} v_i^{\hat\alpha_1} - \frac{1}{\hat\beta_1^{\hat\alpha_1}}\left(\sum_{i=1}^{n}\log(v_i)\, v_i^{\hat\alpha_1}\right) + 2\sum_{i=1}^{n}\log(v_i)$$

$$0 = -\frac{n}{\hat\alpha_1} - n\log\hat\beta_1 + \frac{n\log\hat\beta_1}{\hat\beta_1^{\hat\alpha_1}}\,\hat\beta_1^{\hat\alpha_1} - \frac{1}{\hat\beta_1^{\hat\alpha_1}}\left(\sum_{i=1}^{n}\log(v_i)\, v_i^{\hat\alpha_1}\right) + 2\sum_{i=1}^{n}\log(v_i)$$

$$0 = -\frac{n}{\hat\alpha_1} - n\frac{\sum_{i=1}^{n} v_i^{\hat\alpha_1}\log(v_i)}{\sum_{i=1}^{n} v_i^{\hat\alpha_1}} + 2\sum_{i=1}^{n}\log(v_i) \tag{14}$$

$From\ (14)$ $$\frac{\sum_{i=1}^{n} v_i^{\hat\alpha_1}\log(v_i)}{\sum_{i=1}^{n} v_i^{\hat\alpha_1}} - \frac{1}{\hat\alpha_1} = \frac{2\sum_{i=1}^{n}\log(v_i)}{n}$$

Similarly $$\frac{\sum_{i=1}^{n} v_i^{\hat\alpha_2}\log(v_i)}{\sum_{i=1}^{n} v_i^{\hat\alpha_2}} - \frac{1}{\hat\alpha_2} = \frac{2\sum_{i=1}^{n}\log(v_i)}{n}$$

This equation has a unique solution. For existence, observe that

$h(\alpha_1) = \log(\hat\alpha_1,\beta_1,\alpha_2,\beta_2)$

$$= \text{n}\log\ \alpha_1 - \frac{n}{\alpha_1}\alpha_1\log\left(\sum_{i=1}^{n}\frac{v_i^{\alpha_1}}{n}\right) - n\frac{\sum_{i=1}^{n} v_i^{\alpha_1}}{\sum_{i=1}^{n} v_i^{\alpha_1}} + \sum_{i=1}^{n}(\alpha_1 - 1)\log v_i$$

$$= \text{n}\log\ \alpha_1 - n\log\left(\sum_{i=1}^{n}\frac{v_i^{\alpha_1}}{n}\right) - n + \sum_{i=1}^{n}(\alpha_1 - 1)\log v_i \tag{15}$$

$h(\alpha_2) = \log(\alpha_1,\beta_1,\hat\alpha_2,\beta_2)$

$$= \text{n}\log\ \alpha_2 - \frac{n}{\alpha_2}\alpha_2\log\left(\sum_{i=1}^{n}\frac{v_i^{\alpha_2}}{n}\right) - n\frac{\sum_{i=1}^{n} v_i^{\alpha_2}}{\sum_{i=1}^{n} v_i^{\alpha_2}} + \sum_{i=1}^{n}(\alpha_2 - 1)\log v_i$$

$$= \text{n}\log\ \alpha_2 - n\log\left(\sum_{i=1}^{n}\frac{v_i^{\alpha_2}}{n}\right) - n + \sum_{i=1}^{n}(\alpha_2 - 1)\log v_i \tag{16}$$

Let $\alpha_k = \alpha_1, \alpha_2$ and $\beta_k = \beta_1, \beta_2$ This implies

$h(\alpha_k) = \log(\hat\alpha_k, \beta_k)$

$$= \text{n}\log\ \alpha_k - \frac{n}{\alpha_k}\alpha_k \log\left(\sum_{i=1}^{n}\frac{{v_i}^{\alpha_k}}{n}\right) - n\,\frac{\sum_{i=1}^{n}{v_i}^{\alpha_k}}{\sum_{i=1}^{n}{v_i}^{\alpha_k}} + \sum_{i=1}^{n}(\alpha_k - 1)\log v_i$$

$$= \text{n}\log\ \alpha_k - n\log\left(\sum_{i=1}^{n}\frac{{v_i}^{\alpha_k}}{n}\right) - n + \sum_{i=1}^{n}(\alpha_k - 1)\log v_i \quad (17)$$

It is evident that h$(\alpha_k) \to -\infty$ *as* $\alpha_k \to 0$. we now consider h(α_k) for large values of α_k , in particular $\alpha_k \geq 2$. Since the function $v \to v^{\alpha_k}$ is strictly convex, for $\alpha_k \geq 2$.

$$\sum_{i=1}^{n}\frac{{v_i}^{\alpha_k}}{n} > \left(\sum_{i=1}^{n}\frac{v_i}{n}\right)^{\alpha_k}$$

$$-n\log\left(\sum_{i=1}^{n}\frac{{v_i}^{\alpha_k}}{n}\right) < -n\log\left[\left(\sum_{i=1}^{n}\frac{v_i}{n}\right)^{\alpha_k}\right]$$

$$= -n\,\alpha_k\log\left(\sum_{i=1}^{n}\frac{v_i}{n}\right) \quad (18)$$

Since the function $v \to -\log(v)$ is strictly convex and by our hypothesis that $\min(v_1, v_2, v_{3,\ldots\ldots\ldots,}v_n) < max(v_1, v_2, v_{3,\ldots\ldots\ldots,}v_n)$

$$-n\log\left(\sum_{i=1}^{n}\frac{{v_i}^{\alpha_k}}{n}\right) < -n\alpha_k\log\left(\sum_{i=1}^{n}\frac{v_i}{n}\right) < -n\,\alpha_k\sum_{i=1}^{n}\frac{\log(v_i)}{n} - \alpha_k\sum_{i=1}^{n}\log(v_i)$$

$$\alpha_k\sum_{i=1}^{n}\log(v_i) - n\,log\left(\sum_{i=1}^{n}\frac{{v_i}^{\alpha_k}}{n}\right) < \alpha_k\left(\sum_{i=1}^{n}\log(v_i) - n\log\left(\sum_{i=1}^{n}\frac{v_i}{n}\right)\right) < 0. \quad (19)$$

Hence $\alpha_k\left(\sum_{i=1}^{n}\log(v_i) - n\ \log\left(\sum_{i=1}^{n}\frac{v_i}{n}\right)\right) < 0$. Since $\log(\alpha_k) = O(\alpha_k)$ as $\alpha_k \to \infty$, it follows that $h(\alpha_k) \to -\infty$ as $\alpha_k \to \infty$. Hence there exists α_1, α_2 with $0 < \alpha_1 < \alpha_2 < \infty$, such that $sup_{\alpha_k \in [\alpha_1, \alpha_2]}$, $h(\alpha_k) = sup_{\alpha_k \in [0,\infty]}h(\alpha_k)$. By Weierstrass' extreme value theorem there exists $\hat{\alpha}_k \in [\alpha_1, \alpha_2]$ such that

$h(\hat{\alpha}_k) = sup_{\alpha_k \in [\alpha_1, \alpha_2]}h(\alpha_k) = sup_{\alpha_k \in [0,\infty]}h(\alpha_k)$ which implies $\hat{\alpha}_k$ is a global maximum of $h(\alpha_k)$.

3. Computation of MLE

Let us compute $\hat{\alpha}_k$. We now give a Newton Raphson algorithm for finding $\hat{\alpha}_k$

Define $f : (0, \infty) \to R$, $f(\alpha_k) = \frac{n}{\alpha_k} - n\frac{\sum_{i=1}^{n}{v_i}^{\alpha_k}\log(v_i)}{\sum_{i=1}^{n}{v_i}^{\alpha_k}} + 2\sum_{i=1}^{n}\log(v_i)$ (20)

$$f'(\alpha_k) = \frac{n}{{\alpha_k}^2} - n\frac{\sum_{i=1}^{n}{v_i}^{\alpha_k}(\log(v_i))^2}{\sum_{i=1}^{n}{v_i}^{\alpha_k}} + n\frac{\left(\sum_{i=1}^{n}{v_i}^{\alpha_k}\log(v_i)\right)\left(\sum_{j=1}^{n}{v_j}^{\alpha_k}\log(v_j)\right)}{\left(\sum_{i=1}^{n}{v_i}^{\alpha_k}\right)^2}$$

$$= \frac{n}{{\alpha_k}^2} - n\frac{\sum_{i<j}(v_iv_j)^{\alpha_k}\left(\log(v_i) - log(v_j)\right)^2}{\left(\sum_{i=1}^{n}{v_i}^{\alpha_k}\right)^2} < 0 \quad (21)$$

$$f''(\alpha_k) = 2\frac{n}{{\alpha_k}^3} - n\frac{\left(\sum_{i=1}^{n} {v_i}^{\alpha_k}\right)^2\left(\sum_{i<j}\log(v_i v_j)(v_i v_j)^{\alpha_k}(\log(v_i)-log(v_j))^2\right)}{\left(\sum_{i=1}^{n} {v_i}^{\alpha_k}\right)^4}$$
$$+n^2\frac{\sum_{i<j}(v_i v_j)^{\alpha_k}(\log(v_i)-log(v_j))^2\left(\sum_{i=1}^{n} {v_i}^{\alpha_k}\log(v_i)\right)}{\left(\sum_{i=1}^{n} {v_i}^{\alpha_k}\right)^4} \tag{22}$$

Which is evidently bounded on compact subsets of $(0,\infty)$ being continuous. Fix $0 < a < \hat{\alpha}_k < b < \infty$ such that $f(a) > 0, f(b) < 0$, $0 < \delta \le |f'(\alpha_k)|$ and $|f'(\alpha_k)| \le N$, for some $N > 0, \forall\, \alpha_k \in [a,b]$. Let $\widehat{\alpha_{k0}} \in (\hat{\alpha}_k, b)$ denote initial choice. Then m^{th} iteration of Newton-Raphson method is given by

$$\hat{\alpha}_{km} = \hat{\alpha}_{km-1} - \frac{\frac{n}{\hat{\alpha}_{km-1}} - n\frac{\sum_{i=1}^{n} {v_i}^{\hat{\alpha}_{km-1}}\log(v_i)}{\sum_{i=1}^{n} {v_i}^{\hat{\alpha}_{km-1}}} + \sum_{i=1}^{n}\log(v_i)}{\frac{n}{\hat{\alpha}^2{}_{km-1}} - n\frac{\sum_{i<j}(v_i v_j)^{\hat{\alpha}_{km-1}}(\log(v_i)-log(v_j))^2}{\left(\sum_{i=1}^{n} {v_i}^{\hat{\alpha}_{km-1}}\right)^2}} \tag{23}$$

4. Power distribution

Let us see about the very basic and simplified level of wind turbines.

4.1 Wind turbine

A wind turbine functions with kinetic energy. The power of wind with mass flow in $m\left(\frac{kg}{s}\right)$ *and velocity* $v\left(\frac{m}{s}\right)$ is given by $P = \frac{1}{2}\, m|v|^2$

A wind turbine which converts all the kinetic energy of the wind into mechanical energy would reduce the speed of the wind to 0.

$$P = \frac{1}{2}\,\rho A|v|^3$$

Where ρ is the standard air density $\left(\frac{kg}{m^2}\right)$. Many researchers have assumed that the air density is independent of the wind speed cubed and constant as for the standard atmosphere being equal to 1.293 kg/m^3 and A is the area (m^2) covered by the rotator blades. In practical design for modern high speed two blade turbines, C_P ranges between 0.4 and 0.5, whereas it ranges between 0.2 and 0.4 for slow wind speed turbines with more number of blades. Modern wind turbines with power coefficients C_P is taken as 0.593 , which is its maximum possible theoretical value ≈ 0.5 stated that power output of a wind turbine is given by $\frac{1}{2}\,\rho C_P \eta A|v|^3$, $\eta \in (0,1)$ is efficiency constant Patel (2006).

Proposition 4.1

P has cdf $FF_p(v) = \omega\left\{1 - \exp\left[-\frac{1}{{\beta_1}^{\alpha_1}}\left(\frac{2v}{\rho A}\right)^{\frac{\alpha_1}{3}}\right]\right\} + (1-\omega)\left\{1 - \exp\left[-\frac{1}{{\beta_2}^{\alpha_2}}\left(\frac{2v}{\rho A}\right)^{\frac{\alpha_2}{3}}\right]\right\}$ (24)

And pdf $ff_P(v) =$

$$\left\{\exp\left[-\frac{1}{\beta_1^{\alpha_1}}\left(\frac{2v}{\rho A}\right)^{\frac{\alpha_1}{3}}\right]\right\}\frac{1}{\beta_1^{\alpha_1}}\frac{\alpha_1 v^{\frac{\alpha_1}{3}-1}}{3}\left(\frac{2}{\rho A}\right)^{\frac{\alpha_1}{3}} + (1-\omega)\left\{\exp\left[-\frac{1}{\beta_2^{\alpha_2}}\left(\frac{2v}{\rho A}\right)^{\frac{\alpha_2}{3}}\right]\right\}\frac{1}{\beta_2^{\alpha_2}}\frac{\alpha_2 v^{\frac{\alpha_2}{3}-1}}{3}\left(\frac{2}{\rho A}\right)^{\frac{\alpha_2}{3}} \quad (25)$$

Proof:

$$FF_p(v) = P(P \le v) = P\left(V \le \left(\frac{2v}{\rho A}\right)^{\frac{1}{3}}\right) = FF\left(\left(\frac{2v}{\rho A}\right)^{\frac{1}{3}}\right)$$

$$= \omega\left\{1 - \exp\left[-\frac{1}{\beta_1^{\alpha_1}}\left(\frac{2v}{\rho A}\right)^{\frac{\alpha_1}{3}}\right]\right\} + (1-\omega)\left\{1 - \exp\left[-\frac{1}{\beta_2^{\alpha_2}}\left(\frac{2v}{\rho A}\right)^{\frac{\alpha_2}{3}}\right]\right\}$$

Differentiating both sides with respect to v, we get the pdf

$$ff_P(v) = \omega\left\{\exp\left[-\frac{1}{\beta_1^{\alpha_1}}\left(\frac{2v}{\rho A}\right)^{\frac{\alpha_1}{3}}\right]\right\}\frac{1}{\beta_1^{\alpha_1}}\frac{\alpha_1}{3}\left(\frac{2}{\rho A}\right)\left(\frac{2v}{\rho A}\right)^{\frac{\alpha_1}{3}-1}$$

$$+(1-\omega)\left\{\exp\left[-\frac{1}{\beta_2^{\alpha_2}}\left(\frac{2v}{\rho A}\right)^{\frac{\alpha_2}{3}}\right]\right\}\frac{1}{\beta_2^{\alpha_2}}\frac{\alpha_2}{3}\left(\frac{2}{\rho A}\right)\left(\frac{2v}{\rho A}\right)^{\frac{\alpha_2}{3}-1}$$

$$= \left\{\exp\left[-\frac{1}{\beta_1^{\alpha_1}}\left(\frac{2v}{\rho A}\right)^{\frac{\alpha_1}{3}}\right]\right\}\frac{1}{\beta_1^{\alpha_1}}\frac{\alpha_1 v^{\frac{\alpha_1}{3}-1}}{3}\left(\frac{2}{\rho A}\right)^{\frac{\alpha_1}{3}}$$

$$+(1-\omega)\left\{\exp\left[-\frac{1}{\beta_2^{\alpha_2}}\left(\frac{2v}{\rho A}\right)^{\frac{\alpha_2}{3}}\right]\right\}\frac{1}{\beta_2^{\alpha_2}}\frac{\alpha_2 v^{\frac{\alpha_2}{3}-1}}{3}\left(\frac{2}{\rho A}\right)^{\frac{\alpha_2}{3}}$$

Power distribution of wind turbine

Suppose $\boldsymbol{V}\sim$ WW$(v;\alpha_1,\beta_1,\alpha_2,\beta_2,\omega)$. Let $=\frac{1}{2}\rho A V^3$, where ρ *and* A are constants. In reality , ρ is stochastic ,but we assume the variability of ρ is negligible.

The Wind energy available in the wind cannot be extracted completely by any real wind turbine, as the air mass would be stopped completely in the intercepting rotor area Jowder and Fawzi (2009). For wind turbine machines that has power with maximum efficiency between rated and cut-off speed and an increasing power between cut-in and rated speed, wind speed of wind turbine with power coefficient C_P and efficiency coefficient η producing power P is given by

$$V_{turbine} = \begin{cases} 0 & V_{turbine} < v_{cut-in} \\ V & v_{cut-in} \le V \le v_{rated} \\ v_{rated} & v_{rated} < V < v_{cut-off} \\ 0 & V \ge v_{cut-off} \end{cases} \quad (26)$$

where $v_{cut-in} < v_{rated} < v_{cut-\text{off}\_}$ are specified by the manufacturer. For example, the GE 1:5 MW SLE. wind turbine has cut-in wind speed $v_{cut-in} = 3.5\frac{m}{s}$ rated wind speed $v_{rated} = 14\frac{m}{s}$ and cut-off wind speed $v_{cut-off} = 25\frac{m}{s}$

$$P_{turbine} = \begin{cases} 0 & V_{turbine} < v_{cut-in} \\ \frac{1}{2}\,\rho C_P \eta A V^3 & v_{cut-in} \le V \le v_{rated} \\ \frac{1}{2}\,\rho C_P \eta A {v_{rated}}^3 & v_{rated} < V < v_{cut-off} \\ 0 & V \ge v_{cut-off} \end{cases} \tag{27}$$

It is evident $V_{turbine}$ and $P_{turbine}$ are discontinuous random variable, but we can still compute their distribution functions and moments.

Proposition 4.2. $P_{turbine}$ has cdf

$$FF_{p_{turbine}}(v)$$

$$= \begin{cases} 0 & -\infty < v < 0 \\ \left\{1-\omega\left(e^{-\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}} - e^{-\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}}\right) + \omega\left(e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}} - e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right)\right\} & 0 \le v \le \frac{1}{2}\,\rho C_P \eta A {v_{cut-in}}^3 \\ 1-\omega e^{-\frac{1}{\beta_1^{\alpha_1}}\left(\frac{2v}{C_P\eta\rho A}\right)^{\frac{\alpha_1}{3}}} - (1-\omega) e^{-\frac{1}{\beta_2^{\alpha_2}}\left(\frac{2v}{C_P\eta\rho A}\right)^{\frac{\alpha_2}{3}}} + e^{-\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}} + \omega\left(e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}} - e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right) & \frac{1}{2}\,\rho C_P \eta A {v_{cut-in}}^3 < V < \frac{1}{2}\,\rho C_P \eta A {v_{rated}}^3 \\ 1 & \frac{1}{2}\,\rho C_P \eta A {v_{rated}}^3 \le v < \infty \end{cases}$$

(28)

Proof:

It is evident that $P(P_{turbine} \le 0)$ is

$$P(\{V \le v_{cut-in}\} \cup \{V \le v_{cut-off}\} = FF(v_{cut-in}) + (1 - \left(FF(v_{cut-off})\right)$$

$$= \omega\left\{1-\exp\left[-\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}\right]\right\} + (1-\omega)\left\{1-\exp\left[-\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}\right]\right\}$$
$$\left\{1-\left\{\omega\left\{1-\exp\left[-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}\right]\right\} + (1-\omega)\left\{1-\exp\left[-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}\right]\right\}\right\}\right\}$$

$$= 1-\omega\left(e^{-\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}} - e^{-\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}}\right) + \omega\left(e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}} - e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right) \tag{29}$$

For $0 < v = \frac{1}{2}\,\rho C_P \eta A v^3 \le \frac{1}{2}\,\rho C_P \eta A v$, note that $v \le \frac{1}{2}\,\rho C_P \eta A {v_r}^3 \Leftrightarrow v \le v_r$

$$\rightarrow\ FF_{p_{turbine}}(v) = P(0 < Pturbine \le v) + P(Pturbine = 0)$$

$$= P\left(\frac{1}{2}\,\rho C_P \eta A {v_{cut-in}}^3 \le Pturbine \le v\right) + 1-\omega\left(e^{-\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}} - e^{-\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}}\right) +$$
$$\omega\left(e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}} - e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right)$$

$$=$$

$$\omega\left\{1-e^{-\frac{1}{\beta_1^{\alpha_1}}\left(\frac{2v}{C_P\eta\rho A}\right)^{\frac{\alpha_1}{3}}}\right\} + (1-\omega)\left\{1-e^{-\frac{1}{\beta_2^{\alpha_2}}\left(\frac{2v}{C_P\eta\rho A}\right)^{\frac{\alpha_2}{3}}}\right\} - \omega\left\{1- e^{-\frac{1}{\beta_1^{\alpha_1}}\left(\frac{2\frac{1}{2}\rho C_P\eta A {v_{cut-in}}^3}{C_P\eta\rho A}\right)^{\frac{\alpha_1}{3}}}\right\} -$$

$$(1-\omega)\left\{1-e^{-\frac{1}{\beta_2{}^{\alpha_2}}\left(\frac{2\frac{1}{2}\rho C_P\eta A v_{cut-in}{}^3}{C_P\eta\rho A}\right)^{\frac{\alpha_2}{3}}}\right\}$$

$$+1-\omega\left(e^{-\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}}-e^{-\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}}\right)+\omega\left(e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}}-e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right)$$

$$=\ 1-\omega e^{-\frac{1}{\beta_1{}^{\alpha_1}}\left(\frac{2v}{C_P\eta\rho A}\right)^{\frac{\alpha_1}{3}}}-(1-\omega)\left\{1-e^{-\frac{1}{\beta_2{}^{\alpha_2}}\left(\frac{2v}{C_P\eta\rho A}\right)^{\frac{\alpha_2}{3}}}\right\}+e^{-\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}}+\ \omega\left\{e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}}-\right.$$

$$\left.e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right\} \tag{30}$$

For $\frac{1}{2}\ \rho C_P\eta A v_{rated}{}^3 \le v < \infty$, it is evident that $FF_{p_{turbine}}(v)=1.$ (31)

Eq. (28) is combination of (29), (30) and (31).

Proposition 4.3

$$E[p_{turbine}{}^r]=\left(\frac{1}{2}\ \rho C_P\eta A\right)^r\left\{\beta_1{}^{\ 3r}\omega\left(\Gamma\left(1+\frac{3r}{\alpha_1},\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}\right)-\Gamma\left(1+\ \frac{3r}{\alpha_1},\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}\right)\right)+\right.$$
$$\left.\beta_2{}^{\ 3r}(1-\ \omega)\left(\Gamma\left(1+\frac{3r}{\alpha_2},\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}\right)-\ \Gamma\left(1+\frac{3r}{\alpha_2},\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}\right)\right)\right\}+v_{rated}{}^{3r}\omega$$
$$\left\{e^{-\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}}-\ e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}}\right\}+v_{rated}{}^{3r}(1-\omega)\left\{e^{-\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}}-e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right\}$$

Proof:

$$E[p_{turbine}{}^r]=\int_{-\infty}^{\infty}p_{turbine}{}^r\ d\left(ff_{p_{turbine}}(v)\right)$$

$$=\int_{v_{cut-in}}^{v_{rated}}\left(\frac{1}{2}\ \rho C_P\eta A v^3\right)^r\ \left\{\omega\left\{\frac{\alpha_1}{\beta_1}\left(\frac{v}{\beta_1}\right)^{\alpha_1-1}\exp\left[-\left(\frac{v}{\beta_1}\right)^{\alpha_1}\right]\right\}dv+\right.$$
$$\left.\int_{v_{cut-in}}^{v_{rated}}(1-\omega)\left\{\frac{\alpha_2}{\beta_2}\left(\frac{v}{\beta_2}\right)^{\alpha_2-1}\exp\left[-\left(\frac{v}{\beta_2}\right)^{\alpha_2}\right]\right\}\right\}dv$$
$$+\int_{v_{rated}}^{v_{cut-off}}\left(\frac{1}{2}\ \rho C_P\eta A v_{rated}{}^3\right)^r\ \left\{\omega\left\{\frac{\alpha_1}{\beta_1}\left(\frac{v}{\beta_1}\right)^{\alpha_1-1}\exp\left[-\left(\frac{v}{\beta_1}\right)^{\alpha_1}\right]\right\}dv+\right.$$
$$\left.\int_{v_{rated}}^{v_{cut-off}}(1-\omega)\left(\frac{1}{2}\ \rho C_P\eta A v_{rated}{}^3\right)^r\left\{\frac{\alpha_2}{\beta_2}\left(\frac{v}{\beta_2}\right)^{\alpha_2-1}\exp\left[-\left(\frac{v}{\beta_2}\right)^{\alpha_2}\right]\right\}\right\}dv \tag{32}$$

Let $y_1=\left(\frac{v}{\beta_1}\right)^{\alpha_1}$, $dy_1=\frac{\alpha_1}{\beta_1}\left(\frac{v}{\beta_1}\right)^{\alpha_1-1}dv$, when $v=v_{cut-in}\ \to\ y_1=\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}$,

$$v=v_{rated}\ \to\ y_1=\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1},\ v=v_{cut-off}\ \to\ y_1=\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1} \tag{33}$$

and $y_2 = \left(\frac{v}{\beta_2}\right)^{\alpha_2}$, $dy_2 = \frac{\alpha_2}{\beta_2}\left(\frac{v}{\beta_2}\right)^{\alpha_2 - 1} dv$, when $v = v_{cut-in} \rightarrow y_2 = \left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}$,

$$v = v_{rated} \rightarrow y_2 = \left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2} , v = v_{cut-off} \rightarrow y_2 = \left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2} \tag{34}$$

Using (33) and (34) we get

$$=\left(\frac{1}{2}\rho C_P \eta A\right)^r \beta_1^{3r}\omega \int_{\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}}^{\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}} y_1^{\frac{3r}{\alpha_1}} e^{-y_1} dy_1 + \left(\frac{1}{2}\rho C_P \eta A\right)^r \beta_2^{3r}(1-\omega)\int_{\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}}^{\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}} y_2^{\frac{3r}{\alpha_2}} e^{-y_2} dy_2 + \left(\frac{1}{2}\rho C_P \eta A\right)^r \omega \int_{\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}}^{\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}} v_{rated}^{3r} e^{-y_1} dy_1$$

$$+\left(\frac{1}{2}\rho C_P \eta A\right)^r (1-\omega)\int_{\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}}^{\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}} v_{rated}^{3r} e^{-y_2} dy_2 \tag{35}$$

Let us consider the Gamma function $\int_0^v e^{-y} y^{n-1} dy = \Gamma n.$

Let $t_1 = 1 + \frac{3r}{\alpha_1}$, $t_2 = 1 + \frac{3r}{\alpha_2}$, This implies

$$E[p_{turbine}{}^r] = \left(\frac{1}{2}\rho C_P \eta A\right)^r \beta_1^{3r}\omega \int_{\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}}^{\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}} y_1^{t_1 - 1} e^{-y_1} dy_1 + \left(\frac{1}{2}\rho C_P \eta A\right)^r \beta_2^{3r}(1-\omega)\int_{\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}}^{\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}} y_2^{t_2 - 1} e^{-y_2} dy_2 + \left(\frac{1}{2}\rho C_P \eta A\right)^r \omega \int_{\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}}^{\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}} v_{rated}^{3r} e^{-y_1} dy_1$$

$$+\left(\frac{1}{2}\rho C_P \eta A\right)^r (1-\omega)\int_{\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}}^{\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}} v_{rated}^{3r} e^{-y_2} dy_2$$

$$= \left(\frac{1}{2}\rho C_P \eta A\right)^r \left\{\beta_1^{\ 3r}\omega\left(\Gamma\left(1+\frac{3r}{\alpha_1},\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}\right) - \Gamma\left(1+\frac{3r}{\alpha_1},\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}\right)\right) + \beta_2^{\ 3r}(1-\omega)\left(\Gamma\left(1+\frac{3r}{\alpha_2},\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}\right) - \Gamma\left(1+\frac{3r}{\alpha_2},\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}\right)\right)\right\} + v_{rated}^{3r}\omega\left\{e^{-\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}} - e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}}\right\}$$

$$+v_{rated}^{3r}(1-\omega)\left\{e^{-\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}} - e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right\} \tag{36}$$

In particular , $p_{turbine}$ has mean

$$E[p_{turbine}] = \left(\frac{1}{2}\rho C_P \eta A\right)\left\{\beta_1^{\ 3}\omega\left(\Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}\right) - \Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}\right)\right) + \beta_2^{\ 3}(1-\omega)\left(\Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}\right) - \Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}\right)\right)\right\} + v_{rated}^{3}\omega \left\{e^{-\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}} -\right.$$

$$e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}}\Bigg\} + v_{rated}{}^3(1- \ \omega)\left\{e^{-\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}} - \ e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right\} \quad (37)$$

We can define the capacity factor of a wind turbine to be the ratio Chang and Tu (2007)

$$CF = \frac{E[p_{turbine}]}{\frac{1}{2}\ \rho C_P \eta A v_{rated}{}^3}$$

An expression for CF using (37)

$$CF = \frac{1}{v_{rated}{}^3}\left\{\beta_1{}^3\omega\left(\Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}\right) - \Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}\right)\right) + \beta_2{}^3(1- \ \omega)\left(\Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}\right) - \Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}\right)\right)\right\}$$

$$+v_{rated}{}^3\omega\left\{e^{-\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}} - e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}}\right\} + v_{rated}{}^3(1-\omega)\left\{e^{-\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}} - \ e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right\}$$

$$= \left(\frac{\beta_1}{v_{rated}}\right)^3\left\{\omega\left(\Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}\right) - \Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}\right)\right) + \left(\frac{\beta_2}{v_{rated}}\right)^3(1-\omega)\left(\Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}\right) - \Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}\right)\right)\right\}$$

$$+\omega\ \left\{e^{-\left(\frac{v_{rated}}{\beta_1}\right)^{\alpha_1}} - e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}}\right\} \quad +(1-\omega)\left\{e^{-\left(\frac{v_{rated}}{\beta_2}\right)^{\alpha_2}} - e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right\} \quad (38)$$

4.3 Turbine – Location

Let us suppose that a particular location wind speed data follow $\mathrm{WW}(v; \alpha_1, \beta_1, \alpha_2, \beta_2, \omega)$ distribution. Furthermore, suppose a manufacturer can manufacture wind turbines with fixed cut-in and cut-off speeds v_{cut-in} and $v_{cut-off}$, respectively, but can adjust the rated wind speed $v_r \in [v_{cut-in}; v_{cut-off}]$. Rehman and Ahmad (2004).

Define a function $CF : [v_{cut-in}; v_{cut-off}) \rightarrow R$ by

$$CF = \left(\frac{\beta_1}{v}\right)^3\left\{\omega\left(\Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v}{\beta_1}\right)^{\alpha_1}\right) - \Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}\right)\right) + \left(\frac{\beta_2}{v}\right)^3(1- \ \omega)\left(\Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v}{\beta_2}\right)^{\alpha_2}\right) - \Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}\right)\right)\right\} + \omega\left\{e^{-\left(\frac{v}{\beta_1}\right)^{\alpha_1}} - \ e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}}\right\} + (1-\omega)\left\{e^{-\left(\frac{v}{\beta_2}\right)^{\alpha_2}} - e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right\} \quad (39)$$

Then $CF'(v) = \ -\frac{3\beta_1{}^3}{v^4}\ \omega\left(\Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v}{\beta_1}\right)^{\alpha_1}\right) - \Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}\right)\right)$

$$+\left(\frac{\beta_1}{v}\right)^3 \omega\left[\left(\frac{v}{\beta_1}\right)^3 e^{-\left(\frac{v}{\beta_1}\right)^{\alpha_1}}\left(\frac{\alpha_1}{\beta_1}\right)\left(\frac{v}{\beta_1}\right)^{\alpha_1-1}\right]-\frac{3\beta_2{}^3}{v^4}(1-\omega)\left(\Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v}{\beta_2}\right)^{\alpha_2}\right)-\Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}\right)\right)+\left(\frac{\beta_2}{v}\right)^3(1-\omega)\left[\left(\frac{v}{\beta_2}\right)^3 e^{-\left(\frac{v}{\beta_2}\right)^{\alpha_1}}\left(\frac{\alpha_2}{\beta_2}\right)\left(\frac{v}{\beta_2}\right)^{\alpha_2-1}\right]$$

$$-\omega\left\{e^{-\left(\frac{v}{\beta_1}\right)^{\alpha_1}}\left(\frac{\alpha_1}{\beta_1}\right)\left(\frac{v}{\beta_1}\right)^{\alpha_1-1}\right\}-(1-\omega)\left\{e^{-\left(\frac{v}{\beta_2}\right)^{\alpha_1}}\left(\frac{\alpha_2}{\beta_2}\right)\left(\frac{v}{\beta_2}\right)^{\alpha_2-1}\right\}$$

$$=-\frac{3\beta_1{}^3}{v^4}\omega\left(\Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v}{\beta_1}\right)^{\alpha_1}\right)-\Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}\right)\right)$$

$$-\frac{3\beta_2{}^3}{v^4}(1-\omega)\left(\Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v}{\beta_2}\right)^{\alpha_2}\right)-\Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}\right)\right) \tag{40}$$

We now compute the value v_{rated} which maximizes $E[p_{turbine}]$

Define a function $P_{avg}:[v_{cut-in};\ v_{cut-off}) \to R$ by

$$P_{avg}(v)=\left(\frac{1}{2}\rho C_P\eta A\right)\left\{\beta_1{}^3\omega\left(\Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v}{\beta_1}\right)^{\alpha_1}\right)-\Gamma\left(1+\frac{3}{\alpha_1},\left(\frac{v_{cut-in}}{\beta_1}\right)^{\alpha_1}\right)\right)+\beta_2{}^3(1-\omega)\left(\Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v}{\beta_2}\right)^{\alpha_2}\right)-\Gamma\left(1+\frac{3}{\alpha_2},\left(\frac{v_{cut-in}}{\beta_2}\right)^{\alpha_2}\right)\right)\right\}+v^3\omega\left\{e^{-\left(\frac{v}{\beta_1}\right)^{\alpha_1}}-e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}}\right\}$$

$$+v^3(1-\omega)\left\{e^{-\left(\frac{v}{\beta_2}\right)^{\alpha_2}}-e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right\} \tag{41}$$

$$P_{avg}{}'(v)=\frac{1}{2}\rho C_P\eta A\omega\left\{\left\{\beta_1{}^3 e^{-\left(\frac{v}{\beta_1}\right)^{\alpha_1}}\left(\frac{v}{\beta_1}\right)^3\left(\frac{\alpha_1}{\beta_1}\right)\left(\frac{v}{\beta_1}\right)^{\alpha_1-1}\right\}+3v^2\left\{e^{-\left(\frac{v}{\beta_1}\right)^{\alpha_1}}-e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}}\right\}-v^3\left\{\left(\frac{\alpha_1}{\beta_1}\right)\left(\frac{v}{\beta_1}\right)^{\alpha_1-1}e^{-\left(\frac{v}{\beta_1}\right)^{\alpha_1}}\right\}\right\}$$

$$+\frac{1}{2}\rho C_P\eta A\left\{(1-\omega)\left\{\beta_2{}^3 e^{-\left(\frac{v}{\beta_2}\right)^{\alpha_2}}\left(\frac{v}{\beta_2}\right)^3\left(\frac{\alpha_2}{\beta_2}\right)\left(\frac{v}{\beta_2}\right)^{\alpha_2-1}\right\}+3v^2(1-\omega)\left\{e^{-\left(\frac{v}{\beta_2}\right)^{\alpha_2}}+e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right\}-v^3(1-\omega)\left\{\left(\frac{\alpha_2}{\beta_2}\right)\left(\frac{v}{\beta_2}\right)^{\alpha_2-1}e^{-\left(\frac{v}{\beta_2}\right)^{\alpha_2}}\right\}\right\}$$

$$=3v^2\omega\left\{e^{-\left(\frac{v}{\beta_1}\right)^{\alpha_1}}-e^{-\left(\frac{v_{cut-off}}{\beta_1}\right)^{\alpha_1}}\right\}+3v^2(1-\omega)\left\{e^{-\left(\frac{v}{\beta_2}\right)^{\alpha_2}}-e^{-\left(\frac{v_{cut-off}}{\beta_2}\right)^{\alpha_2}}\right\} \tag{42}$$

Setting the RHS equal to 0, we see that P_{avg} has no critical points in the domain $[v_{cut-in};\ v_{cut-off})$. Rather, since $P_{avg} > 0$ on $[v_{cut-in};\ v_{cut-off})$, we only conclude that

$sup_{v\in[v_{cut-in};\, v_{cut-off})}$, $P_{avg}(v) = \frac{1}{2}\,\rho C_P \eta A v_{cut-off}{}^3$ .These derivations will give the expected value of the Wind power and the estimation of capacity factor.

5. Conclusions

Wind power stochastic characteristics play a vital role in planning, design and operation of the wind turbines. So installation of a ideal wind turbine is very intrinsic. The capacity factor is a very significant index of productivity of a wind turbine. In this paper a mathematical formulation using multiple parameter Weibull distribution has been derived to compute the capacity factor and expected power output of wind turbines with précised cut-in, cut-off and rated speed. The parameter estimation is done using the maximum likelihood method. The revealed model could be applied in particular regions where the wind speed distribution presents a bimodal pdf.

Acknowledgements

This research is supported by Modelling and simulation laboratory of Nano Science and Technology and Research centre of Sri Ramakrishna Engineering College, Coimbatore, India.

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