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Graphical technique for the flutter analysis of flexible bridge

Tzen Chin Lee† and Cheer Germ Go‡

Department of Civil Engineering, National Chung-Hsing University, Taichung, Taiwan-40227, R.O.C.

Abstract. The flutter of a bridge is induced by self-excited force factors such as lift, drag and aerodynamic moment. These factors are associated with flutter derivatives in the analysis of wind engineering. The flutter derivatives are the function of structure configuration, wind velocity and response circular frequency. Therefore, the governing equations for the interaction between the wind and dynamic response of the structure are complicated and highly nonlinear. Herein, a numerical algorithm through graphical technique for the solution of wind at flutter is presented. It provides a concise approach to the solution of wind velocity at flutter.

Key words: flutter; aerodynamic instability; cable-suspended bridges.

1. Introduction

In 1940, the flutter failure of Tacoma Narrows bridge invoked the heed on the study of the aerodymanic response of the bridge. Physically, the flutter is a kind of self-excited oscillation. If the structure system is given a small disturbance, its motion will taps off the energy exceeds the dissipated energy by the system through the mechanical damping (Robert 1977, Simiu and Scanlan 1996). This phenomenon of aerodynamic instability is witnessed when a critical wind velocity is reached. In design practice, the analysis of the lowest wind velocity that initiates the instability have to be performed. The wind velocity at flutter should be higher than meteorological possible wind velocities at the bridge site. Researches have shown that the flutter is stimulated by self-excited force factors such as lift, drag and aerodynamic moment. These factors are associated with flutter derivatives in the analysis of wind engineering. The flutter derivatives are the function of structure configuration, wind velocity and response circular frequency. Therefore, the governing equations for the interaction between the wind and dynamic response of the structure are complicated and highly nonlinear. Herein, a numerical algorithm through graphical technique for the solution of wind at flutter is presented.

The researches on the solution of wind influences on the structure may be traced back to the year of 1971. The governing equation is set up as a function of aerodynamic variables and flutter derivatives (Scanlan and Tomko 1971, Beliveau, Vaicaitis and Shinozuka 1977).

† Ph.D. Candidate
‡ Professor

Afterward, *pK-F* method was developed to analysis the flutter phenomena of different problems of structure subjected to wind (Scanlan and Nicholas 1990, Namini, Albrecht and Bosch 1992, Miyata, Yamada and Kazama 1995). Although this method has its advantages, the calculating process is quite complicated. Herein, an alternative approach of graphical technique is proposed to find the wind velocity at flutter.

2. Aerodynamic forces

On the structural element, the aerodynamic forces (Namini, Albrecht and Bosch 1992) due to wind flow may be expressed in the following Eqs. of (1), (2), and (3) are shown in Fig. 1.

$$L_y = \frac{1}{2}\rho U^2(2B)\left(KH_1^*\frac{\dot{y}}{U} + KH_2^*\frac{B\dot{\alpha}}{U} + K^2H_3^*\alpha\right)\Delta L \tag{1}$$

$$D_z = \frac{1}{2}\rho U^2(2B)\left(KP_1^*\frac{\dot{z}}{U} + KP_2^*\frac{B\dot{\alpha}}{U} + K^2P_3^*\alpha\right)\Delta L \tag{2}$$

$$M_\alpha = \frac{1}{2}\rho U^2(2B^2)\left(KA_1^*\frac{\dot{y}}{U} + KA_2^*\frac{B\dot{\alpha}}{U} + K^2A_3^*\alpha\right)\Delta L \tag{3}$$

Where L_y represents the lift force,
D_z the drag force,
M_α the torsional moment,
ρ air mass density,
U wind velocity,
B the width of bridge deck,
K reduced frequency,
ΔL the length of the element,
α the torsional angle in x direction,
H_i^*、P_i^*、A_i^* ($i = 1, 2, 3$) nondimensional flutter derivatives.

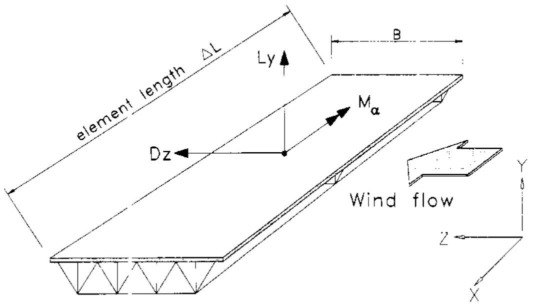

Fig. 1 Aerodynamic forces : Lift force L_y, Drag force D_z and aerodynamic moment M_α

At the use of the finite element method to analyze the problem, coordinates of nodes and nodal restrictions are shown in Fig. 2. Eqs. (1), (2) and (3) may be written in matrix form [A]. This aerodynamic load matrix [A] may be categorized into two parts, i.e., stiffness matrix [A_S] and damping matrix [A_D],

$$[A\,] = \frac{1}{2}\rho U^2 \left([A_S\,]\ \{u\} + \frac{1}{U}\,[A_D\,]\{\dot{u}\} \right) \tag{4}$$

Where

$\{u\}$ 、 $\{\dot{u}\}$ represent the element displacement and velocity matrix, respectively.
[A_S] 、 [A_D] represent aerodynamic stiffness and damping matrix, respectively.
[A_S] 、 [A_D] are shown in Eqs. (5) and (6) :

$$[A_S]= \begin{bmatrix} 0\ 0\ 0 & 0 & 0\ 0\ 0\ 0\ 0 & 0 & 0\ 0 \\ 0\ 0\ 0 & -BK^2H_3^* & 0\ 0\ 0\ 0\ 0 & 0 & 0\ 0 \\ 0\ 0\ 0 & BK^2P_3^* & 0\ 0\ 0\ 0\ 0 & 0 & 0\ 0 \\ 0\ 0\ 0 & B^2K^2A_3^* & 0\ 0\ 0\ 0\ 0 & 0 & 0\ 0 \\ 0\ 0\ 0 & 0 & 0\ 0\ 0\ 0\ 0 & 0 & 0\ 0 \\ 0\ 0\ 0 & 0 & 0\ 0\ 0\ 0\ 0 & 0 & 0\ 0 \\ 0\ 0\ 0 & 0 & 0\ 0\ 0\ 0\ 0 & 0 & 0\ 0 \\ 0\ 0\ 0 & 0 & 0\ 0\ 0\ 0\ 0 & -BK^2H_3^* & 0\ 0 \\ 0\ 0\ 0 & 0 & 0\ 0\ 0\ 0\ 0 & BK^2P_3^* & 0\ 0 \\ 0\ 0\ 0 & 0 & 0\ 0\ 0\ 0\ 0 & B^2K^2A_3^* & 0\ 0 \\ 0\ 0\ 0 & 0 & 0\ 0\ 0\ 0\ 0 & 0 & 0\ 0 \\ 0\ 0\ 0 & 0 & 0\ 0\ 0\ 0\ 0 & 0 & 0\ 0 \end{bmatrix} \Delta L \tag{5}$$

$$[A_D]= \begin{bmatrix} 0 & 0 & 0 & 0 & 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0 \\ 0 & BKH_1^* & 0 & -B^2KH_2^* & 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0 \\ 0 & 0 & BKP_1^* & B^2KP_2^* & 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0 \\ 0 & -B^2KA_1^* & 0 & B^3KA_2^* & 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0 \\ 0 & 0 & 0 & 0 & 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0 \\ 0 & 0 & 0 & 0 & 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0 \\ 0 & 0 & 0 & 0 & 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0 \\ 0 & 0 & 0 & 0 & 0\ 0\ 0 & BKH_1^* & 0 & -B^2KH_2^* & 0\ 0 \\ 0 & 0 & 0 & 0 & 0\ 0\ 0 & 0 & BKP_1^* & B^2KP_2^* & 0\ 0 \\ 0 & 0 & 0 & 0 & 0\ 0\ 0 & -B^2KA_1^* & 0 & B^3KA_2^* & 0\ 0 \\ 0 & 0 & 0 & 0 & 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0 \\ 0 & 0 & 0 & 0 & 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0 \end{bmatrix} \Delta L \tag{6}$$

Where the flutter derivatives (H_1^* 、 H_2^* 、 H_3^* 、 P_1^* 、 P_2^* 、 P_3^* 、 A_1^* 、 A_2^* 、 A_3^*) are related to the geometrical nature of bridge sections. H_1^* 、 H_2^* 、 H_3^* are crosswind variables which are related to lift force, P_1^* 、 P_2^* 、 P_3^* are alongwind variables which are related to drag force, A_1^*

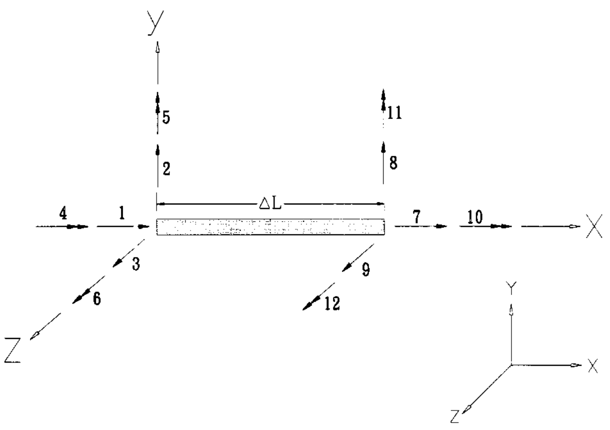

Fig. 2 Coordinates for nodal dispacements

、A_2^* 、A_3^* are torsional variables which are related to torsional moment.

Their definitions are stated as follows :

H_1^* : The crosswind velocity disturbance produced by the movement in the crosswind direction on a structure.
H_2^* : The torsional velocity disturbance produced by the movement in the crosswind direction on a structure.
H_3^* : The torsional displacement disturbance produced by the movement in the crosswind direction on a structure.
P_1^* : The alongwind velocity disturbance produced by the movement in the alongwind direction on a structure.
P_2^* : The torsional velocity disturbance produced by the movement in the alongwind direction on a structure.
P_3^* : The torsional displacement disturbance produced by the movement in the alongwind direction on a structure.
A_1^* : The crosswind velocity disturbance produced by the movement in the torsional direction on a structure.
A_2^* : The torsional velocity disturbance produced by the movement in the torsional direction on a structure.
A_3^* : The torsional displacement disturbance produced by the movement in the torional direction on a structure.

3. Graphical technique for flutter analysis

As the Eq. (4), the dynamical equation of a bridge excited by wind flow is expressed in the following(Scanlan 1978) :

$$[M]\{\ddot{u}\}+[C]\{\dot{u}\}+[K]\{u\}=\frac{1}{2}\rho U^2\left([A_S]\{u\}+\frac{1}{U}[A_D]\{\dot{u}\}\right) \tag{7}$$

where

$[M]$, $[C]$, $[K]$ represent the mass, damping and stiffness matrices of the element, respectively.

$[A_S]$ 、 $[A_D]$ represent the aerodymanic stiffness, and damping matrices, respectively.

$\{u\}$, $\{\dot{u}\}$, $\{\ddot{u}\}$ represent displacement, velocity and acceleration matrices of the element, respectively.

Make $\{u\}=[\phi]\{v\}$, and then,

$$[M][\phi]\{\ddot{v}\}+[C][\phi]\{\dot{v}\}+[K][\phi]\{v\}=\frac{1}{2}\rho U^2\left([A_S][\phi]\{v\}+\frac{1}{U}[A_D][\phi]\{\dot{v}\}\right) \tag{8}$$

Where

$[\phi]$ is natural mode shape matrix.

$\{v\}$ is generalized coordinates vector.

Pre-multiplying Eq. (8) by the matrix of $[\phi]^T$,

$$\begin{aligned}&[\phi]^T[M][\phi][\ddot{v}]+[\phi]^T[C][\phi]\{\dot{v}\}+[\phi]^T[K][\phi]\{v\}\\&=\frac{1}{2}\rho U^2\left([\phi]^T[A_S][\phi]\{v\}+\frac{1}{U}[\phi]^T[A_D][\phi]\{\dot{v}\}\right)\end{aligned} \tag{9}$$

Assuming

$[M^g]=[\phi]^T[M][\phi]$
$[C^g]=[\phi]^T[C][\phi]$
$[K^g]=[\phi]^T[K][\phi]$
$[A_S^g]=[\phi]^T[A_S][\phi]$
$[A_D^g]=[\phi]^T[A_D][\phi]$

Eq. (9) becomes

$$[M^g]\{\ddot{v}\}+[C^g]\{\dot{v}\}+[K^g]\{v\}=\frac{1}{2}\rho U^2\left([A_S^g]\{v\}+\frac{1}{U}[A_D^g]\{\dot{v}\}\right) \tag{10}$$

Where $[M^g]$ 、 $[C^g]$ 、 $[K^g]$ represent the generalized mass, generalized damping, generalized stiffness matrices of the structure, respectively.

And that, $[M^g]$ · $[C^g]$ · $[K^g]$ are diagonal matrices.

$[A_S^g]\cdot[A_D^g]$ are nondiagonal matrices.

Let $\{v\}=\{R\}e^{\lambda t}$, and

substituting into the Eq. (10)

$$\left([M^g]\lambda^2+[C^g]\lambda+[K^g]-\frac{1}{2}\rho U^2\left([A_S^g]+\frac{1}{U}[A_D^g]\lambda\right)\right)\{R\}e^{\lambda t}=0 \tag{11}$$

For the nontrivial solution $\{R\}$, the determinant of the matrix addition inside the parenthesis at left-hand side has to be zero :

$$\left| [M^g]\lambda^2+[C^g]\lambda+[K^g]-\frac{1}{2}\rho U^2\left([A_S^g]+\frac{1}{U}[A_D^g]\lambda\right)\right| =0 \tag{12}$$

For the vibration system discussed herein, the convergence is guaranteed when all the roots λ_v of Eq. (12) containing negative real parts, i.e., $Re(\lambda_v)<0$. Referring to Eq. (12), a characteristic polynominal $f(\lambda)$ is defined as

$$f(\lambda)=\left| [M^g]\lambda^2+[C^g]\lambda+[K^g]-\frac{1}{2}\rho U^2\left([A_S^g]+\frac{1}{U}[A_D^g]\lambda\right)\right| \tag{13}$$

Considering two complex plane, i.e., the plane of the variable $\lambda=x+iy$ and the plane of $f(\lambda)=u+iv$. All the roots λ_v with negative real part of Eq. (12) are clustered in the left-half of complex plane λ as shown in Fig. 3. These roots are also mapped into the origin in the $f(\lambda)$ plane (Dym 1967, Hahn 1967, Leipholz 1970). In the complex plane λ, it is noted that all the roots λ_v with negative real part stay to the left of iy axis. Then it follows that the origin $f(\lambda_v)=0$ is always stay to the left of curve $f(iy)$ from $y=-\infty$ .to $y=+\infty$ on the $f(\lambda)$ plane.
Where

$$f(iy)=\left| -[M^g]y^2+[C^g]iy+[K^g]-\frac{1}{2}\rho U^2\left([A_S^g]+\frac{1}{U}[A_D^g]iy\right)\right| \tag{14}$$

Therefore, given a starting value of wind velocity U, the curve of $f(iy)$ from $y=-\infty$ to $y=+\infty$ is plotted. At the condition of the origin being on the left-hand side of the curve $f(iy)$ on the $f(\lambda)$ plane, which means the $Re(\lambda_v)<0$, the system vibration is convergent. This means that the structure system under the effect of the wind velocity is stable. Then gradually increase the wind velocity U, repeart the above steps until the original point fallen on the right-hand side of curve $f(iy)$ on the $f(\lambda)$ plane, which means $Re(\lambda_v)>0$, the system is divergent, and the structure under

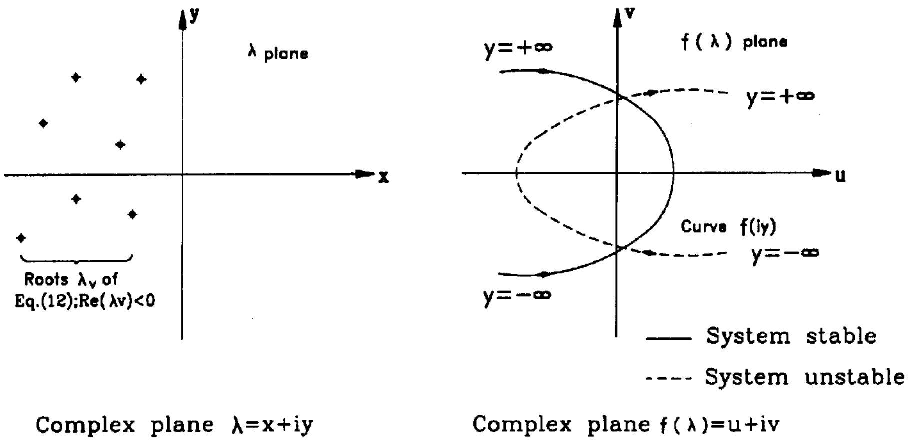

Fig. 3 Graphical technique for flutter analysis

the effect of the wind velocity is unstable. The critical wind is determined as the system is turning from being stable into unstable.

4. Numerical examples

As an example, a symmetrical single span structure (Bucher and Lin 1988) is analyzed. Its girder has 490 m span, 32.8 m wide bridge deck, and is 10 m high. The structure is simulated as a beam element model with section properties as follows : the unit length mass as 1.36×10^4 kg/m, polar moment of inertia as 8.94×10^5 kgm^2/m, bending moment of inertia $I = 40.7$ m^4, torsion moment of inertia $J = 1.482$ m^4. The aerodynamic variables (H_1^*、H_2^*、H_3^*、P_1^*、P_2^*、P_3^*、A_1^*、A_2^*、A_3^*) of the subject bridge section are determined from the reference (Scanlan and Tomko 1971, Bucher and Lin 1988). By the use of critical damping 1% and air density 1.25 kg/m^3, the wind velocity at flutter is found to be 56.1 m/sec from frequency domain analysis and 55.0 m/sec from time domain analysis (Bucher and Lin 1988). By the application of propose technique, the origin is jumping over to the right-hand side of the trend of curve $f(iy)$ when wind velocity is approaching to 55.9 m/sec as shown in Fig. 4. This result is in a very good agreement with that of the reference (Bucher and Lin 1988).

The second example is the analysis of the flutter failure of Tacoma Narrows Bridge. The span of the bridge is 853 m with a cross sectional area of 0.654 m^2. The deck width and deck depth are 11.9 m and 2.541 m, respectively. The bending moment of inertia of the section is 0.122 m^4 and the torsional moment of inertia is 1.146×10^{-7}m^4. The Properties are 2.1×10^{10} kgf/m^2 for elastic modulus, 0.8×10^{10} kgf/m^2 for shear modulus, and 5.05×10^3 kg/m for mass per unit length. With the damping ratio of 1% and air density of 1.29 kg/m^3, the aerodynamic variables (H_1^*、H_2^*、H_3^*、P_1^*、P_2^*、P_3^*、A_1^*、A_2^*、A_3^*) of the subject bridge section are determined from the reference (Scanlan and Tomko 1971, Scanlan, Beliveau, and Budlong 1974). The wind velocity for flutter phenomenon is 17.7 m/sec as shown in Fig. 5. This result is in a good agreement with that of references (Vlasov 1961, Hirai 1942, Hirai, Okauchi and

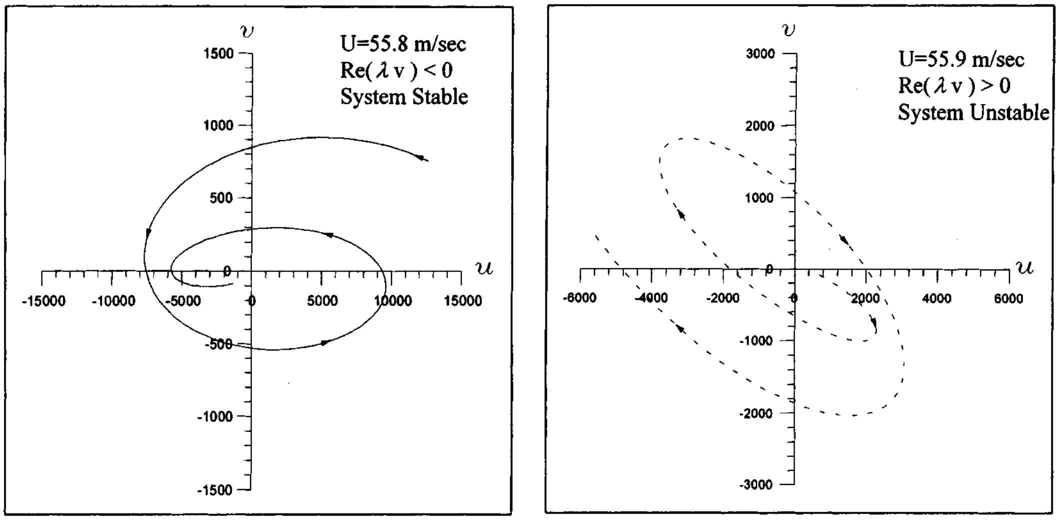

Fig. 4 Example for flutter analysis using graphical technique

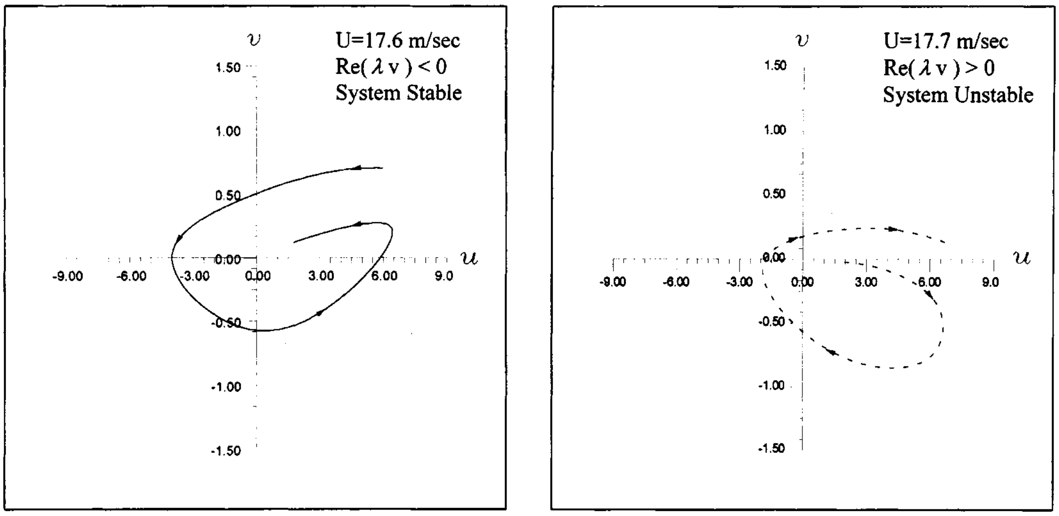

Fig. 5 Analysis of the flutter failure of Tacoma Narrows Bridge

Table 1 The critical wind velocity of Tacoma Narrows Bridge

Reference	Critical wind velocity U_{cr} (m/sec)
Vlasov (1961)	16.9
Hirai, Okauchi and Miyata (1968)	16.2
Hirai (1942)	20.5
Proposed graphical technique	17.7
The measurement data (Vlasov 1961)	18.6

Miyata 1968) and listed in the Table 1.

5. Conclusions

For the self-excited vibration problem such as a flutter phenomenon, the governing equation is complicated and highly nonlinear. To solve this problem, the application of numerical method is necessary. By the use of numerical approach, the possibility of skipping over the desired solution and convergence problem has to be noted and taken into consideration. Proposed approach by using graphical technique to solve the problem may greatly reduce these difficulties. This graphical technique can be also consicely and easily programmed.

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