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An efficient and simple shear deformation theory for free vibration of functionally graded rectangular plates on Winkler–Pasternak elastic foundations

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Abstract. This work presents a simple hyperbolic shear deformation theory for analysis of functionally graded plates resting on elastic foundation. The proposed model contains fewer number of unknowns and equations of motion than the first-order shear deformation model, but the transverse shear stresses account for a hyperbolic variation and respect the tangential stress-free boundary conditions on the plate boundary surface without introducing shear correction factors. Equations of motion are obtained from Hamilton's principle. The Navier-type analytical solutions for simply-supported plates are compared with the existing solutions to demonstrate the accuracy of the proposed theory.

Keywords: shear deformation theory; vibration; functionally graded plate; elastic foundation

1. Introduction

Functionally graded material is a class of heterogeneous composite material that presents a continuous distribution of mechanical characteristics from one point to another. This material is obtained by mixing two or more materials in a certain volume ratio (for example, metal and ceramic). Classical composites structures such as fiber reinforced plastics (FRPs) suffer from discontinuity of material characteristics at the interface of the layers and constituents. Since the concept of FGMs has been proposed in 1980s, these novel types of materials have been utilized in many engineering application fields, such as aircrafts, space vehicles, defense industries, electronics and biomedical sectors, to eliminate stress concentrations, to relax residual stresses, and to enhance bonding strength. Due to the wide material variations and applications of FGMs, it is important to investigate the behaviors of FGM structures to mechanical and other loadings (Bounouara *et al*. 2016, Hamidi *et al*. 2015, Bakora and Tounsi 2015, Ait Yahia *et al*. 2015, Arefi 2015, Ebrahimi and Dashti 2015, Hadji *et al*. 2015, Hadji and Adda Bedia 2015, Sallai *et al*. 2015,

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Larbi Chaht *et al*. 2015, Darılmaz 2015, Akbaş 2015, Bouchafa *et al*. 2015, Meradjah *et al*. 2015, Yaghoobi *et al*. 2014, Bousahla *et al*. 2014, Zidi *et al*. 2014, Fekrar *et al*. 2014, Tounsi *et al*. 2013, Bouderba *et al*. 2013, Bourada *et al*. 2012).

Several works have been carried out to investigate the vibration of FG graded plates. Vel and Batra (2004) proposed a 3D exact solution for free and forced vibrations of simply supported FG rectangular plates. By employing a global collocation technique, the first and the cubic shear deformation plate theories, Ferreira *et al.* (2006), studied the free vibrations of FG plates. Qian *et al*. (2004) examined bending deformations, and free and forced vibrations of a thick FG graded elastic plate by utilizing a higher order shear and normal deformation plate model. Matsunaga (2008) investigated free vibration and buckling behaviors of FG plates by considering the influences of transverse shear and normal deformations and rotatory inertia. Lu *et al.* (2009) analyzed the free vibration response of FG thick plates supported by elastic foundation using a three-dimensional elasticity. By utilizing the element-free kp-Ritz method, Zhao *et al*. (2009) discussed the free vibration of FG plates. Chen *et al.* (2009) studied the vibration and buckling of FG plates based on a higher-order deformation theory. Malekzadeh (2009) examined the free vibration behavior of thick FG plates on elastic foundation using the three dimensional elasticity theory. Ait Atmane *et al.* (2010) studied the free vibration behavior of simply supported FG plates resting on a Winkler–Pasternak elastic foundation by proposing a new higher shear deformation theory. Using a four variable refined plate theory, Benachour *et al*. (2011) discussed the free vibration response of FG plates with arbitrary gradient. Neves *et al.* (2012ab) proposed a trigonometric shear deformation model and a hybrid quasi-3D hyperbolic shear deformation theory for bending and free vibration analysis of FG plates. Akavci (2014) presented the free vibration analysis of thick FG plates supported on two-parameter elastic foundation based on a higher order hyperbolic shear deformation theory. Belabed *et al.* (2014) developed an efficient and simple higher order shear and normal deformation theory for FG plates. Hebali *et al.* (2014) considered the static and dynamic analysis of FG thick plates with a new quasi-3D hyperbolic shear deformation theory. Ait Amar Meziane *et al*. (2014) presented an efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions. Mahi *et al.* (2015) proposed a new hyperbolic shear deformation theory for bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates. Ait Atmane *et al*. (2015) presented a variationally consistent shear deformation theory for free vibration analysis of thick FG beams having porosities. Bourada *et al*. (2015) investigated the bending and vibration behaviors of FG thick beams using a new simple shear and normal deformations theory. Nguyen *et al*. (2015) presented a refined higher-order shear deformation theory for bending, vibration and buckling analysis of FG sandwich plates. Pradhan and Chakraverty (2015) discussed the free vibration behavior of FG thin elliptic plates with various edge supports. Attia *et al*. (2015) investigated the free vibration response of FG plates with temperature-dependent properties using various four variable refined plate theories. Belkorissat *et al*. (2015) discussed the vibration properties of FG nanoplates using a new nonlocal refined four variable theory. Bennai *et al.* (2015) presented a new higher-order shear and normal deformation theory for FG sandwich beams. Kar and Panda (2015) studied the nonlinear flexural vibration of shear deformable FG spherical shell panel. Bennoun et al. (2016) proposed a novel five variable refined plate theory for vibration analysis of FG sandwich plates. Ait Atmane et al. (2016) studied the effect of thickness stretching and porosity on mechanical response of a FG beams resting on elastic foundations.

The purpose of this work to examine the efficiency of an improved version of a hyperbolic shear deformation theory developed by Mahi *et al*. (2015) for free vibration analysis of FG plates. By making a further supposition to the conventional hyperbolic shear deformation theory (Mahi *et al*. 2015), the present theory contains only four unknowns and its governing equations are therefore reduced. Thus, the novelty of this paper is the use of four variable refined plate theory for free vibration analysis of FG plates, resulting in considerably lower computational effort when compared with the other higher-order theories reported in the literature having more number of unknown functions. Equations of motion are obtained from Hamilton's principle. Navier solution is utilized to determine the closed form solutions for simply supported FG plates. Comparison studies are established to check the accuracy of the present results.

2. Mathematical formulations

In the current work, a FG simply supported rectangular plate with length, width and uniform thickness equal to a, b and h respectively is considered. The geometry of the plate and coordinate system are illustrated in Fig. 1. The material characteristics of FG plate are considered to vary continuously within the thickness of the plate in according to the power law distribution as follows

$$P(z) = P_m + \left(P_c - P_m\right)\left(\frac{1}{2} + \frac{z}{h}\right)^p \tag{1}$$

where P presents the effective material characteristic such as Young's modulus E and mass density ρ, P_m and P_c presents the property of the top and the bottom faces of the plate, respectively, and p is the power law exponent. The Poisson's ratio ν is supposed to be constant.

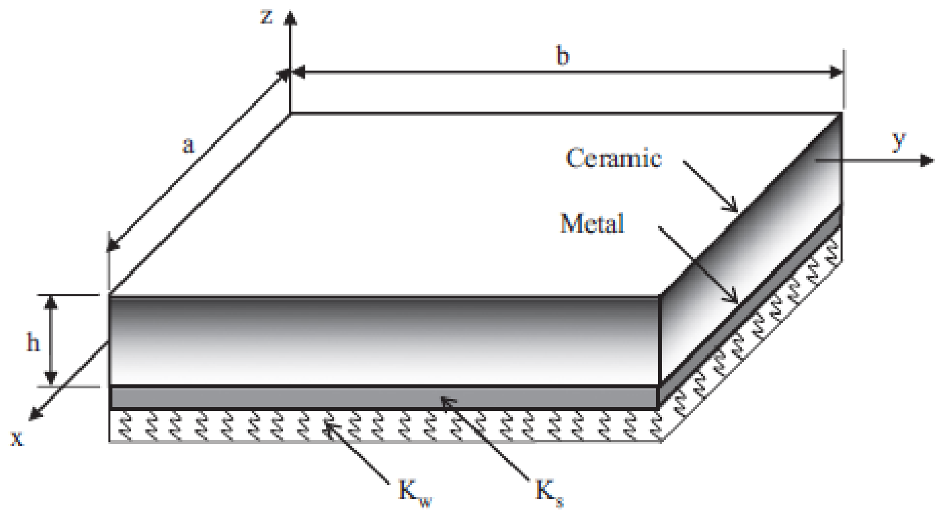


Fig. 1 Schematic representation of a rectangular FG plate resting on elastic foundation

2.1 Kinematics and strains

In this investigation, further simplifying supposition are made to the conventional higher shear deformation theory (HSDT) so that the number of unknowns is reduced. The displacement field of the conventional HSDT is expressed by (Mahi *et al*. 2015)

$$u(x,y,z,t)=u_0(x,y,t) \quad z\frac{\partial w_0}{\partial x}+\Psi(z)\theta_x(x,y,t) \tag{2a}$$

$$v(x,y,z,t)=v_0(x,y,t) \quad z\frac{\partial w_0}{\partial y}+\Psi(z)\theta_y(x,y,t) \tag{2b}$$

$$w(x,y,z,t)=w_0(x,y,t) \tag{2c}$$

where u_0; v_0; w_0, θ_x, θ_y are five unknown displacements of the mid-plane of the plate, $\Psi(z)$ denotes shape function representing the variation of the transverse shear strains and stresses within the thickness. By dividing the deflection w_0 into bending and shear parts (i.e., $w_0=w_b+w_s$) and making further assumptions given by $\theta_x=-\partial w_b(x,y)/\partial x$ and $\theta_y=-\partial w_s(x,y)/\partial y$, the displacement field of the novel refined theory can be expressed in a simpler form as

$$u(x,y,z,t)=u_0(x,y,t)-z\frac{\partial w_b}{\partial x}-f(z)\frac{\partial w_s}{\partial x} \tag{3a}$$

$$v(x,y,z,t)=v_0(x,y,t)-z\frac{\partial w_b}{\partial y}-f(z)\frac{\partial w_s}{\partial y} \tag{3b}$$

$$w(x,y,z,t)=w_0(x,y,t)+w_s(x,y,t) \tag{3b}$$

where the shape function $f(z)$ is given as

$$f(z)=z-\Psi(z)=z-\frac{h}{2}\tanh\left(2\frac{z}{h}\right)+\frac{4}{3\cosh^2(1)}\left(\frac{z^3}{h^2}\right) \tag{4}$$

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix}+z\begin{Bmatrix}k_x^b\\ k_y^b\\ k_{xy}^b\end{Bmatrix}+f(z)\begin{Bmatrix}k_x^s\\ k_y^s\\ k_{xy}^s\end{Bmatrix},\quad \begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}=g(z)\begin{Bmatrix}\gamma_{yz}^0\\ \gamma_{xz}^0\end{Bmatrix} \tag{5}$$

where

$$\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u_0}{\partial x}\\ \dfrac{\partial v_0}{\partial x}\\ \dfrac{\partial u_0}{\partial y}+\dfrac{\partial v_0}{\partial x}\end{Bmatrix},\quad \begin{Bmatrix}k_x^b\\ k_y^b\\ k_{xy}^b\end{Bmatrix}=\begin{Bmatrix}-\dfrac{\partial^2 w_b}{\partial x^2}\\ -\dfrac{\partial^2 w_b}{\partial y^2}\\ -2\dfrac{\partial^2 w_b}{\partial x\partial y}\end{Bmatrix},\quad \begin{Bmatrix}k_x^s\\ k_y^s\\ k_{xy}^s\end{Bmatrix}=\begin{Bmatrix}-\dfrac{\partial^2 w_s}{\partial x^2}\\ -\dfrac{\partial^2 w_s}{\partial y^2}\\ -2\dfrac{\partial^2 w_s}{\partial x\partial y}\end{Bmatrix},\quad \begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}=g(z)\begin{Bmatrix}\gamma_{yz}^s\\ \gamma_{xz}^s\end{Bmatrix} \tag{6a}$$

and

$$g(z) = 1 - \frac{df(z)}{dz} \tag{6b}$$

For elastic and isotropic FGMs, the constitutive relations can be expressed as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{7}$$

where (σ_x , σ_y , τ_{xy} , τ_{yz} , τ_{xz}) and (ε_x , ε_y , γ_{xy} , γ_{yz} , γ_{xz}) are the stress and strain components, respectively. Using the material properties defined in Eq. (1), stiffness coefficients, C_{ij} , can be written as

$$C_{11} = C_{22} = \frac{E(z)}{1-\nu^2}, \quad C_{12} = \frac{\nu\, E(z)}{1-\nu^2}, \quad C_{44} = C_{55} = C_{66} = \frac{E(z)}{2(1+\nu)}, \tag{8}$$

2.2 Equations of motion

Hamilton's principle is herein employed to determine the equations of motion (Ould Larbi *et al*. 2013, Draiche *et al*. 2014, Tagrara *et al*. 2015)

$$0 = \int_0^t (\delta U + \delta V - \delta K)\, dt \tag{9}$$

where δU is the variation of strain energy; δV is the variation of work done; and δK is the variation of kinetic energy.

The variation of strain energy of the plate is expressed by

$$\begin{aligned} \delta U &= \int_V \left[\sigma_x \delta\, \varepsilon_x + \sigma_y \delta\, \varepsilon_y + \tau_{xy} \delta\, \gamma_{xy} + \tau_{yz} \delta\, \gamma_{yz} + \tau_{xz} \delta\, \gamma_{xz} \right] dV \\ &= \int_A \left[N_x \delta\, \varepsilon_x^0 + N_y \delta\, \varepsilon_y^0 + N_{xy} \delta\, \gamma_{xy}^0 + M_x^b \delta\, k_x^b + M_y^b \delta\, k_y^b + M_{xy}^b \delta\, k_{xy}^b \right. \\ &\quad \left. + M_x^s \delta\, k_x^s + M_y^s \delta\, k_y^s + M_{xy}^s \delta\, k_{xy}^s + S_{yz}^s \delta\, \gamma_{yz}^s + S_{xz}^s \delta\, \gamma_{xz}^s \right] dA = 0 \end{aligned} \tag{10}$$

where A is the top surface and the stress resultants N , M , and S are defined by

$$\left(N_i, M_i^b, M_i^s\right) = \int_{-h/2}^{h/2} (1, z, f)\sigma_i\, dz, \quad (i = x, y, xy) \quad \text{and} \quad \left(S_{xz}^s, S_{yz}^s\right) = \int_{-h/2}^{h/2} g\left(\tau_{xz}, \tau_{yz}\right) dz \tag{11}$$

The variation of the potential energy of elastic foundation can be calculated by

$$\delta V = \int_A f_e \delta (w_b + w_s) dA \tag{12}$$

where f_e is the density of reaction force of foundation. For the Pasternak foundation model (Tebboune *et al.* 2015; Besseghier *et al*. 2015)

$$f_e = K_W w - K_{S1} \frac{\partial^2 w}{\partial x^2} - K_{S2} \frac{\partial^2 w}{\partial y^2} \tag{13}$$

where K_W is the modulus of subgrade reaction (elastic coefficient of the foundation) and K_{S1} and K_{S2} are the shear moduli of the subgrade (shear layer foundation stiffness). If foundation is homogeneous and isotropic, we will get $K_{S1} = K_{S2} = K_S$. If the shear layer foundation stiffness is neglected, Pasternak foundation becomes a Winkler foundation.

The variation of kinetic energy of the plate can be written as

$$\begin{aligned}
\delta K &= \int_V \left[\dot{u} \delta \dot{u} + \dot{v} \delta \dot{v} + \dot{w} \delta \dot{w} \right] \rho(z)\, dV \\
&= \int_A \Big\{ I_0 \left[\dot{u}_0 \delta \dot{u}_0 + \dot{v}_0 \delta \dot{v}_0 + (\dot{w}_b + \dot{w}_s)(\delta \dot{w}_b + \delta \dot{w}_s) \right] \\
&- I_1 \left(\dot{u}_0 \frac{\partial \delta \dot{w}_b}{\partial x} + \frac{\partial \dot{w}_b}{\partial x} \delta \dot{u}_0 + \dot{v}_0 \frac{\partial \delta \dot{w}_b}{\partial y} + \frac{\partial \dot{w}_b}{\partial y} \delta \dot{v}_0 \right) \\
&- J_1 \left(\dot{u}_0 \frac{\partial \delta \dot{w}_s}{\partial x} + \frac{\partial \dot{w}_s}{\partial x} \delta \dot{u}_0 + \dot{v}_0 \frac{\partial \delta \dot{w}_s}{\partial y} + \frac{\partial \dot{w}_s}{\partial y} \delta \dot{v}_0 \right) \\
&+ I_2 \left(\frac{\partial \dot{w}_b}{\partial x} \frac{\partial \delta \dot{w}_b}{\partial x} + \frac{\partial \dot{w}_b}{\partial y} \frac{\partial \delta \dot{w}_b}{\partial y} \right) + K_2 \left(\frac{\partial \dot{w}_s}{\partial x} \frac{\partial \delta \dot{w}_s}{\partial x} + \frac{\partial \dot{w}_s}{\partial y} \frac{\partial \delta \dot{w}_s}{\partial y} \right) \\
&+ J_2 \left(\frac{\partial \dot{w}_b}{\partial x} \frac{\partial \delta \dot{w}_s}{\partial x} + \frac{\partial \dot{w}_s}{\partial x} \frac{\partial \delta \dot{w}_b}{\partial x} + \frac{\partial \dot{w}_b}{\partial y} \frac{\partial \delta \dot{w}_s}{\partial y} + \frac{\partial \dot{w}_s}{\partial y} \frac{\partial \delta \dot{w}_b}{\partial y} \right) \Big\} dA
\end{aligned} \tag{14}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; $\rho(z)$ is the mass density given by Eq. (1); and (I_0, I_1, J_1, I_2, J_2, K_2) are mass inertias defined as

$$(I_0, I_1, J_1, I_2, J_2, K_2) = \int_{-h/2}^{h/2} (1, z, f, z^2, z\, f, f^2) \rho(z) dz \tag{15}$$

Substituting Eqs. (10), (12), and (14) into Eq. (9), integrating by parts, and collecting the coefficients of δu_0, δv_0, δw_b and δw_s; the following equations of motion are obtained

$$\begin{aligned}
&\delta u_0: \ \frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=I_0\ddot{u}_0-I_1\frac{\partial \ddot{w}_b}{\partial x}-J_1\frac{\partial \ddot{w}_s}{\partial x}\\
&\delta v_0: \ \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=I_0\ddot{v}_0-I_1\frac{\partial \ddot{w}_b}{\partial y}-J_1\frac{\partial \ddot{w}_s}{\partial y}\\
&\delta w_b: \ \frac{\partial^2 M_x^b}{\partial x^2}+2\frac{\partial^2 M_{xy}^b}{\partial x\partial y}+\frac{\partial^2 M_y^b}{\partial y^2}-f_e=I_0(\ddot{w}_b+\ddot{w}_s)+I_1\left(\frac{\partial \ddot{u}_0}{\partial x}+\frac{\partial \ddot{v}_0}{\partial y}\right)-I_2\nabla^2\ddot{w}_b-J_2\nabla^2\ddot{w}_s\\
&\delta w_s: \ \frac{\partial^2 M_x^s}{\partial x^2}+2\frac{\partial^2 M_{xy}^s}{\partial x\partial y}+\frac{\partial^2 M_y^s}{\partial y^2}+\frac{\partial S_{xz}^s}{\partial x}+\frac{\partial S_{yz}^s}{\partial y}-f_e=I_0(\ddot{w}_b+\ddot{w}_s)+J_1\left(\frac{\partial \ddot{u}_0}{\partial x}+\frac{\partial \ddot{v}_0}{\partial y}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -J_2\nabla^2\ddot{w}_b-K_2\nabla^2\ddot{w}_s
\end{aligned} \tag{16}$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2$ is the Laplacian operator in two-dimensional Cartesian coordinate system.

Substituting Eq. (5) into Eq. (7) and the subsequent results into Eqs. (11), the stress resultants are obtained in terms of strains as following compact form

$$\begin{Bmatrix} N \\ M^b \\ M^s \end{Bmatrix}=\begin{bmatrix} A & B & B^s \\ B & D & D^s \\ B^s & D^s & H^s \end{bmatrix}\begin{Bmatrix} \varepsilon \\ k^b \\ k^s \end{Bmatrix}, \quad S=A^s\gamma \tag{17}$$

in which

$$N=\{N_x,N_y,N_{xy}\}^t, \quad M^b=\{M_x^b,M_y^b,M_{xy}^b\}^t, \quad M^s=\{M_x^s,M_y^s,M_{xy}^s\}^t \tag{18a}$$

$$\varepsilon=\{\varepsilon_x^0,\varepsilon_y^0,\gamma_{xy}^0\}^t, \quad k^b=\{k_x^b,k_y^b,k_{xy}^b\}^t, \quad k^s=\{k_x^s,k_y^s,k_{xy}^s\}^t \tag{18b}$$

$$A=\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \quad B=\begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix}, \quad D=\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \tag{18c}$$

$$B^s=\begin{bmatrix} B_{11}^s & B_{12}^s & 0 \\ B_{12}^s & B_{22}^s & 0 \\ 0 & 0 & B_{66}^s \end{bmatrix}, \quad D^s=\begin{bmatrix} D_{11}^s & D_{12}^s & 0 \\ D_{12}^s & D_{22}^s & 0 \\ 0 & 0 & D_{66}^s \end{bmatrix}, \quad H^s=\begin{bmatrix} H_{11}^s & H_{12}^s & 0 \\ H_{12}^s & H_{22}^s & 0 \\ 0 & 0 & H_{66}^s \end{bmatrix} \tag{18d}$$

$$S=\{S_{xz}^s,S_{yz}^s\}^t, \quad \gamma=\{\gamma_{xz}^0,\gamma_{yz}^0\}^t, \quad A^s=\begin{bmatrix} A_{44}^s & 0 \\ 0 & A_{55}^s \end{bmatrix} \tag{18e}$$

and stiffness components are given as

$$\begin{Bmatrix} A_{11} & B_{11} & D_{11} & B_{11}^s & D_{11}^s & H_{11}^s \\ A_{12} & B_{12} & D_{12} & B_{12}^s & D_{12}^s & H_{12}^s \\ A_{66} & B_{66} & D_{66} & B_{66}^s & D_{66}^s & H_{66}^s \end{Bmatrix}=\int_{-h/2}^{h/2} C_{11}\left(1,z,z^2,f(z),z\,f(z),f^2(z)\right)\begin{Bmatrix} 1 \\ \nu \\ \dfrac{1-\nu}{2} \end{Bmatrix}dz \tag{19a}$$

$$\left(A_{22}, B_{22}, D_{22}, B^s_{22}, D^s_{22}, H^s_{22}\right) = \left(A_{11}, B_{11}, D_{11}, B^s_{11}, D^s_{11}, H^s_{11}\right) \tag{19b}$$

$$A^s_{44} = A^s_{55} = \int_{-h/2}^{h/2} C_{44}\left[g(z)\right]^2 dz, \tag{19c}$$

Introducing Eq. (17) into Eq. (16), the equations of motion can be expressed in terms of displacements (δu_0, δv_0, δw_b, δw_s) and the appropriate equations take the form

$$\begin{aligned} &A_{11}d_{11}u_0 + A_{66}d_{22}u_0 + \left(A_{12}+A_{66}\right)d_{12}v_0 - B_{11}d_{111}w_b - \left(B_{12}+2B_{66}\right)d_{122}w_b \\ &\quad - \left(B^s_{12}+2B^s_{66}\right)d_{122}w_s - B^s_{11}d_{111}w_s = I_0\ddot{u}_0 - I_1 d_1\ddot{w}_b - J_1 d_1\ddot{w}_s, \end{aligned} \tag{20a}$$

$$\begin{aligned} &A_{22}d_{22}v_0 + A_{66}d_{11}v_0 + \left(A_{12}+A_{66}\right)d_{12}u_0 - B_{22}d_{222}w_b - \left(B_{12}+2B_{66}\right)d_{112}w_b \\ &\quad - \left(B^s_{12}+2B^s_{66}\right)d_{112}w_s - B^s_{22}d_{222}w_s = I_0\ddot{v}_0 - I_1 d_2\ddot{w}_b - J_1 d_2\ddot{w}_s, \end{aligned} \tag{20b}$$

$$\begin{aligned} &B_{11}d_{111}u_0 + \left(B_{12}+2B_{66}\right)d_{122}u_0 + \left(B_{12}+2B_{66}\right)d_{112}v_0 + B_{22}d_{222}v_0 \\ &\quad - D_{11}d_{1111}w_b - 2\left(D_{12}+2D_{66}\right)d_{1122}w_b - D_{22}d_{2222}w_b - D^s_{11}d_{1111}w_s \\ &\quad - 2\left(D^s_{12}+2D^s_{66}\right)d_{1122}w_s - D^s_{22}d_{2222}w_s - f_e = I_0\left(\ddot{w}_b+\ddot{w}_s\right) \\ &\quad + I_1\left(d_1\ddot{u}_0 + d_2\ddot{v}_0\right) - I_2\left(d_{11}\ddot{w}_b + d_{22}\ddot{w}_b\right) - J_2\left(d_{11}\ddot{w}_s + d_{22}\ddot{w}_s\right) \end{aligned} \tag{20c}$$

$$\begin{aligned} &B^s_{11}d_{111}u_0 + \left(B^s_{12}+2B^s_{66}\right)d_{122}u_0 + \left(B^s_{12}+2B^s_{66}\right)d_{112}v_0 + B^s_{22}d_{222}v_0 - D^s_{11}d_{1111}w_b \\ &\quad - 2\left(D^s_{12}+2D^s_{66}\right)d_{1122}w_b - D^s_{22}d_{2222}w_b - H^s_{11}d_{1111}w_s \\ &\quad - 2\left(H^s_{12}+2H^s_{66}\right)d_{1122}w_s - H^s_{22}d_{2222}w_s + A^s_{44}d_{11}w_s + A^s_{55}d_{22}w_s - f_e = I_0\left(\ddot{w}_b+\ddot{w}_s\right) \\ &\quad + J_1\left(d_1\ddot{u}_0 + d_2\ddot{v}_0\right) - J_2\left(d_{11}\ddot{w}_b + d_{22}\ddot{w}_b\right) - K_2\left(d_{11}\ddot{w}_s + d_{22}\ddot{w}_s\right) \end{aligned} \tag{20d}$$

where d_{ij}, d_{ijl} and d_{ijlm} are the following differential operators

$$d_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \quad d_{ijl} = \frac{\partial^3}{\partial x_i \partial x_j \partial x_l}, \quad d_{ijlm} = \frac{\partial^4}{\partial x_i \partial x_j \partial x_l \partial x_m}, \quad d_i = \frac{\partial}{\partial x_i}, \quad (i, j, l, m = 1,2). \tag{21}$$

2.3 Analytical solution for simply-supported FG plates

Based on Navier technique, the following expansions of generalized displacements are considered to automatically respect the simply supported boundary conditions

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_b \\ w_s \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn}e^{i\omega t}\cos(\alpha x)\sin(\beta y) \\ V_{mn}e^{i\omega t}\sin(\alpha x)\cos(\beta y) \\ W_{bmn}e^{i\omega t}\sin(\alpha x)\sin(\beta y) \\ W_{smn}e^{i\omega t}\sin(\alpha x)\sin(\beta y) \end{Bmatrix} \tag{22}$$

where $\alpha = m\pi / a$ and $\beta = n\pi / b$, ω is the frequency of free vibration of the plate, $\sqrt{i} = -1$

the imaginary unit.

Substituting Eqs. (22) into Eq. (20) and collecting the displacements and acceleration for any values of m and n, the following problem is obtained

$$\left(\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33} & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & 0 & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix}\right) \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{bmn} \\ W_{smn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{23}$$

where

$$S_{11} = -(\alpha^2 A_{11} + \beta^2 A_{66}),\, S_{12} = -\alpha\beta(A_{12} + A_{66}),\, S_{13} = \alpha[\alpha^2 B_{11} + \beta^2 (B_{12} + 2B_{66})],$$
$$S_{14} = -\alpha[\alpha^2 B_{11}^S + \beta^2 (B_{12}^S + 2B_{66}^S)] \quad , S_{22} = -(\alpha^2 A_{66} + \beta^2 A_{22}),$$
$$S_{23} = \beta[\beta^2 B_{22} + \alpha^2 (B_{12} + 2B_{66})]$$
$$S_{24} = -\beta[\beta^2 B_{22}^s + \alpha^2 (B_{12}^s + 2B_{66}^s)],$$
$$S_{33} = -[D_{11}(\alpha^4 + \beta^4) + 2\alpha^2\beta^2 (D_{12} + 2D_{66})] + K_w - K_s(\alpha^2 + \beta^2)$$
$$S_{34} = [D_{11}^s(\alpha^4 + \beta^4) + 2\alpha^2\beta^2 (D_{12}^s + 2D_{66}^s)] + K_w - K_s(\alpha^2 + \beta^2),$$
$$S_{44} = -[H_{11}^s(\alpha^4 + \beta^4) + 2\alpha^2\beta^2 (H_{12}^s + 2H_{66}^s) +] + A_{11}^s(\alpha^2 + \beta^2) + K_w - K_s(\alpha^2 + \beta^2)$$
$$m_{11} = -I_0,\, m_{13} = \beta I_1,\, m_{14} = -\beta J_1,\, m_{22} = m_{11},\, m_{23} = -\alpha I_1,\, m_{24} = -\alpha J_1,$$
$$m_{33} = -(I_0 + I_2(\alpha^2 + \beta^2))$$
$$m_{34} = -I_0 + J_2(\alpha^2 + \beta^2),\, m_{41} = -\beta J_1,\, m_{42} = -\alpha J_1,\, m_{44} = -(I_0 + K_2(\alpha^2 + \beta^2)) \tag{24}$$

3. Numerical examples and discussions

In this section various numerical examples are examined to check the accuracy of the present formulation in predicting the free vibration behaviors of simply supported FG plates resting on elastic foundation. Two types of FG plates of Al/Al_2O_3 and Al/ZrO_2 are employed in this investigation. The material characteristics of FG plates are presented in Table 1. For convenience, the following non-dimensional parameters are employed

$$\bar{\omega} = \omega h \sqrt{\rho_m / E_m}, \quad \tilde{\omega} = \omega \frac{a^2}{h} \sqrt{\rho_m / E_m}, \quad \beta = \omega h \sqrt{\rho_c / E_c} \quad k_w = \frac{K_w a^4}{D}, \quad k_s = \frac{K_s a^2}{D}$$
$$D = \frac{h^3}{12(1-\nu^2)} \frac{[p(8 + 3p + p^2)E_m + 3(2 + p + p^2)E_c]}{[(1+p)(2+p)(3+p)]} \tag{25}$$

In Table 2, non-dimensional fundamental frequencies of Al/ZrO_2 FG square plates are examined for three different power law exponent and compared with 3D exact solution of Vel and Batra (2004), quasi 3D sinusoidal and hyperbolic shear deformation theories of Neves *et al*. (2012a,b) and 2D higher order shear deformation models of Matsunaga (2008) and Hosseini-Hashemi *et al*. (2011) and Akavci (2014). It can be demonstrated from the table that the results of the present formulation agree with the results of other 2D and 3D deformation theories.

The non-dimensional natural frequencies computed by the proposed are compared in Table 3 with the 3D theory of Vel and Batra (2004) and 2D higher order shear deformation theories of Matsunaga (2008) and Akavci (2014). It can be observed from the table that a good agreement is achieved between the obtained results and those reported by other theories.

To check the higher order modes, the first eight frequencies of the Al/Al_2O_3 FG square and rectangular plates are calculated and illustrated in Tables 4. Table 4 shows a comparison between the first eight non-dimensional natural frequencies of FG square plates computed using the present theory and those given by Matsunaga (2008) and by Akavci (2014). It can be seen from Table 4 that for both thin and thick plates, a good agreement between the results is demonstrated.

Table 1 Material properties employed in the FG plates

Material	Properties		
	Young's modulus (GPa)	Poisson's ratio	Mass density kg/m^3
Aluminium (Al)	70	0.3	2702
Alumina (Al_2O_3)	380	0.3	3800
Zirconia (ZrO_2)	200	0.3	5700

Table 2 Comparison of non dimensional frequencies $\bar{\omega} = \omega h \sqrt{\rho_m / E_m}$ of Al/ZrO_2 of FG square plates ($a/h = 5$, $k_w = k_s = 0$)

Source	$p = 2$	$p = 3$	$p = 5$
Vel and Batra (2004)	0.2197	0.2211	0.2225
Neves *et al*. (2012a) ($\varepsilon_z = 0$)	0.2189	0.2202	0.2215
Neves *et al*. (2012a) ($\varepsilon_z \neq 0$)	0.2198	0.2212	0.2225
Neves *et al.* (2012b) ($\varepsilon_z = 0$)	0.2191	0.2205	0.2220
Neves *et al.* (2012b) ($\varepsilon_z \neq 0$)	0.2201	0.2216	0.2230
Matsunaga (2008)	0.2264	0.2270	0.2280
Hosseini-Hashemi *et al.* (2011)	0.2264	0.2276	0.2291
Akavci (2014)	0.2263	0.2268	0.2277
Present study	0.2258	0.2266	0.2276

Table 3 Comparison of non dimensional frequencies $\tilde{\omega} = \omega \dfrac{a^2}{h}\sqrt{\rho_m / E_m}$ of Al/ZrO_2 of FG square plates ($m = n = 1, k_w = k_s = 0$)

Mode no.	Source	p=0 [(a)]	p=1				a/h =5		
		$a/h=\sqrt{10}$	$a/h=10$	$a/h=5$	$a/h=10$	$a/h=20$	$p=2$	$p=3$	$p=5$
1	Vel and Batra (2004)	4.6582	5.7769	5.4806	5.9609	6.1076	5.4923	5.5285	5.5632
	Matsunaga (2008)	4.6582	5.7769	5.7123	6.1932	6.3390	5.6599	5.6757	5.7020
	Akavci (2014)	4.6569	5.7754	5.7110	6.1924	6.3388	5.6593	5.6718	5.6941
	Present study	4.6274	5.7705	5.6955	6.1876	6.3373	5.6475	5.6640	5.6885
2	Vel and Batra (2004)	8.7132	27.554	14.558	29.123	58.250	14.278	14.150	14.026
	Matsunaga (2008)	8.7132	27.554	15.339	30.685	61.374	14.970	14.742	14.476
	Akavci (2014)	8.7132	27.554	15.341	30.686	61.374	14.972	14.743	14.477
	Present study	8.7130	27.554	15.344	30.686	61.375	14.978	14.750	14.483
3	Vel and Batra (2004)	14.463	46.503	24.381	49.013	98.145	23.909	23.696	23.494
	Matsunaga (2008)	14.463	46.503	25.776	51.795	103.71	25.140	24.741	24.278
	Akavci (2014)	14.728	46.574	25.926	51.866	103.74	25.296	24.909	24.461
	Present study	14.728	46.574	25.924	51.866	103.74	25.296	24.908	24.460
4	Vel and Batra (2004)	24.830	201.34	57.620	212.22	828.78	54.685	53.179	52.068
	Matsunaga (2008)	24.830	201.34	61.509	227.29	888.60	57.576	55.237	53.288
	Akavci (2014)	25.427	203.98	62.886	231.52	904.25	58.993	56.373	54.067
	Present study	25.347	202.92	62.635	230.40	899.60	59.135	56.805	54.615

(a) $\tilde{\omega} = \omega \dfrac{a^2}{h}\sqrt{\rho_c / E_c}$

Table 5 presents non-dimensional fundamental frequencies of Al/ZrO_2 FG rectangular plates resting on elastic foundation. The results of the present formulation are compared with the results of the first order shear deformation theory (FSDT) of Hosseini-Hashemi *et al*. (2010) and higher order shear deformation theories (HSDTs) of Hasani Baferani *et al*. (2011) and Akavci (2014). It can be observed from the Table 5 that, the results of present formulation with only four unknowns are in good agreement with the results of other theories with five unknowns.

Table 6 illustrated non-dimensional fundamental frequencies of Al/Al_2O_3 FG plates resting on elastic foundation. The results are obtained for different aspect ratios and compared with those obtained by Hasani Baferani *et al.* (2011) and Akavci (2014) by employing HSDTs and Hosseini-Hashemi *et al*. (2010) by utilizing FSDT. It can be observed that, the proposed theory agrees well with the other shear deformation theories.

Table 7 presents the comparison of non-dimensional fundamental frequencies of Al/Al_2O_3 FG plates resting on elastic foundation with those reported by Akavci (2014) using HSDT. It can be confirmed from the Table 7 that, the results of the proposed theory are in good agreement with the results of Akavci (2014).

Table 4 The first eight non-dimensional natural frequencies $\beta = \omega h\sqrt{\rho_c / E_c}$ of Al/Al_2O_3 of FG square plates ($k_w = k_s = 0$)

h/b	p	Source	Mode no							
			1	2	3	4	5	6	7	8
0.1		Mode	1, 0, 1	1, 1, 1	2,0 , 1	1, 2, 1	1, 0, 2	2, 2, 1	3, 0, 1	1, 3, 1
	0	Matsunaga (2008)	0.02936	0.0577	0.1120	0.1381	0.1948	0.2121	0.2357	0.2587
		Akavci (2014)	0.02936	0.0577	0.1119	0.1379	0.1948	0.2120	0.2355	0.2585
		Present study	0.02934	0.0577	0.1118	0.1377	0.1948	0.2114	0.2347	0.2576
	0.5	Matsunaga (2008)	0.0249	0.0491	0.0956	0.1180	0.1749	0.1819	0.2022	0.2222
		Akavci (2014)	0.0249	0.0490	0.0954	0.1176	0.1749	0.1813	0.2016	0.2214
		Present study	0.02489	0.04901	0.09524	0.1174	0.1749	0.1808	0.2010	0.2207
	1	Matsunaga (2008)	0.0224	0.0442	0.0861	0.1063	0.1620	0.1640	0.1824	0.2004
		Akavci (2014)	0.0224	0.0442	0.0860	0.1061	0.1620	0.1636	0.1819	0.1999
		Present study	0.02244	0.04420	0.08591	0.1059	0.1621	0.1632	0.1814	0.1993
	4	Matsunaga (2008)	0.0194	0.0381	0.0735	0.0904	0.1308	0.1383	0.1534	0.1681
		Akavci (2014)	0.0194	0.0380	0.0734	0.0902	0.1308	0.1379	0.1529	0.1677
		Present study	0.01941	0.03810	0.07352	0.09037	0.1309	0.1381	0.1532	0.1679
	10	Matsunaga (2008)	0.0186	0.0364	0.0699	0.0858	0.1153	0.1306	0.1446	0.1583
		Akavci (2014)	0.0186	0.0364	0.0699	0.0858	0.1153	0.1305	0.1445	0.1582
		Present study	0.01860	0.03639	0.06994	0.08582	0.1153	0.1304	0.1445	0.1581
		Mode	1,0,1	1,1,1	2,0,1	1,0,2	1,2,1	1,1,2	1,0,3	2,2,1
0.2	0	Matsunaga (2008)	0.1120	0.2121	0.3874	0.3897	0.4658	0.5511	0.6566	0.6753
		Akavci (2014)	0.1119	0.2120	0.3872	0.3897	0.4657	0.5510	0.6587	0.6759
		Present study	0.1118	0.2114	0.3851	0.3896	0.4626	0.5511	0.6586	0.6697
	0.5	Matsunaga (2008)	0.0956	0.1819	0.3343	0.3497	0.4040	0.4941	0.5878	0.5891
		Akavci (2014)	0.0954	0.1813	0.3330	0.3495	0.4015	0.4940	0.5905	0.5856
		Present study	0.09524	0.1808	0.3314	0.3497	0.3990	0.4945	0.5903	0.5809
	1	Matsunaga (2008)	0.0861	0.1640	0.3020	0.3236	0.3644	0.4567	0.5325	0.5444
		Akavci (2014)	0.0860	0.1636	0.3009	0.3236	0.3629	0.4569	0.5461	0.5302
		Present study	0.08591	0.1632	0.2996	0.3241	0.3611	0.4585	0.5461	0.5258
	4	Matsunaga (2008)	0.0735	0.1383	0.2502	0.2607	0.3000	0.3668	0.4325	0.4362
		Akavci (2014)	0.0734	0.1379	0.2493	0.2606	0.2987	0.3668	0.4381	0.4304
		Present study	0.07352	0.1381	0.2496	0.2619	0.2990	0.3703	0.4383	0.4303
	10	Matsunaga (2008)	0.0699	0.1306	0.2300	0.2337	0.2790	0.3243	0.3855	0.3981
		Akavci (2014)	0.0699	0.1305	0.2337	0.2300	0.2792	0.3245	0.3878	0.3991
		Present study	0.06994	0.1304	0.2333	0.2306	0.2785	0.3263	0.3877	0.3971

Table 5 Comparison of non dimensional frequencies $\beta = \omega h \sqrt{\rho_c / E_c}$ of Al/ZrO_2 of rectangular FG plates ($a/b = 1.5$)

(k_w, k_s)	a/h	p	Source			
			Hasani Baferani *et al.* (2011)	Hosseini-Hashemi *et al.* (2010)	Akavci (2014)	Present study
(0, 0)	0.05	0	–	0.02392	0.02393	0.02392
		0.25	–	0.02269	0.02309	0.02308
		1	–	0.02156	0.02202	0.02201
		5	–	0.02180	0.02244	0.02244
	0.1	0	–	0.09188	0.09203	0.09191
		0.25	–	0.08603	0.08895	0.08884
		1	–	0.08155	0.08489	0.08479
		5	–	0.08171	0.08576	0.08573
	0.2	0	–	0.32284	0.32471	0.32328
		0.25	–	0.31003	0.31531	0.31396
		1	–	0.29399	0.30152	0.30026
		5	–	0.29099	0.31860	0.29710
(250,25)	0.05	0	0.03421	0.03421	0.03422	0.03421
		0.25	0.03321	0.03285	0.03312	0.03311
		1	0.03249	0.03184	0.03213	0.03213
		5	0.03314	0.03235	0.03277	0.03277
	0.1	0	0.13365	0.13365	0.13375	0.13366
		0.25	0.13004	0.12771	0.12959	0.12952
		1	0.12749	0.12381	0.12585	0.12578
		5	0.12950	0.12533	0.12778	0.12776
	0.2	0	0.43246	0.49945	0.50044	0.49967
		0.25	0.42868	0.48327	0.48594	0.48522
		1	0.46406	0.46997	0.47298	0.47233
		5	0.44824	0.47400	0.47637	0.47610

The variation of non-dimensional fundamental frequencies in terms of the power law exponent and side-to-thickness ratio is presented in Fig. 2. It can be observed from the figure that the increase of the power law exponent leads to a decrease in the fundamental frequency. It is due to the fact that a higher value of p corresponds to lower value of volume fraction of the ceramic phase, and thus makes the plates become the softer ones. Fig. 2 demonstrates also that with a decrease of the side-to-thickness ratio, the shear deformation influence becomes very effective.

Fig. 3 presents the effect of the elastic foundation parameters on the variations of non-dimensional natural frequencies of simply supported Al/Al_2O_3 FG square plates versus the power law exponent. It can be seen that the presence of elastic foundation makes the plate becomes stiffer. It can be confirmed from the results that, increasing value of Winkler and Pasternak parameters amplifies the natural frequency. The results show also, Pasternak parameter of foundation has more important impact than Winkler parameter on the fundamental frequency of plate.

The variations of non-dimensional fundamental frequency of simply supported Al/Al_2O_3 FG square plate are shown in Figs. 4(a) and 4(b) with respect to Winkler parameter of foundation. It is observed from the results that, increasing the power law exponent reduces the fundamental frequency. It is also concluded from results that, increasing value of power law exponent increases the impact of elastic foundation on natural frequency.

Table 6 Comparison of non dimensional frequencies $\beta = \omega h \sqrt{\rho_c / E_c}$ of Al/Al_2O_3 of rectangular FG plates ($h/a = 0.15$)

(k_w, k_s)	a/b	p	Source			
			Hasani Baferani *et al.* (2011)	Hosseini-Hashemi *et al.* (2010)	Akavci (2014)	Present study
(0, 0)	0.5	0	–	0.08006	0.08018	0.08009
		0.25	–	0.07320	0.07335	0.07327
		1	–	0.06335	0.06148	0.06142
		5	–	0.05379	0.05215	0.05221
	1	0	–	0.12480	0.12508	0.12486
		0.25	–	0.11354	0.11457	0.11439
		1	–	0.09644	0.09613	0.09599
		5	–	0.08027	0.08089	0.08102
	2	0	–	0.28513	0.28659	0.28547
		0.25	–	0.25555	0.26356	0.26260
		1	–	0.20592	0.22189	0.22115
		5	–	0.16315	0.18232	0.18277
(100,10)	0.5	0	0.12869	0.12870	0.12876	0.12871
		0.25	0.11885	0.11842	0.11847	0.11842
		1	0.10498	0.10519	0.10388	0.10384
		5	0.09227	0.09223	0.09098	0.09101
	1	0	0.17020	0.17020	0.17039	0.17024
		0.25	0.15734	0.15599	0.15665	0.15652
		1	0.13854	0.13652	0.13592	0.13583
		5	0.12077	0.11786	0.11774	0.11782
	2	0	0.31449	0.32768	0.32889	0.32796
		0.25	0.30484	0.29612	0.30297	0.30190
		1	0.26966	0.24674	0.25901	0.25841
		5	0.22932	0.20359	0.21785	0.21819

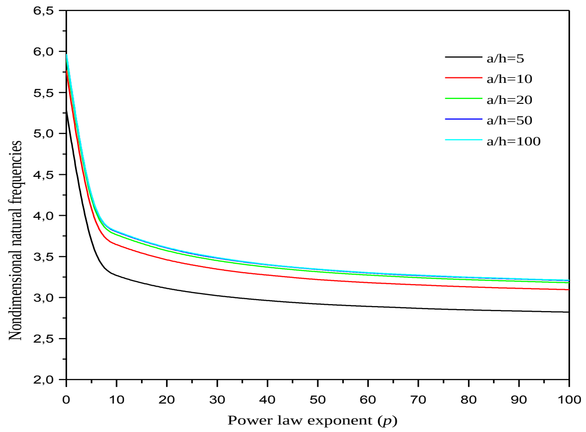


Fig. 2 Variation of non-dimensional fundamental frequency $\tilde{\beta} = \omega \frac{a^2}{h} \sqrt{\rho_c / E_c}$ of Al/Al_2O_3 FG square plates with power law exponent

Table 7 Comparison of non dimensional frequencies $\tilde{\omega}$ of Al/Al_2O_3 of rectangular FG plates

$(k_w, k_s)^{(a)}$	a/b	a/h	p = 0 Akavci (2014)	0 Present	1 Akavci (2014)	1 Present	5 Akavci (2014)	5 Present	10 Akavci (2014)	10 Present
(0,0)	0.5	5	6.7771	6.7640	5.2122	5.2035	4.3763	4.3839	4.2153	4.2149
		10	7.1794	7.1758	5.4918	5.4892	4.6986	4.7014	4.5432	4.5434
		20	7.2948	7.2938	5.5712	5.5704	4.7943	4.7950	4.6411	4.6412
	1	5	10.4133	10.382	8.0368	8.0165	6.6705	6.6855	6.4099	6.4075
		10	11.3468	11.337	8.6899	8.6836	7.4033	7.4100	7.1521	7.1522
		20	11.6338	11.631	8.8879	8.8864	7.6393	7.6413	7.3934	7.3935
	2	5	22.8734	22.728	17.8289	17.732	14.3625	14.394	13.7120	13.678
		10	27.1085	27.056	20.8487	20.814	17.5051	17.536	16.8613	16.860
		20	28.7174	28.703	21.9670	21.957	18.7946	18.806	18.1727	18.174
(0, 100)	0.5	5	11.1237	11.116	11.8489	10.846	10.9925	10.994	11.0818	11.081
		10	11.4503	11.448	11.0940	11.093	11.2538	11.254	11.3313	11.331
		20	11.5474	11.546	11.1660	11.166	11.3343	11.334	11.4093	11.409
	1	5	15.2095	15.190	14.3923	14.384	14.3071	14.310	14.3829	14.381
		10	15.9813	15.974	14.9443	14.941	14.8693	14.872	14.9193	14.919
		20	16.2285	16.226	15.1189	15.118	15.0607	15.062	15.1056	15.106
	2	5	28.6623	28.558	25.6912	25.638	24.3625	24.368	24.3109	24.294
		10	32.3444	32.300	28.2316	28.208	26.7223	26.738	26.5586	26.556
		20	33.8076	33.795	29.2272	29.220	27.7770	27.784	27.5919	27.592
(100, 0)	0.5	5	7.2276	7.2150	5.8746	5.8670	5.2360	5.2420	5.1288	5.1285
		10	7.6153	7.6120	6.1393	6.1370	5.5276	5.5298	5.4199	5.4200
		20	7.7272	7.7262	6.2152	6.2146	5.6156	5.6162	5.5087	5.5088
	1	5	10.7082	10.678	8.4748	8.4560	7.2560	7.2690	7.0373	7.0350
		10	11.6262	11.617	9.1107	9.1048	7.9520	7.9580	7.7356	7.7356
		20	11.9909	11.906	9.3044	9.3024	8.1789	8.1808	7.9658	7.9658
	2	5	23.0053	22.862	18.0231	17.927	14.6363	14.668	14.0098	13.977
		10	27.2246	27.172	21.0241	20.990	17.7396	17.769	17.1126	17.111
		20	28.8295	28.815	22.1378	22.128	19.0187	19.030	18.4115	18.412
(100, 100)	0.5	5	11.4036	11.396	11.1817	11.178	11.3598	11.360	11.4581	11.458
		10	11.7285	11.726	11.4284	11.427	11.6243	11.625	11.7103	11.710
		20	11.8253	11.825	11.5008	11.500	11.7054	11.706	11.7888	11.789
	1	5	15.4127	15.394	14.6407	14.632	14.5862	14.588	14.6702	14.668
		10	16.1808	16.174	15.1927	15.189	15.1498	15.152	15.2075	15.207
		20	16.4271	16.425	15.3674	15.366	15.3414	15.342	15.3938	15.394
	2	5	28.7674	28.664	25.8251	25.772	24.5206	24.526	24.4759	24.458
		10	32.4417	32.398	28.3613	28.338	26.8763	26.892	26.7186	26.716
		20	33.9029	33.890	29.3557	29.350	27.9292	27.935	27.7497	27.749

[a] $k_w = K_w a^4 / D_m$, $k_s = K_s a^2 / D_m$ where $D_m = E_m h^3 / 12(1-\nu^2)$

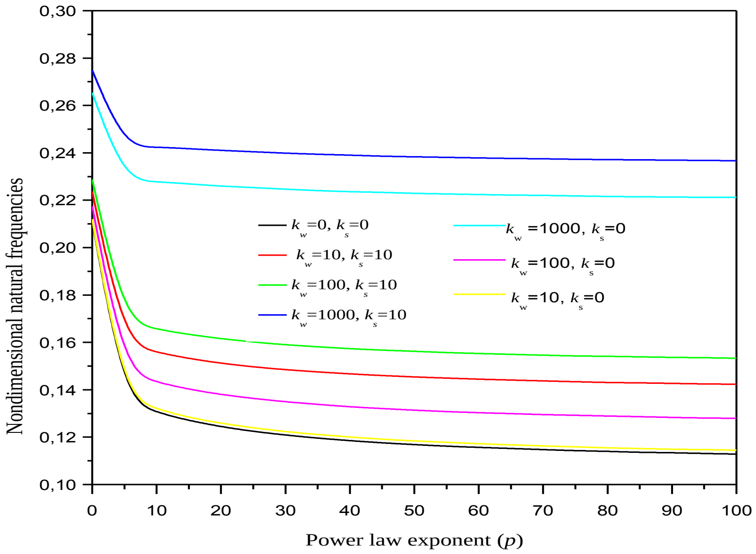
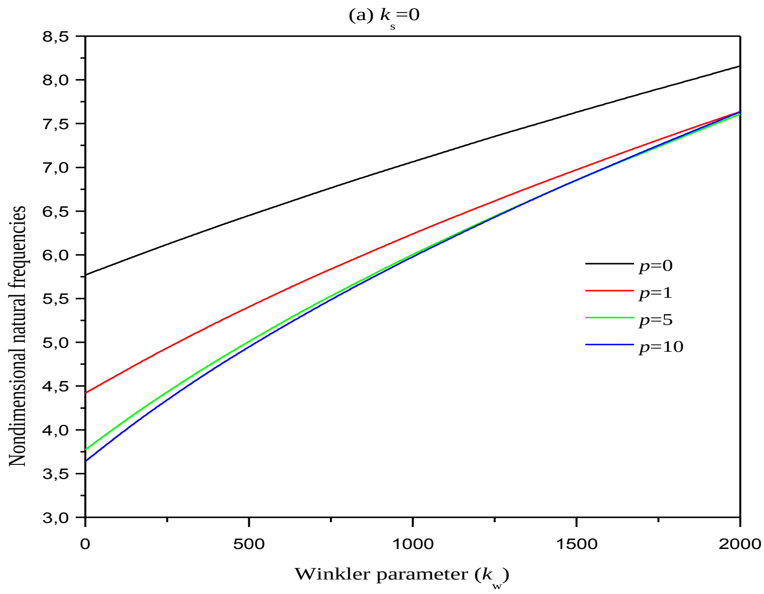


Fig. 3 Variation of non dimensional natural frequencies $\beta = \omega h \sqrt{\rho_c / E_c}$ of Al_2O_3 FG square plates resting on elastic foundation with power law exponent ($a / h = 5$)



Continued-

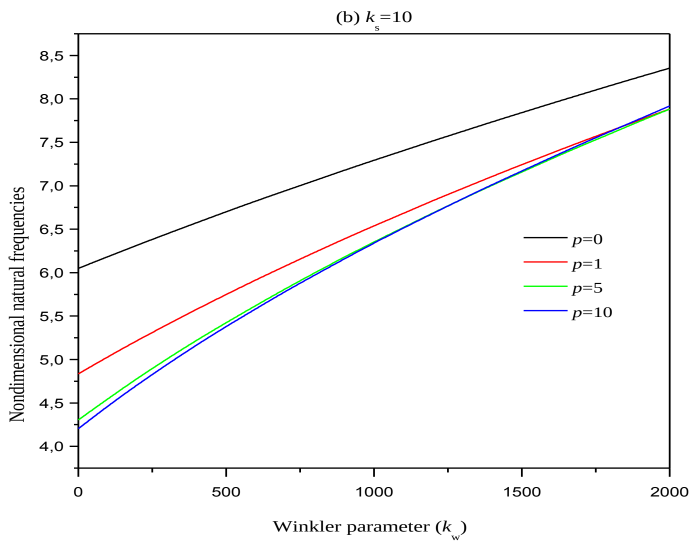


Fig. 4 Variation of non dimensional natural frequencies $\tilde{\beta} = \omega \frac{a^2}{h}\sqrt{\rho_c / E_c}$ of Al_2O_3 FG square plates resting on elastic foundation with Winkler coefficient ($a/h = 10$): (a) $k_s = 0$ and (b) $k_s = 10$

4. Conclusions

In the present work, a higher-order hyperbolic shear deformation theory is proposed for free vibration analysis of FG plates resting on elastic foundation. The model use only four unknowns, but accounts for shear deformation effect without employing any shear correction factor. Equations of motion are obtained from Hamilton's principle. Analytical solutions for free vibration problems are illustrated for a simply supported plate resting on elastic foundation. The following main points can be outlined from the current investigation:

- The proposed theory contains four unknowns, but gives results comparable with those predicted by existing shear deformation theories having more number of unknowns.
- The increase of power law exponent leads to reducing of the natural frequencies of plate.
- The increase of the values of Winkler and Pasternak parameters causes to increase in the natural frequency of FG plate.
- The Pasternak parameter of foundation has more important influence on increasing natural frequency of FG plate than the Winkler parameter.

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