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A semi-active stochastic optimal control strategy for nonlinear structural systems with MR dampers

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Abstract. A non-clipped semi-active stochastic optimal control strategy for nonlinear structural systems with MR dampers is developed based on the stochastic averaging method and stochastic dynamical programming principle. A nonlinear stochastic control structure is first modeled as a semi-actively controlled, stochastically excited and dissipated Hamiltonian system. The control force of an MR damper is separated into passive and semi-active parts. The passive control force components, coupled in structural mode space, are incorporated in the drift coefficients by directly using the stochastic averaging method. Then the stochastic dynamical programming principle is applied to establish a dynamical programming equation, from which the semi-active optimal control law is determined and implementable by MR dampers without clipping in terms of the Bingham model. Under the condition on the control performance function given in section 3, the expressions of nonlinear and linear non-clipped semi-active optimal control force components are obtained as well as the non-clipped semi-active LQG control force, and thus the value function and semi-active nonlinear optimal control force are actually existent according to the developed strategy. An example of the controlled stochastic hysteretic column is given to illustrate the application and effectiveness of the developed semi-active optimal control strategy.

Keywords: nonlinear stochastic optimal control; semi-active optimal control law; MR damper; stochastic averaging; stochastic dynamical programming.

1. Introduction

The vibration control of structural systems under strong random excitations such as severe wind or earthquake ground motion is a significant research subject (Housner, *et al.* 1997). The linear quadratic Gaussian (LQG) control strategy was used frequently for structural response mitigation. In recent years, the optimal control of multi-degree-of-freedom stochastic systems has been studied and a hybrid solution to the dynamical programming equation was presented (Dimentberg, *et al*. 2000, 2002, Bratus, *et al*. 2000). A nonlinear stochastic optimal control strategy based on the stochastic averaging method and stochastic dynamical programming principle has been proposed and applied to structural systems, which can achieve better control efficacy (Zhu, *et al.* 2001, Ying and Zhu 2003). As a semi-active smart

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control device, the magneto-rheological (MR) damper was designed with attractive features such as simplicity, reliability and small power source requirement, and was studied on the dynamic behavior and potential application to structural systems (Symans and Constantinou 1999, Maiti, *et al*. 2006, Samali, *et al*. 2006, Jung, *et al*. 2007). The clipped semi-active LQG control strategy was presented for linear stochastic systems with MR dampers, in which the clipping treatment is necessary (Jansen and Dyke 2000, Batterbee and Sims 2005). However, a semi-active nonlinear stochastic optimal control strategy for MR dampers without clipping has been proposed through the improvement of the active nonlinear stochastic optimal control strategy (Ying, *et al.* 2003, 2004a). It was concluded that a semi-active MR damper can implement the active optimal control law. This semi-active stochastic optimal control strategy has been applied to a wind-excited linear building structure with MR dampers (Zhu, *et al.* 2004). In that study, only the linear non-clipped semi-active optimal control force components of MR dampers were obtained. The passive linear control force components of MR dampers were incorporated in the uncontrolled structural system, and then the combined system was decoupled by using the uncontrolled structural modes. In general, the linear damping coefficient matrix due to the passive control force components of MR dampers cannot be diagonalized by the uncontrolled structural mode matrix.

The present paper focuses mainly on the development of a semi-active stochastic optimal control strategy for nonlinear structural systems with MR dampers. A nonlinear multi-degree-of-freedom structure installed with MR dampers is modeled as a semi-actively controlled, stochastically excited and dissipated Hamiltonian system. Under the analysis of a linear structural system, it is demonstrated that the passive control force components of MR dampers cannot be decoupled by using the original structural modes. Then the stochastic averaging method is directly applied to the quasi-Hamiltonian system to yield averaged Itô stochastic differential equations, so that the passive control force components are incorporated in the drift coefficients. Applying the stochastic dynamical programming principle to the averaged control system yields a dynamical programming equation, from which the semi-active optimal control law is determined. And the non-clipped nonlinear and linear semi-active optimal control forces are obtained for the structural system with MR dampers under the condition given in section 3. Finally, the proposed semi-active optimal control strategy is applied to a stochastically excited hysteretic column with an MR damper to illustrate the control efficacy.

2. Structural system and averaging equations

Consider a nonlinear n-degree-of-freedom (DOF) structure with MR dampers under stochastic excitations (Zhu, *et al.* 2004, Ying, *et al.* 2004b). The differential equation of motion of the structure is of the form

$$\boldsymbol{M}\ddot{\boldsymbol{X}} + \boldsymbol{C}(\boldsymbol{X}, \dot{\boldsymbol{X}})\dot{\boldsymbol{X}} + \frac{\partial V_{\mathrm{s}}(\boldsymbol{X})}{\partial \boldsymbol{X}} = \boldsymbol{F}\boldsymbol{W}(t) + \boldsymbol{B}\boldsymbol{U} \tag{1}$$

where $\boldsymbol{X}$ is the n-dimensional displacement vector; $\boldsymbol{M}$ and $\boldsymbol{C}(\boldsymbol{X}, \dot{\boldsymbol{X}})$ are the $n\times n$-dimensional symmetric positive-definite mass matrix and quasi-linear damping coefficient matrix, respectively; $V_{\mathrm{s}}(\boldsymbol{X})$ is the non-negative structural potential energy; $\boldsymbol{F}$ is the $n\times n_e$-dimensional excitation amplitude matrix; $\boldsymbol{W}(t)$ is the n_e-dimensional stochastic excitation vector, e.g. due to ground motion or wind loading; $\boldsymbol{B}$ is the $n\times m$-dimensional damper-placement coefficient matrix and $\boldsymbol{U}$ is the m-dimensional control force vector produced by MR dampers. The control force $\boldsymbol{U}$ can be split into a passive part $\boldsymbol{U}_{\mathrm{p}}$ independent of applied voltage and a semi-active part $\boldsymbol{U}_{\mathrm{s}}$ dependent on applied voltage (Ying, *et al.* 2003), i.e.,

$\boldsymbol{U}=\boldsymbol{U}_{\mathrm{p}}+\boldsymbol{U}_{\mathrm{s}}$. It is assumed that all MR dampers are installed in the structure, e.g., the r-th damper between the i-DOF and j-DOF. According to the Bingham model, the passive and semi-active control force components are expressed as

$$\begin{aligned} U_{\mathrm{p}r} &= -C_{\mathrm{d}r}(\dot{X}_j-\dot{X}_i) = -C_{\mathrm{d}r}(B_{ir}\dot{X}_i+B_{jr}\dot{X}_j) \\ U_{sr} &= -F_{\mathrm{d}r}\mathrm{sgn}(\dot{X}_j-\dot{X}_i) = -F_{\mathrm{d}r}\mathrm{sgn}(B_{ir}\dot{X}_i+B_{jr}\dot{X}_j) \end{aligned} \tag{2}$$

where $C_{\mathrm{d}r}$ and $F_{\mathrm{d}r}$ are respectively the viscous damping coefficient and controllable force amplitude of the damper; $B_{ir}=-1$ and $B_{jr}=1$ are elements of matrix $\boldsymbol{B}$. Then the sum of elements in the corresponding each column of $\boldsymbol{B}$ is equal to zero. By denoting diagonal matrices $\boldsymbol{C}_{\mathrm{d}}=\mathrm{diag}(C_{\mathrm{d}1}, C_{\mathrm{d}2},..., C_{\mathrm{d}m})$ and $\boldsymbol{F}_{\mathrm{d}}=\mathrm{diag}(F_{\mathrm{d}1}, F_{\mathrm{d}2},..., F_{\mathrm{d}m})$, assembling $U_{\mathrm{p}r}$ and $U_{\mathrm{s}r}$ into vectors yields

$$\boldsymbol{U}_{\mathrm{p}} = -\boldsymbol{C}_{\mathrm{d}}\boldsymbol{B}^T\dot{\boldsymbol{X}}, \quad \boldsymbol{U}_{\mathrm{s}} = -\boldsymbol{F}_{\mathrm{d}}\mathrm{sgn}(\boldsymbol{B}^T\dot{\boldsymbol{X}}) \tag{3}$$

The nonlinear stochastic control structure (1) can be modeled as a semi-actively controlled, stochastically excited and dissipated Hamiltonian system by rewriting Eq. (1) into

$$\dot{\boldsymbol{Q}} = \frac{\partial H}{\partial \boldsymbol{P}}, \quad \dot{\boldsymbol{P}} = -\frac{\partial H}{\partial \boldsymbol{Q}} - \boldsymbol{C}\frac{\partial H}{\partial \boldsymbol{P}} + \boldsymbol{F}\boldsymbol{W}(t) + \boldsymbol{B}\boldsymbol{U} \tag{4}$$

where $\boldsymbol{Q}=\boldsymbol{X}$; $\boldsymbol{P}=\boldsymbol{M}\dot{\boldsymbol{X}}$ is the momentum vector; and $H=\boldsymbol{P}^T\boldsymbol{M}^{-1}\boldsymbol{P}/2+V_{\mathrm{s}}(\boldsymbol{X})$ is the total structural energy. As many engineering structures uncouplable in mode space, the Hamiltonian system associated with (4) is assumed to be integrable. However, the passive control force component $\boldsymbol{U}_{\mathrm{p}}$ cannot be decoupled generally. For instance, $V_{\mathrm{s}}(\boldsymbol{X})=\boldsymbol{X}^T\boldsymbol{K}\boldsymbol{X}/2$, where the stiffness matrix

$$\boldsymbol{K} = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -k_n & k_n \end{bmatrix} \tag{5}$$

The structural mode matrix $\boldsymbol{\Phi}$ can be determined by stiffness $\boldsymbol{K}$ and mass $\boldsymbol{M}$. If $n-1$ MR dampers are installed respectively between adjacent DOFs and an MR damper is between the first DOF and its support, the placement matrix is

$$\boldsymbol{B} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{6}$$

The overall passive control force vector of MR dampers is expressed as $\boldsymbol{B}\boldsymbol{U}_{\mathrm{p}}=-\boldsymbol{B}\boldsymbol{C}_{\mathrm{d}}\boldsymbol{B}^T\dot{\boldsymbol{X}}$ in terms of Eqs. (1) and (3), where the damping coefficient matrix

$$\boldsymbol{B}\boldsymbol{C}_{\mathrm{d}}\boldsymbol{B}^T = \begin{bmatrix} C_{\mathrm{d}1}+C_{\mathrm{d}2} & -C_{\mathrm{d}2} & 0 & \cdots & 0 & 0 \\ -C_{\mathrm{d}2} & C_{\mathrm{d}2}+C_{\mathrm{d}3} & -C_{\mathrm{d}3} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -C_{\mathrm{d}n} & C_{\mathrm{d}n} \end{bmatrix} \tag{7}$$

It seems that matrix (7) is similar to matrix (5) in form. In fact, the number of MR dampers installed is usually less than that of structural DOFs, e.g., without MR damper placed between the first and second DOFs. Then C_{d2}=0 but $k_2\neq 0$. The damping coefficient matrix $\boldsymbol{BC}_{\mathrm{d}}\boldsymbol{B}^T$ due to MR dampers cannot be diagonalized generally by structural mode $\boldsymbol{\Phi}$. Therefore, the modal transform technique is replaced by the stochastic averaging method used directly.

In general, the Hamiltonian system corresponding to structural system (1) is integrable completely. Under this assumption, applying the stochastic averaging method for quasi-integrable Hamiltonian systems (Zhu, *et al.* 1997) to system (4) yields the Itô stochastic differential equation

$$d\boldsymbol{H} = \left[\boldsymbol{m}(\boldsymbol{H}) + \left\langle\left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{P}}\right)^T \boldsymbol{B}\boldsymbol{U}_s\right\rangle\right]\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{H})d\boldsymbol{B}_s(t) \tag{8}$$

where $\boldsymbol{H}=[H_1, H_2, ..., H_n]^T$ and H_i is the first integral generally representing structural mode energy; $\boldsymbol{B}_{\mathrm{s}}(t)$ is the unit Wiener process vector; <·> represents the averaging operation; $\boldsymbol{m}(\boldsymbol{H})$ is the drift coefficient vector and $\boldsymbol{\sigma}(\boldsymbol{H})$ is the diffusion coefficient matrix. For Gaussian white noise excitation $\boldsymbol{W}(t)$ with intensity $2\boldsymbol{D}$, the drift and diffusion coefficients are

$$\boldsymbol{m}(\boldsymbol{H}) = \frac{1}{T(\boldsymbol{H})}\oint\left\{\left[-\left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{P}}\right)^T \boldsymbol{C}\boldsymbol{M}^{-1}\boldsymbol{P} + \left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{P}}\right)^T \boldsymbol{B}\boldsymbol{U}_{\mathrm{p}} + \boldsymbol{D}_\partial \boldsymbol{H}\right]\Big/\Pi_{\mathrm{d}}\left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{p}}\right)\right\}\Pi(\mathrm{d}\boldsymbol{q}) \tag{9}$$

$$\boldsymbol{\sigma}(\boldsymbol{H})\boldsymbol{\sigma}^T(\boldsymbol{H}) = \frac{1}{T(\boldsymbol{H})}\oint(2D_\partial \boldsymbol{H})\Pi(\mathrm{d}\boldsymbol{q}) \tag{10}$$

$$T(\boldsymbol{H}) = \oint\left[1\Big/\Pi_{\mathrm{d}}\left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{p}}\right)\right]\Pi(\mathrm{d}\boldsymbol{q}) \tag{11}$$

where $\boldsymbol{D}_\partial = (\partial/\partial \boldsymbol{P})^T \boldsymbol{FDF}^T(\partial/\partial \boldsymbol{P})$ is the operator matrix; $\Pi_{\mathrm{d}}(\partial \boldsymbol{H}/\partial \boldsymbol{p})=(\partial H_1/\partial p_1)(\partial H_2/\partial p_2)...(\partial H_n/\partial p_n)$ and $\Pi(\mathrm{d}\boldsymbol{q})=\mathrm{d}q_1\mathrm{d}q_2...\mathrm{d}q_n$. Here the passive control force component $\boldsymbol{U}_{\mathrm{p}}$ of MR dampers is separated form **U** and incorporated in the drift coefficient $\boldsymbol{m}(\boldsymbol{H})$. Its element is $-[\boldsymbol{\Phi}^T\boldsymbol{BC}_{\mathrm{d}}\boldsymbol{B}^T\boldsymbol{\Phi}]_{ii}H_i$ for the linear structure with mode $\boldsymbol{\Phi}$. From system (4) to (8), the state ($\boldsymbol{Q}$, $\boldsymbol{P}$) control problem is converted into the energy ($\boldsymbol{H}$) control problem and the dimension $2n$ is reduced to n.

3. Semi-active stochastic optimal control law

The stochastic optimal control of system (8) is to minimize a performance index by designing the semi-active control force component $\boldsymbol{U}_{\mathrm{s}}$ of MR dampers. The performance index of infinite time-interval ergodic control is

$$J = \lim_{t_f\to\infty}\frac{1}{t_f}\int_0^{t_f}\langle L(\boldsymbol{H}(t),\ \ \boldsymbol{U}_{\mathrm{s}}(t))\rangle\,\mathrm{d}t \tag{12}$$

where $L(\boldsymbol{H}, \boldsymbol{U}_{\mathrm{s}})$ is a continuous differentiable convex function. Applying the stochastic dynamical programming principle (Kushner 1978) to control problem (8) and (12) yields the dynamical programming equation

$$\min_{\boldsymbol{U}_{\mathrm{s}}}\left\{L(\boldsymbol{H}, \boldsymbol{U}_{\mathrm{s}}) + \left[\boldsymbol{m}(\boldsymbol{H}) + \left\langle\left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{P}}\right)^T \boldsymbol{B}\boldsymbol{U}_{\mathrm{s}}\right\rangle\right]^T\frac{\partial V}{\partial \boldsymbol{H}} + \frac{1}{2}\mathrm{tr}\left[\frac{\partial^2 V}{\partial \boldsymbol{H}^2}\boldsymbol{\sigma}(\boldsymbol{H})\boldsymbol{\sigma}^T(\boldsymbol{H})\right]\right\} = \lambda \tag{13}$$

where V is the value function; λ is a constant and tr[$\cdot$] represents the trace operation of a square matrix. The optimal control law $\boldsymbol{U}_s^*$ can be determined by minimizing the left side of Eq. (13). Let $L=g(\boldsymbol{H})+\boldsymbol{U}_s^T\boldsymbol{R}\boldsymbol{U}_s$ with $g(\boldsymbol{H})\geq 0$ and $\boldsymbol{R}$ symmetric positive-definite. Then

$$\boldsymbol{U}_s^* = -\frac{1}{2}\boldsymbol{R}^{-1}\boldsymbol{B}^T\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{P}}\frac{\partial V}{\partial \boldsymbol{H}} \tag{14}$$

The value function V can be obtained by solving the following equation with $\boldsymbol{m}^u(\boldsymbol{H})=\langle(\partial\boldsymbol{H}/\partial\boldsymbol{P})^T\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T(\partial\boldsymbol{H}/\partial\boldsymbol{P})\rangle$:

$$\frac{1}{2}\mathrm{tr}\left[\frac{\partial^2 V}{\partial \boldsymbol{H}^2}\boldsymbol{\sigma}(\boldsymbol{H})\boldsymbol{\sigma}^T(\boldsymbol{H})\right]+\boldsymbol{m}^T(\boldsymbol{H})\frac{\partial V}{\partial \boldsymbol{H}}-\frac{1}{4}\left(\frac{\partial V}{\partial \boldsymbol{H}}\right)^T\boldsymbol{m}^u(\boldsymbol{H})\frac{\partial V}{\partial \boldsymbol{H}}+g(\boldsymbol{H})-\lambda = 0 \tag{15}$$

Note that the Hamiltonian H is quadratic in system momentum $\boldsymbol{P}$. It is assumed that there exists a transform $\boldsymbol{P}=\boldsymbol{T}_s\boldsymbol{p}$ and the first integral $H_i=(\boldsymbol{p}^T\boldsymbol{A}\boldsymbol{p}/2)_{ii}+v_i(\boldsymbol{q})$ with $\boldsymbol{A}=\boldsymbol{T}_s^T\boldsymbol{M}^{-1}\boldsymbol{T}_s$. For the structure with $V_s(\boldsymbol{X})=\boldsymbol{X}^T\boldsymbol{K}\boldsymbol{X}/2$ and mode $\boldsymbol{\Phi}$, the transform matrix $\boldsymbol{T}_s=\boldsymbol{M}\boldsymbol{\Phi}$ and the first integral $H_i=(A_{ii}\dot{q}_i+G_{ii}q_i)/2$, where q_i is the modal displacement, A_{ii} and G_{ii} are constants. By taking $\boldsymbol{R}$ as diagonal positive-definite and $g(\boldsymbol{H})$ such that $\partial V/\partial H_1=\partial V/\partial H_2=\cdots=\partial V/\partial H_n\geq 0$, the optimal control law (14) becomes.

$$\boldsymbol{U}_s^* = -\frac{1}{2}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{T}_s^{-1}\boldsymbol{A}\boldsymbol{p}\frac{\partial V}{\partial H_1} = -\frac{1}{2}\boldsymbol{R}^{-1}\boldsymbol{B}^T\dot{\boldsymbol{X}}\frac{\partial V}{\partial H_1} = -\boldsymbol{F}_d\mathrm{sgn}(\boldsymbol{B}^T\dot{\boldsymbol{X}}) \tag{16}$$

where the r-th element of diagonal matrix $\boldsymbol{F}_d$ is

$$F_{dr} = \frac{1}{2R_{rr}}\frac{\partial V}{\partial H_1}\left|(\boldsymbol{B}^T\dot{\boldsymbol{X}})\right|_r \geq 0, \qquad r=1,2,\ldots,m \tag{17}$$

According to the Bingham model, the optimal control law (16) meets entirely the dynamics of MR dampers (3) and then is always implementable by MR dampers without clipping. Eq. (16) represents a non-clipped semi-active stochastic optimal control force, which can be determined finally by solving Eq. (15) and substituting $\partial V/\partial H_1$ into (16).

4. Non-clipped optimal control forces

To explore non-clipped linear and nonlinear semi-active optimal control forces (16), consider the structural system with $V_s(\boldsymbol{X})=\boldsymbol{X}^T\boldsymbol{K}\boldsymbol{X}/2$, damping coefficient $\boldsymbol{C}$ and mode $\boldsymbol{\Phi}$. Eq. (15) becomes

$$\sum_{i=1}^{n}\left[\frac{1}{2}\sigma_i^2(H_i)\frac{\partial^2 V}{\partial H_i^2}+m_i(H_i)\frac{\partial V}{\partial H_i}-\frac{1}{4}m_{ii}^u\left(\frac{\partial V}{\partial H_i}\right)^2\right]+g(\boldsymbol{H})-\lambda = 0 \tag{18}$$

For diagonal positive-definite $\boldsymbol{R}$ and $g(\boldsymbol{H})=s_0+\boldsymbol{s}_1^T\boldsymbol{H}$ (s_0 is a constant and $\boldsymbol{s}_1$ is a constant vector), the following polynomial solution of the value function to Eq. (18) is obtained:

$$V = \alpha_0+\alpha_1\sum_{i=1}^{n}H_i \tag{19}$$

$$\alpha_1 = \frac{2}{\Delta_i}\left(\sqrt{4\zeta_j^2\omega_i^2 + \Delta_i s_{1i}} - 2\zeta_i\omega_i\right) \tag{20}$$

where α_0 is a constant; $\Delta_i = [\boldsymbol{\Phi}^T\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{\Phi}]_{ii}$; ω_i and ζ_i are the i-th modal frequency and damping ratio, respectively. Obviously, the value function (19) satisfies the condition $\partial V/\partial H_1 = \partial V/\partial H_2 = \cdots = \partial V/\partial H_n = \alpha_1$. Substituting Eq. (19) into (16) yields the semi-active optimal control force

$$U_{sr}^* = -\frac{\alpha_1}{2R_{rr}}\sum_{i=1}^{n} B_{ir}\dot{X}_i, \qquad r = 1, 2, \ldots, m \tag{21}$$

which is a linear damping force and implementable by an MR damper without clipping in terms of the Bingham model since $s_{1i} \geq 0$ and $\Delta_i \geq 0$ or $\alpha_1 \geq 0$.

In fact, a linear non-clipped semi-active optimal control force can be obtained by using the LQG control strategy too. For system (4) and performance index (12) with $\langle L \rangle = \boldsymbol{Z}^T\boldsymbol{S}_L\boldsymbol{Z} + \boldsymbol{U}^T\boldsymbol{R}_L\boldsymbol{U}$ ($\boldsymbol{Z} = [\boldsymbol{Q}^T, \boldsymbol{P}^T]^T$ is the system state vector), the value function $V = \boldsymbol{Z}^T\boldsymbol{G}_L\boldsymbol{Z}$ and the control force

$$\boldsymbol{U}^L = -\boldsymbol{R}_L^{-1}\boldsymbol{B}_L^T\boldsymbol{G}_L\boldsymbol{Z} \tag{22}$$

where matrix $\boldsymbol{G}_L$ is obtained by solving the following Riccati equation:

$$\boldsymbol{A}_L^T\boldsymbol{G}_L + \boldsymbol{G}_L\boldsymbol{A}_L - \boldsymbol{G}_L\boldsymbol{B}_L\boldsymbol{R}_L^{-1}\boldsymbol{B}_L^T\boldsymbol{G}_L + \boldsymbol{S}_L = 0 \tag{23}$$

$$\boldsymbol{A}_L = \begin{bmatrix} 0 & \boldsymbol{M}^{-1} \\ -\boldsymbol{K} & -\boldsymbol{C}\boldsymbol{M}^{-1} \end{bmatrix}, \quad \boldsymbol{B}_L = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{B} \end{Bmatrix} \tag{24}$$

Dividing the matrices of Eq. (23) into $n \times n$-dimensional sub-matrices and taking the sub-matrices of $\boldsymbol{S}_L$ as $\boldsymbol{S}_{11}^L = \boldsymbol{S}_{12}^L = \boldsymbol{S}_{21}^L = 0$ yield

$$\boldsymbol{G}_{11}^L = \boldsymbol{K}\boldsymbol{G}_{22}^L\boldsymbol{M}, \; \boldsymbol{C}\boldsymbol{G}_{22}^L\boldsymbol{M} + \boldsymbol{M}\boldsymbol{G}_{22}^L\boldsymbol{C} + \boldsymbol{M}\boldsymbol{G}_{22}^L\boldsymbol{B}\boldsymbol{R}_L^{-1}\boldsymbol{B}^T\boldsymbol{G}_{22}^L\boldsymbol{M} - \boldsymbol{M}\boldsymbol{S}_{22}^L\boldsymbol{M} = 0 \tag{25}$$

where $\boldsymbol{S}_{22}^L$ is the sub-matrix of $\boldsymbol{S}_L$; $\boldsymbol{G}_{11}^L$ and $\boldsymbol{G}_{22}^L$ are the sub-matrices of $\boldsymbol{G}_L$. Let matrix $\boldsymbol{R}_L$ be diagonal positive-definite and $\boldsymbol{S}_{22}^L = \boldsymbol{M}^{-1}(2\beta\boldsymbol{C} + \beta^2\boldsymbol{B}\boldsymbol{R}_L^{-1}\boldsymbol{B}^T)\boldsymbol{M}^{-1}$ (β is a constant). A solution of $\boldsymbol{G}_{22}^L = \beta\mathbf{M}^{-1}$ to the second equation of (25) is obtained and then Eq. (22) becomes

$$\boldsymbol{U}^L = -\boldsymbol{R}_L^{-1}\boldsymbol{B}^T\boldsymbol{G}_{22}^L\boldsymbol{P} = -\beta\boldsymbol{R}_L^{-1}\boldsymbol{B}^T\dot{\boldsymbol{X}} \tag{26}$$

The semi-active LQG control force

$$U_{sr}^L = -\frac{\beta}{R_{rr}^L}\sum_{i=1}^{n} B_{ir}\dot{X}_i - U_{pr} = -\left(\frac{\beta}{R_{rr}^L} - C_{dr}\right)\left|\sum_{i=1}^{n} B_{ir}\dot{X}_i\right| \mathrm{sgn}\left(\sum_{i=1}^{n} B_{ir}\dot{X}_i\right), \quad r = 1, 2, \ldots, m \tag{27}$$

which is similar to Eq. (21) implementable by an MR damper without clipping in terms of the Bingham model. The non-clipped semi-active LQG control law (27), agreeing with the non-clipped semi-active optimal control law (21), demonstrates the existence of the value function solution and semi-active optimal control force (16) and (15) in this case.

The nonlinear optimal control is generally more effective than the linear optimal control (Zhu, *et al.* 2001, Ying, *et al.* 2003). Let further function $g(\boldsymbol{H})$ be of the form

$$g(\boldsymbol{H}) = s_0 + \boldsymbol{s}_1^T\boldsymbol{H} + (\boldsymbol{s}_2^T\boldsymbol{H})H + (\boldsymbol{s}_3^T\boldsymbol{H})H^2 \tag{28}$$

where s_0 is a constant; $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ and $\boldsymbol{s}_3$ are constant vectors; $H=H_1+H_2+\ldots+H_n$. The corresponding value function is obtained from Eq. (18) as

$$V(H) = \alpha_0 + \alpha_1 H + \alpha_2 H^2 \tag{29}$$

$$\alpha_1 = \frac{2}{\Delta_i}\left\{\sqrt{4\zeta_i^2\omega_i^2 + \Delta_i\left[s_{1i} + 2\alpha_2\left(\Gamma_i + \sum_{j=1}^{n}\Gamma_j\right)\right]} - 2\zeta_i\omega_i\right\},\quad \alpha_2 = \frac{s_{2i}}{4\zeta_i\omega_i + \Delta_i\alpha_1} = \sqrt{\frac{s_{3i}}{\Delta_i}} \tag{30}$$

where $\Gamma_i = [\boldsymbol{\Phi}^T\boldsymbol{FDF}^T\boldsymbol{\Phi}]_{ii}$. The value function (29) also satisfies the condition $\partial V/\partial H_1=\partial V/\partial H_2= \cdots =\partial V/\partial H_n=\alpha_1+2\alpha_2 H$. Substituting Eq. (29) into (16) yields the semi-active nonlinear optimal control force

$$U_{sr}^* = -\frac{\alpha_1 + 2\alpha_2 H}{2R_{rr}}\sum_{i=1}^{n} B_{ir}\dot{X}_i,\qquad r = 1, 2,..., m \tag{31}$$

which is also implementable by an MR damper without clipping in terms of the Bingham model since s_{1i}, s_{2i}, $s_{3i}\geq 0$ and Δ_i, $\Gamma_i\geq 0$ or α_1, $\alpha_2\geq 0$. The semi-active linear optimal control force (21) is just the linear part of the semi-active nonlinear optimal control force (31), which can be generalized further.

5. Example

Consider a nonlinear hysteretic column under stochastic support excitations and a semi-active control force. The differential equation of motion of the system is (Zhu, *et al.* 2000).

$$\ddot{X} + 2\zeta_0\dot{X} + (\alpha - k_1)X + (1-\alpha)Z = W_1(t) + k_2 X W_2(t) + u \tag{32}$$

where X is the dimensionless displacement; ζ_0 is the viscous damping ratio; α is the ratio of stiffness after yield to stiffness before yield; k_1 and k_2 are constants; $W_1(t)$ and $W_2(t)$ are the transverse and longitudinal support motion excitations, and for simplicity, are idealized as Gaussian white noises with intensities $2D_1$ and $2D_2$, respectively; u is the semi-active control force produced by an MR damper. According to the Bingham model, the semi-active control force can be split and expressed as $u = -2\zeta_1\dot{X} + u_s$. Z represents the hysteretic component of the restoring force, modeled as

$$\dot{Z} = A\dot{X} - \beta\dot{X}|Z|^n - \gamma|\dot{X}|Z|Z|^{n-1} \tag{33}$$

where A, β, γ and n are the hysteresis parameters.

Applying the stochastic averaging method to system (32) with (33) yields the averaged Itô equation

$$\mathrm{d}H = \left[m(H) + \langle u_s\frac{\partial H}{\partial P}\rangle\right]\mathrm{d}t + \sigma(H)\mathrm{d}B_e(t) \tag{34}$$

where the passive control force component $-2\zeta_1\dot{X}$ of the MR damper is incorporated in the drift coefficient; $B_e(t)$ is the unit Wiener process;

$$H = \dot{x}^2/2 + V_s(x),\quad \sigma^2(H) = \frac{2}{T(H)}\int_{-a}^{a}(2D_1 + 2k_2^2 D_2 x^2)\sqrt{2H - 2V_s(x)}\,\mathrm{d}x$$

$$m(H) = \frac{1}{T(H)}\left[-A_r - 4\zeta\int_{-a}^{a}\sqrt{2H-2V_s(x)}\,\mathrm{d}x + 2k_2^2 D_2\int_{-a}^{a}\frac{x^2\mathrm{d}x}{\sqrt{2H-2V_s(x)}}\right] + D_1$$

$$T(H) = 2\int_{-a}^{a}\frac{\mathrm{d}x}{\sqrt{2H-2V_s(x)}} \tag{35}$$

in which $\zeta = \zeta_0+\zeta_1$; $V_s(x)$ is the equivalent potential energy of the system; A_r is the area of hysteresis loop; and a is the amplitude of displacement and related to H by $H=V_s(\pm a)$. Their expressions are as given in the reference (Zhu, *et al.* 2000).

For the optimal control of system (34) with performance index (12), applying the stochastic dynamical programming principle yields the dynamical programming equation. Then the semi-active optimal control force can be obtained, which is for $L=g(H)+ru_s^2$ as

$$u_s^* = -\frac{1}{2r}\dot{X}\frac{\mathrm{d}V}{\mathrm{d}H} = -F_d\,\mathrm{sgn}(\dot{X}) \tag{36}$$

where r is a positive weighting parameter; $g(H)=s_0+s_1H+s_2H^2+s_3H^3$ and $F_d=(\mathrm{d}V/\mathrm{d}H)|\dot{X}|/(2r)$. $\mathrm{d}V/\mathrm{d}H$ can be obtained by solving the following equation:

$$\frac{1}{2}\sigma^2(H)\frac{\mathrm{d}^2V}{\mathrm{d}H^2} + m(H)\frac{\mathrm{d}V}{\mathrm{d}H} - \frac{1}{4r}G(H)\left(\frac{\mathrm{d}V}{\mathrm{d}H}\right)^2 + g(H) - \lambda = 0 \tag{37}$$

with

$$G(H) = \frac{2}{T(H)}\int_{-a}^{a}\sqrt{2H-2V_s(x)}\,\mathrm{d}x \tag{38}$$

To make the controlled system stable, parameters s_0, s_1, s_2, s_3 and r are selected so that $\mathrm{d}V/\mathrm{d}H\geq 0$. Then $F_d\geq 0$ and the semi-active optimal control force (36) is always implementable by the MR damper without clipping in terms of the Bingham model.

To evaluate the control efficacy, the fully averaged Itô equation is derived after substituting the semi-active optimal control force (36) into Eq. (34) as

$$\mathrm{d}H = \left[m(H) - \frac{1}{2r}G(H)\frac{\mathrm{d}V}{\mathrm{d}H}\right]\mathrm{d}t + \sigma(H)\mathrm{d}B_e(t) \tag{39}$$

The Fokker-Planck-Kolmogorov (FPK) equation associated with this Itô equation can be established. A stationary probability solution to the FPK equation is

$$p_s(H) = C_0\exp\left\{-\int_0^H\left[\left(-2m(y) + \frac{G(y)}{r}\frac{\mathrm{d}V}{\mathrm{d}y} + \frac{\mathrm{d}\sigma^2(y)}{\mathrm{d}y}\right)/\sigma^2(y)\right]\mathrm{d}y\right\} \tag{40}$$

where C_0 is the normalization constant. By using Eq. (40), the mean-square displacement of the controlled hysteretic column under stochastic excitations and the mean-square semi-active optimal control force are obtained respectively as

$$E[X_c^2] = \int_0^\infty \frac{p_s(H)}{T(H)} \mathrm{d}H \int_{-a}^{a} \frac{2x^2 dx}{\sqrt{2H - 2V_s(x)}} \tag{41}$$

$$E[u_s^{*2}] = \int_0^\infty \frac{G(H)}{4r^2} \left(\frac{\mathrm{d}V}{\mathrm{d}H}\right)^2 p_s(H) \mathrm{d}H \tag{42}$$

The mean-square displacement of the uncontrolled hysteretic column $E[X_u^2]$ can be calculated similarly. Then the proposed semi-active optimal control efficacy of the hysteretic column is evaluated by using two performance criteria

$$K = \frac{\sqrt{E[X_u^2]} - \sqrt{E[X_c^2]}}{\sqrt{E[X_u^2]}} \times 100\% \tag{43}$$

$$\mu = \frac{K}{\sqrt{E[u_s^{*2}]}/\sqrt{2D_1}} \tag{44}$$

The ratio K measures the percentage reduction of the root-mean-square displacements of the controlled and uncontrolled hysteretic columns, or the control effectiveness. The ratio μ measures the percentage displacement reduction per normalized root-mean-square control force, or the control efficiency. The higher the value of K and μ are, the better the semi-active optimal control law is.

Numerical results are obtained for the stochastically excited and controlled hysteretic column with structural parameters ζ=0.025, k_1=0.04, k_2=0.1, α=β=γ=0.5, A=n=1, and control parameters r=1, s_1=s_3=0, s_2=1, dV(0)/dH=3.5. Figs. 1 and 2 show the control effectiveness (K) and efficiency (μ) varying with external excitation intensity (D_1) and parametric excitation intensity (D_2), respectively. It is seen that the high control effectiveness (e.g. more than 70% percentage reduction of the root-mean-square displacement) and control efficiency (e.g. more than 55% percentage reduction per normalized

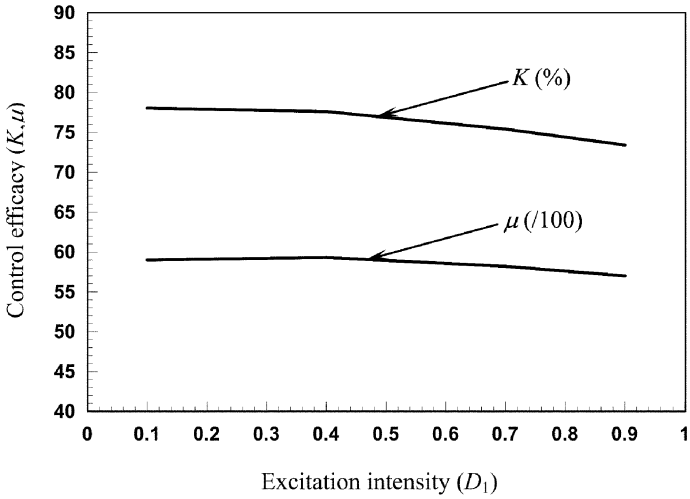

Fig. 1 Control effectiveness (K) and control efficiency (μ) versus external excitation intensity (D_1)

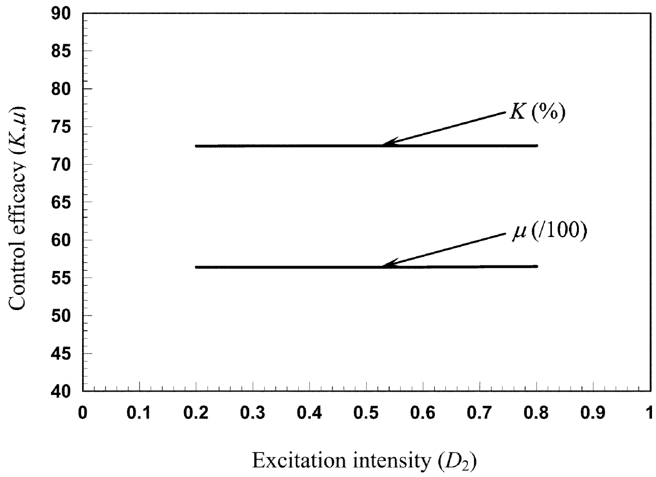

Fig. 2 Control effectiveness (K) and control efficiency (μ) versus parametric excitation intensity (D_2)

root-mean-square control force) of the hysteretic column under the proposed semi-active optimal control can be achieved.

6. Conclusions

In the present paper, a non-clipped semi-active stochastic optimal control strategy has been developed for nonlinear structural systems with MR dampers. The passive control force components of MR dampers are incorporated in the drift coefficients by directly using the stochastic averaging method. The semi-active optimal control law, determined based on the stochastic dynamical programming principle, is implementable by MR dampers without clipping in terms of the Bingham model. The nonlinear non-clipped semi-active optimal control force components of MR dampers as well as the linear non-clipped semi-active optimal control force and LQG control force components, obtained under a certain condition, demonstrate theoretically the existence of the value function solution and semi-active nonlinear optimal control forces according to the developed strategy. Also the presently developed semi-active control strategy has some advantages as the previously proposed nonlinear stochastic optimal control strategy.

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