Smart Structures and Systems, *Vol. 26, No. 3 (2020) 361-371*
DOI: https://doi.org/10.12989/sss.2020.26.3.361

Exact solution for nonlinear vibration of clamped-clamped functionally graded buckled beam

Abdellatif Selmi*[1,2]

[1] *Department of Civil Engineering, College of Engineering in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia*
[2] *Ecole Nationale d'Ingénieurs de Tunis (ENIT), Civil Engineering Laboratory. B.P. 37, Le belvédère 1002, Tunis, Tunisia*

(*Received January 3, 2020, Revised March 7, 2020, Accepted March 27, 2020*)

Abstract. Exact solution for nonlinear behavior of clamped-clamped functionally graded (FG) buckled beams is presented. The effective material properties are considered to vary along the thickness direction according to exponential-law form. The in-plane inertia and damping are neglected, and hence the governing equations are reduced to a single nonlinear fourth-order partial-integral-differential equation. The von Kármán geometric nonlinearity has been considered in the formulation. Galerkin procedure is used to obtain a second order nonlinear ordinary equation with quadratic and cubic nonlinear terms. Based on the mode of the corresponding linear problem, which readily satisfy the boundary conditions, the frequencies for the nonlinear problem are obtained using the Jacobi elliptic functions. The effects of various parameters such as the Young's modulus ratio, the beam slenderness ratio, the vibration amplitude and the magnitude of axial load on the nonlinear behavior are examined.

Keywords: buckled beam; exact solution; functionally graded material; nonlinear vibration

1. Introduction

Structures made of composite materials have been widely used to satisfy high performance demands. In such structures, stress discontinuities may occur at the interface between two different materials. In contrast, in FG structures the smooth and continuous variation of the properties from one surface to the other eliminates abrupt changes in the stress and displacement distributions (Udupa *et al*. 2014). Nowadays, structures made of FGMs have a great potential for practical applications in engineering and industrial fields (Beldjelili *et al*. 2016, Fourn *et al*. 2018, Karami *et al*. 2018, Zaoui *et al*. 2019).

The linear vibration of FG beams, plates and shells has been extensively investigated by using different methods (Hosseini *et al*. 2011, Damanpack *et al*. 2013a, Houari *et al*. 2016, Tounsi *et al*. 2016, Tufekci *et al*. 2016, Belabed *et al*. 2018, Bourada *et al*. 2019). Fundamental frequency analysis of FG beams using different higher-order beam theories was investigated by Şimşek (2010). Based on the first-order shear deformation theory, Hosseini *et al*. (2011) presented an exact closed-form solution for free vibration of moderately thick rectangular FG plates. Kamarian *et al*. (2016) studied the natural frequency of non-uniform nanocomposite beams with surface-bonded piezoelectric layers. The equation of motion was derived employing Hamilton's principle. The generalized differential quadrature technique was used to analyze the free vibration of the structures. Bouafia *et al*. (2017) investigated the free flexural vibration behaviors of FG nano-beams using a nonlocal quasi-3D theory. The nonlocal elastic behavior is described by the differential constitutive model of Eringen. The governing equations were derived using the principal of minimum total potential energy. Belabed *et al*. (2018) developed a hyperbolic plate theory for the free vibration analysis of FG sandwich plates. The equation of motion for the FG sandwich plates was obtained based on Hamilton's principle. The closed form solutions were derived by using the Navier technique. The fundamental frequencies are found by solving the eigenvalue problems. The vibration characteristics of rotating FG cylindrical shell resting on Winkler and Pasternak elastic foundations have been investigated by Hussain *et al*. (2018a). Shell dynamical equations were derived by using the Hamilton variational principle and the Langrangian functional. The wave propagation approach in standard eigenvalue form has been employed in order to derive the characteristic frequency equation describing the natural frequencies of vibration in rotating FG cylindrical shell. The wave propagation approach was used (Hussain *et al*. 2018b, Hussain and Naeem 2019a) to analyze the vibration of rotating FG cylindrical shells and zigzag and chiral rotating FG carbon nanotubes. The governing shell equations were obtained from Love's shell theory. Vibrations of rotating zigzag and chiral FG carbon nanotubes with ring supports have been performed by Hussain and Naeem (2019b). To discretize the governing equations of the developed model, Galerkin's method was utilized for frequency equations of single-walled carbon nanotubes. The unknown axial functions have been assumed by characteristic beam functions, which fulfill boundary conditions applied at the tube ends. Berghouti *et al*. (2019) studied the dynamic behavior of FG porous nano-beams using a nonlocal nth-order shear

*Corresponding author, Ph.D., Assistant Professor,
E-mail: selmi_fr2016@yahoo.com

http://www.techno-press.org/?journal=sss&subpage=7
ISSN: 1738-1584 (Print), 1738-1991 (Online)

deformation theory. The adopted theory takes into account the effect of shear deformation without considering shear correction factors. Mahmoudi *et al*. (2019) developed a refined quasi-three-dimensional shear deformation theory for thermo-mechanical analysis of FG sandwich plates resting on a two-parameter elastic foundation. The theory takes into account the stretching effect due to its quasi-three-dimensional nature. The boundary conditions in the top and bottom surfaces of the sandwich FG plate are satisfied and no correction factor was required. The governing equations and boundary conditions were derived using the principle of virtual displacements. Belbachir *et al*. (2019) described the response of anti-symmetric cross-ply laminated plates subjected to a uniformly distributed nonlinear thermo-mechanical loading. The principle of virtual work was used to obtain the governing equations and boundary conditions. Bourada *et al*. (2019) investigated the free vibration analysis of simply supported perfect and imperfect (porous) FG beams using a high order trigonometric deformation theory. This theory has a parabolic shear deformation distribution across the thickness. The Hamilton's principle was applied to determine the equations of motion. Balubaid *et al*. (2019) examined the free vibrational behavior of simply supported FG nano-plate using nonlocal refined plate theory. The equations of motion of the system were determined and resolved via Hamilton's principle and Navier procedure. Boutaleb *et al*. (2019) studied the dynamic analysis of the FG rectangular nano-plates. The theory of non-local elasticity based on the quasi 3D high shear deformation theory has been employed to determine the natural frequencies. The theory of nonlocal elasticity was utilized to examine the impact of the small scale on the natural frequency of the FG rectangular nano-plate. The equations of motion were deduced by implementing Hamilton's principle. Addou *et al*. (2019) investigated the effect of Winkler/Pasternak/Kerr foundation and porosity on the dynamic behavior of FG plates using a simple quasi-3D hyperbolic theory. The used theory was demonstrated to be simple and easy to apply because it considers only four-unknown variables to determine the four coupled vibration responses.

To obtain a more accurate and reliable structural analysis and design, the geometrical nonlinearity of FG structures becomes very important and should be taken into consideration. In these years, the research effort committed to making sense of the nonlinear behaviors of FG structures had attracted increasing attention.

Yaghoobi and Yaghoobi (2013) proposed an analytical investigation on the buckling analysis of symmetric sandwich plates with FG face sheets. The sandwich plates are considered resting on a nonlinear elastic foundation and subjected to mechanical, thermal, and thermo-mechanical loads. Many authors (Ke *et al*. 2010, Daouadji and Tounsi 2013, Sofiyev *et al*. 2016) studied the vibration of FG beams including the linear and nonlinear analyses. Euler-Bernoulli beam theory and Hamilton's principle was used while the rotary inertia of the cross section was neglected. Ke *et al*. (2010) investigated the nonlinear free vibrations of FG beams based on Euler-Bernoulli beam theory and von Kármán geometric nonlinearity. The governing equation was solved by direct numerical integration. The effects of material property distribution and different end supports on nonlinear dynamic behavior were discussed. Ansari *et al*. (2011) studied the vibration of a finite Euler-Bernoulli beam traversed by a moving load. The solution was obtained using the Galerkin method in conjunction with the Multiple Scales Method. Nonlinear active control of dynamic response of FG beams with rectangular cross-section subjected to mechanical and thermal loadings was presented by Bodaghi *et al*. (2012). The first-order shear deformation theory and the von Kármán geometrical nonlinearity were used to derive the nonlinear equations of motion of the beam. The nonlinear differential equations were solved based on the generalized differential quadrature technique together with the Newmark-beta scheme. Damanpack *et al*. (2013b) addressed an active control of geometrically nonlinear dynamic response of sandwich beams impacted by blast pulses with integrated piezoelectric sensor/actuator patches. The first-order shear deformation theory was used for the face sheets and piezoelectric patches, and the extended high-order sandwich theory was used for the flexible core. The von Kármán geometrical nonlinearity and Hamilton's principle was employed in the analysis. The nonlinear equations were solved by Newmark and the modified Newton-Raphson methods for dynamic analysis. The geometrically nonlinear static and dynamic analysis of FG beams under thermal fields and mechanical excitations was investigated by Bodaghi *et al*. (2014). The von Kármán type geometric nonlinearity, the first-order shear deformation theory and the Hamilton principle were used to formulate the governing equations of motion. The solution of nonlinear differential equations was obtained using the hybrid generalized differential quadrature method-Newmark algorithm-Newton-Raphson iterative scheme. Ansari *et al*. (2015) developed an exact solution for the nonlinear forced vibration of FG nano-beams in thermal environment based on surface elasticity theory. Abdelghany *et al*. (2015) obtained the nonlinear dynamic response of an axially FG Euler-Bernoulli simply supported beam. The beam is subjected to moving load and rested on a nonlinear visco-elastic foundation. The influences of power index, linear and non-linear stiffness of foundation and the velocity of passing load on free vibration and nonlinear dynamic response were studied. Duc *et al*. (2015) developed an analytical approach on the nonlinear response of thick FG circular cylindrical shells surrounded by elastic foundations. The shell is subjected to mechanical and thermal loads. The amplitude-time curves for nonlinear dynamic analysis of the circular cylindrical shells were obtained. Nonlinear vibration of FG beams based on the Euler-Bernoulli beam theory and von Kármán's geometric nonlinearity was studied by Ding *et al*. (2018). Ali *et al*. (2018) investigated the nonlinear free and forced vibration responses of sandwich nano-beams with three various FG patterns of reinforced carbon nanotubes. The sandwich nano-beam is resting on nonlinear visco-elastic foundation and is subjected to thermal and electrical loads. The nonlinear governing equations of motion were derived for an Euler-

Bernoulli beam based on Hamilton principle and von Kármán nonlinear relation. Nonlinear buckling and post-buckling of imperfect piezoelectric FG circular cylindrical shells with metal-ceramic-metal layers in thermal environment using Reddy's third-order shear deformation shell theory were studied by Khoa *et al*. (2019).

Exact solution for nonlinear deferential equation is very limited. For this reason, approximate methods are inevitable to solve nonlinear deferential equations. He and Wu (2007) described a new kind of analytical technique for nonlinear problems called Variational Iteration Method (VIM). They reviewed trends and developments in the use of the VIM and its applications to nonlinear problems arising in various engineering applications. Rafei *et al*. (2007) applied the VIM to nonlinear oscillators with discontinuities and showed that the VIM is an effective and convenient method leading to high accuracy solutions in the first iteration. Yazdi (2013) used the Homotopy perturbation method to analyze the geometrically nonlinear vibrations of thin rectangular laminated FG plates. He investigated the effect of initial deflection, aspect ratio and material properties on frequency ratio. Şimşek (2015) proposed a novel size-dependent beam model for nonlinear free vibration of a FG nano-beam based on the nonlocal strain gradient theory and Euler-Bernoulli beam theory in conjunction with von Kármán geometric nonlinearity. The partial nonlinear differential equation describing the nonlinear vibration of FG nano-beam was reduced to an ordinary nonlinear differential equation with cubic nonlinearity, and a closed-form solution was obtained.

Praveen and Reddy (1998) analyzed the nonlinear dynamic response of FG plates subjected to pressure loads and thickness varying temperature fields by using the first-order shear deformation plate theory and the finite element method. Sundararajan *et al*. (2005) studied the nonlinear free vibration of both rectangular and skew FG plates by using finite element method. The variation of nonlinear frequency ratio with amplitude was highlighted considering various parameters such as gradient index, temperature, thickness, aspect ratio, and skew angle. Ganapathi and Prakash (2006) investigated the thermal buckling of a simply supported FG skew plate using first order shear deformation theory in conjunction with a finite element approach. The effects of aspect and thickness ratio, gradient index, and skew angle on the critical buckling temperature are brought out.

Approximate methods for studying nonlinear vibrations of beams are important for investigating and designing purposes. In recent years, some promising approximate methods have been suggested, such as Frequency Amplitude Formulation (Fereidoon *et al*. 2011), Variational Iteration (Barari *et al*. 2008, Fouladi *et al*. 2010), Homotopy Analysis (Pirbodaghi *et al*. 2009), Homotopy-Perturbation (Moeenfard *et al*. 2011), Parametrized-Perturbation (He 2006), Max-Min (Ganji *et al*. 2011, Ibsen *et al*. 2010), Differential Transform (Ganji *et al*. 2012), Classical Balance (Lee *et al*. 2012), Multiple Time-Scale (Younesian *et al*. 2014), Incremental Harmonic Balance method (Huang *et al*. 2011), etc.

The approximate methods have their own limitations. For example, the perturbation methods, which are the most extensively used analytical techniques, are generally restricted to the case of weak nonlinearity and are implemented on the basis of a single small parameter in the equation. Most of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all (Pirbodaghi *et al*. 2009). As second example of limitation, the Harmonic Balance methods are extremely time consuming to construct higher order analytical approximations (Leung *et al*. 2010).

From the literature review, it is clear that most of the researchers are interested in the free and forced nonlinear vibrations of FG beams with more focus on free vibration, however, to the best of the authors' knowledge, there is no reported work on the exact solution for nonlinear vibration of FG buckled beams.

This paper seeks to address this research gap by obtaining the closed form exact solution, instead of approximate solutions, to the problem of nonlinear vibrations of clamped-clamped FG buckled beams. A detailed analysis of the influence of the material property distribution, the beam slenderness ratio, the vibration amplitude and the magnitude of axial load on the nonlinear behavior of FG beams is carried out.

2. Governing equations

Consider a straight Euler-Bernoulli FG beam of length L, width b and thickness h (Fig. 1).

The volume fractions of the constituents of FG beam are assumed to vary from the bottom (z = -h/2) to the top surface (z = h/2) of the beam according to exponential-law

$$\begin{cases} E(z) = E_0 e^{\beta z} \\ \rho(z) = \rho_0 e^{\beta z} \end{cases} \tag{1}$$

where the subscript 0 denotes the midplane (z = 0) and β is a constant characterizing the distributions of material properties. $\beta = 0$ corresponds to an isotropic homogeneous beam. The Material property (E, ρ) at the top and bottom surfaces of the FG beam are assumed to be (E_1, ρ_1) and (E_2, ρ_2), respectively. The Poisson's ratio is taken constant.

According to the Euler-Bernoulli beam theory, the axial and transverse displacements of an arbitrary point in the beam (Aydogdu and Taskin 2007), denoted by $\widetilde{U}(x, z, t)$

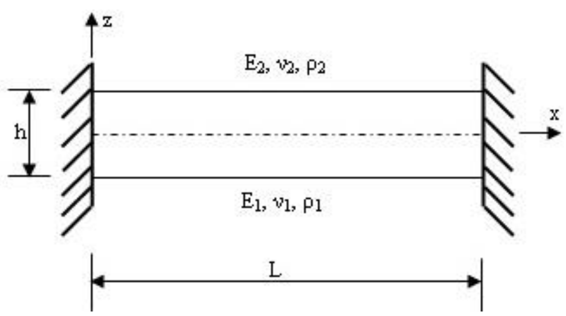


Fig. 1 Geometry of FG beam

and $\widetilde{W}(x,z,t)$ respectively, can be expressed as

$$\begin{cases}\widetilde{U}(x,z,t) = U(x,t) + z\dfrac{\partial W}{\partial x} \\ \widetilde{W}(x,z,t) = W(x,t)\end{cases} \tag{2}$$

Where t is time, $U(x,t)$ and $W(x,t)$ are the displacement components in the mid-plane along x and z direction, respectively.

The von Kármán type nonlinear strain-displacement relationship gives (Ke *et al*. 2010)

$$\varepsilon_x = \frac{\partial U}{\partial x} + z\frac{\partial^2 W}{\partial^2 x} + \frac{1}{2}(\frac{\partial W}{\partial x})^2 \tag{3}$$

The normal stress σ_x is given by linear elastic constitutive law as

$$\sigma_x = \frac{E(z)}{1-\nu^2}\left[\frac{\partial U}{\partial x} + z\frac{\partial^2 W}{\partial^2 x} + \frac{1}{2}(\frac{\partial W}{\partial x})^2\right] \tag{4}$$

In terms of beam notations, the total induced axial force N and bending moment M are related to the stress resultants as follows

$$N = bA_{11}\left[\frac{\partial U}{\partial x} + \frac{1}{2}(\frac{\partial W}{\partial x})^2\right] + bB_{11}\frac{\partial^2 W}{\partial^2 x} \tag{5}$$

$$M = bB_{11}\left[\frac{\partial U}{\partial x} + \frac{1}{2}(\frac{\partial W}{\partial x})^2\right] + bD_{11}\frac{\partial^2 W}{\partial^2 x} \tag{6}$$

Where A_{11}, B_{11} and D_{11} are the stiffness components defined as

$$\{A_{11}, B_{11}, D_{11}\} = \int_{-h/2}^{h/2}\frac{E(z)}{1-\nu^2}\{1, z, z^2\}dz \tag{7}$$

By using Hamilton's principle, the equations of motion can be derived as

$$I_1\frac{\partial^2 U}{\partial t^2} - \frac{\partial N}{\partial x} = 0 \tag{8}$$

$$I_1\frac{\partial^2 W}{\partial t^2} + \frac{\partial^2 M}{\partial x^2} - N\frac{\partial^2 W}{\partial x^2} = 0 \tag{9}$$

Where I_1 is the inertia related term defined as

$$I_1 = b\int_{-h/2}^{h/2}\rho(z)\,dz \tag{10}$$

It is assumed that the in-plane inertia and damping are negligible and the distributed axial force is zero. It follows from Eq. (9) that the induced axial force N is a constant. Substituting Eqs. (5)-(6) into Eqs. (8)-(9), one can obtain

$$\frac{\partial}{\partial x}\left\{bA_{11}\left[\frac{\partial U}{\partial x} + \frac{1}{2}(\frac{\partial W}{\partial x})^2\right] + bB_{11}\frac{\partial^2 W}{\partial x^2}\right\} = 0 \tag{11}$$

$$I_1\frac{\partial^2 W}{\partial t^2} + \frac{\partial}{\partial x}\left\{bB_{11}\left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2}\right] + bD_{11}\frac{\partial^3 W}{\partial x^3}\right\}$$
$$-\frac{\partial^2 W}{\partial x^2}\left\{bA_{11}\left[\frac{\partial U}{\partial x} + \frac{1}{2}(\frac{\partial W}{\partial x})^2\right] + bB_{11}\frac{\partial^2 W}{\partial^2 x}\right\} = 0 \tag{12}$$

Integrating Eq. (11) with respect to the spatial coordinate, x, one can obtain

$$A_{11}\left[\frac{\partial U}{\partial x} + \frac{1}{2}(\frac{\partial W}{\partial x})^2\right] + B_{11}\frac{\partial^2 W}{\partial x^2} + c_1(t) = 0 \tag{13}$$

which can be rewritten as

$$\frac{\partial U}{\partial x} = -\frac{1}{2}(\frac{\partial W}{\partial x})^2 - \frac{B_{11}}{A_{11}}\frac{\partial^2 W}{\partial x^2} - \frac{1}{A_{11}}c_1(t) = 0 \tag{14}$$

Integrating Eq. (14) once more yields

$$\begin{aligned} U = &-\frac{1}{2}\int(\frac{\partial W}{\partial x})^2\,dx - \frac{B_{11}}{A_{11}}\frac{\partial W}{\partial x} \\ &-\frac{1}{A_{11}}c_1(t)x + c_2(t) = 0\end{aligned} \tag{15}$$

For beams with immovable ends (i.e., $U = 0$, at $x = 0$ and L) subjected to an external compressive axial load, P, applied at x = L, Eq. (15) yields

$$c_2(t) = a \tag{16}$$

$$c_1(t) = \frac{P}{b} - \frac{A_{11}}{L}\int_0^L(\frac{1}{2}(\frac{\partial W}{\partial x})^2 - \frac{B_{11}}{A_{11}}\frac{\partial^2 W}{\partial x^2})dx \tag{17}$$

Where a is a constant.

Substituting Eq. (17) into Eq. (14) and differentiating the obtained equation with respect to x yields

$$\frac{\partial^2 U}{\partial x^2} = -\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2} - \frac{B_{11}}{A_{11}}\frac{\partial^3 W}{\partial x^3} \tag{18}$$

Then, from Eq. (12), the following equation is obtained

$$\begin{aligned} &I_1\frac{\partial^2 W}{\partial t^2} + b(D_{11} - \frac{B_{11}^2}{A_{11}})\frac{\partial^4 W}{\partial x^4} \\ &+\left[P - b\frac{A_{11}}{L}\int_0^L(\frac{1}{2}(\frac{\partial W}{\partial x})^2 - \frac{B_{11}}{A_{11}}\frac{\partial^2 W}{\partial x^2})dx\right]\frac{\partial^2 W}{\partial x^2} = 0\end{aligned} \tag{19}$$

Introducing the following quantities

$$\begin{cases}\tilde{E} = b(D_{11} - \dfrac{B_{11}^2}{A_{11}}) \\ \tilde{D} = b\dfrac{A_{11}}{L} \\ \tilde{C} = \dfrac{B_{11}}{A_{11}}\end{cases} \tag{20}$$

Eq. (19) can be rewritten as

$$\begin{aligned} &\tilde{E}\frac{\partial^4 W}{\partial x^4} + I_1\frac{\partial^2 W}{\partial t^2} - \tilde{D}\int_0^L\frac{1}{2}(\frac{\partial W}{\partial x})^2\,dx\frac{\partial^2 W}{\partial x^2} \\ &+P\frac{\partial^2 W}{\partial x^2} + \tilde{D}\tilde{C}\int_0^L\frac{\partial^2 W}{\partial x^2}dx\frac{\partial^2 W}{\partial x^2} = 0\end{aligned} \tag{21}$$

For nonlinear vibration analysis, the transverse displacement is assumed to be in the following form

$$W(x,t) = a\varphi(x)\psi(t) \tag{22}$$

Where $\psi(t)$ is an arbitrary function of time and is $\varphi(x)$ is the linear fundamental vibration mode.

The linear modes shape of fixed-fixed beam is

$$\begin{aligned}\varphi(x) = cos(\frac{\gamma x}{L}) - cosh(\frac{\gamma x}{L}) \\ +R\left[sin(\frac{\gamma x}{L}) - sinh(\frac{\gamma x}{L})\right]\end{aligned} \tag{23}$$

Where a is an arbitrary constant that represents the amplitude of the deflection The first mode has a value of $\gamma = 4.73$ and R = (sin(γ) + sinh(γ)) / (cos(γ) – cosh(γ)).

Inserting $W(x,t)$ into Eq. (21) and applying Galerkin's procedure yields a second order nonlinear ordinary differential equation

$$\ddot{\psi} + \chi_1\psi + \chi_2\psi^2 + \chi_3\psi^3 = 0 \tag{24}$$

where a super dot denotes differentiation with respect to time, and

$$\chi_1 = (\frac{\tilde{E}}{I_1})\Phi_4 + (\frac{P}{I_1})\Phi_2,\ \chi_2 = \frac{\tilde{D}\tilde{C}a}{I_1}\Phi_1\Phi_2,$$

$$\chi_3 = -\frac{\tilde{D}a^2}{2I_1}\Phi_2\Phi_3$$

$$\text{With} \begin{cases} \Phi_1 = \int_0^L \varphi'' dx, & \Phi_2 = \frac{\int_0^L \varphi''\varphi dx}{\int_0^L \varphi^2 dx}, \\ \Phi_3 = \int_0^L (\varphi')^2 dx, & \Phi_4 = \frac{\int_0^L \varphi^{(4)}\varphi dx}{\int_0^L \varphi^2 dx} \end{cases},$$

Eq. (24) contains a quadratic nonlinear term due to the presence of bending-extension coupling effect in FG beams. This term, however, vanishes and Eq. (24) reduces to a Duffing equation for homogeneous beams and clamped-clamped FGM beams because $\chi_2 = 0$ in both cases. The parameters χ_1 and χ_3 are regrouped regrouped in the following way

$$\begin{cases} p^2 = \chi_1 + \chi_3 \\ k^2 = \dfrac{\chi_3}{2p^2} \end{cases} \tag{25}$$

Integrating Eq. (24) with respect to time, with the initial conditions at t = 0, $\psi = 1$ and $\frac{d\psi}{dt} = 0$ gives

$$\left(\frac{d\psi}{dt}\right)^2 = \chi_1(1-\psi^2) + \frac{\chi_2}{2}(1-\psi^4) \tag{26}$$

which can be written as follow

$$\left(\frac{d\psi}{dt}\right)^2 = p^2(1 - k^2 - (1-2k^2)\psi^2 - k^2\psi^4) \tag{27}$$

and it reduces to

$$\left(\frac{d\psi}{d(pt)}\right)^2 = (1-\psi^2)(k^2\psi^2 - k^2 + 1) \tag{28}$$

Assuming $\psi = cos(\phi)$ we can obtain Jacobi elliptic function (Byrd and Friedman 1971) with the modulus k

$$pt = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 sin^2\phi}} \tag{29}$$

From the inversion of Eq. (29), the solution for ψ can be obtained as follow

$$\psi = cn[pt,k] \tag{30}$$

The period of the function $cn[pt,k]$ is 4 K defined by the complete elliptic integral

$$4K = 4\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 sin^2\phi}} \tag{31}$$

The corresponding frequency of the nonlinear problem is defined by using the following equation

$$\omega_{nl} = \frac{\pi\sqrt{\chi_1 + \chi_3}}{2K} \tag{32}$$

Then, the solution of the free vibration of a buckled beam with fixed end restrains can be expressed as follow

$$W(x,t) = a\begin{Bmatrix} cos(\frac{\gamma x}{L}) - cosh(\frac{\gamma x}{L}) \\ +R\left[sin(\frac{\gamma x}{L}) - sinh(\frac{\gamma x}{L})\right] \end{Bmatrix} cn[pt,k] \tag{33}$$

3. Validation of the analysis

For the purpose of validation of the proposed method, the dimensionless linear frequency $\bar{\omega}_l = \omega\sqrt{D_0/I_{10}}$ for clamped-clamped FG beams is compared with solutions given by Ke *et al.* (2010) and Yang and Chen (2008). $D_0 = D_{110} - B_{110}^2/A_{110}$. A_{110}, B_{110}, D_{110} and I_{10} denote the values of A_{11}, B_{11}, D_{11} and I_1 of an isotropic homogeneous beam of material properties (E_1, ν_1, ρ_1). The material properties change exponentially along beam thickness as described in Eq. (1) with E_1 = 70 GPa, ν_1 = 0.33, ρ_1 = 2780 kg/m^3, L/h = 20. The Young's modulus ratio is taken E_2/E_1 = 0.2, 1.0 and 5.0.

Table 1 shows that the results delivered by the developed method are close to the results given by other methods.

Table 1 Comparison of dimensionless fundamental frequencies of FG beams

E_2/E_1	Present	Ke *et al.* (2010)	Yang and Chen (2008)
0,2	5,255	5,255	5,25
1	5,593	5,5933	5,59
5	5,255	5,255	5,25

4. Influence of axial load, vibration amplitude, beam slenderness ratio and Young's modulus ratio on the nonlinear behavior of FG beams

In this section, the effects of various parameters such as axial load, vibration amplitude, Young's modulus ratio ($r_E = E_2/E_1$) and beam slenderness ratio (L/h) on the nonlinear behavior are examined.

We consider a FG beam with bottom surface made of epoxy: Young's modulus E_1 = 4.2 GPa, Poisson's ratio ν_1 = 0.34, and mass density ρ_1 = 1272 kg/m^3. The beam thickness and width are assumed to be 5 mm and 30 mm, respectively.

4.1 Influence of the axial load

In order to explain the axial load effect, we consider a beam slenderness ratio L/h = 20.

Figs. 2(a)-(c) show the evolution of the dimensionless nonlinear frequency ω_{nl}/ω_l (the nonlinear frequency normalized by the linear natural frequency) as a function of the dimensionless axial load P/Pcr (Pcr is the buckling load of the clamped-clamped FG beam calculated as $P_{cr} = -\tilde{E}\Phi_4/\Phi_2$). In each figure, going from the innermost curve to the outermost one, the dimensionless amplitude a/r (r is the radius of gyration defined as $r = \sqrt{I/A}$ where I is the moment of inertia and A is the cross-section area) increases from 0 to 2. From one figure to another, it is the Young's modulus ratio which varies from 1 to 20.

One property in common for all Young's modulus ratios, namely that increasing the vibration amplitude leads to swelling of the curve. The swelling is clearly visible for r_E = 20 and a/r = 2.

Going from a/r = 0 to 2, the space between the represented curves widens in the same manner.

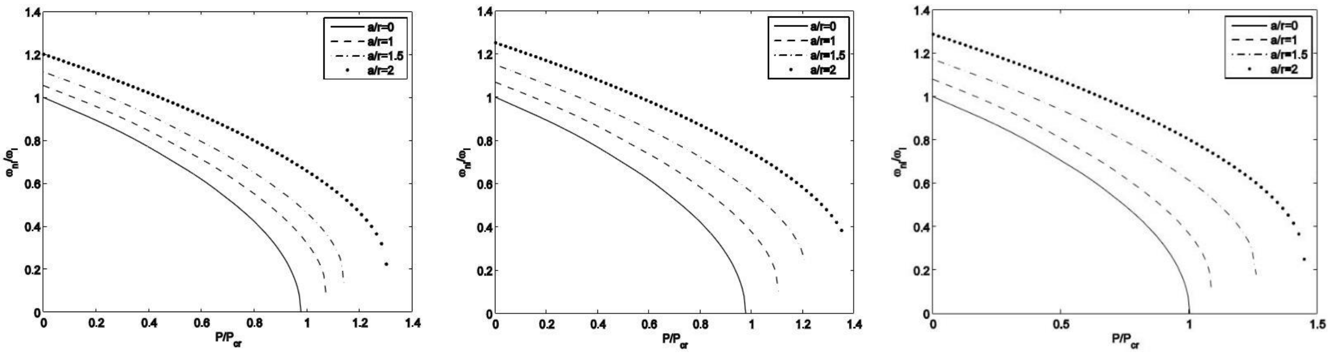
From Figs. 2(a)-(c), it can be seen that the slope at the end is equal to zero for various Young's modulus ratios and dimensionless amplitudes.

4.2 Influence of the vibration amplitude

Several FG beams with slenderness ratio L/h = 20 and different Young's modulus ratios are studied. The dimensionless amplitude effect on the dimensionless nonlinear frequency is depicted in Figs. 3(a)-(c). From one figure to another, it is the Young's modulus ratio which varies from 1 to 20. For each chosen Young's modulus ratio a set of dimensionless axial loads are considered.

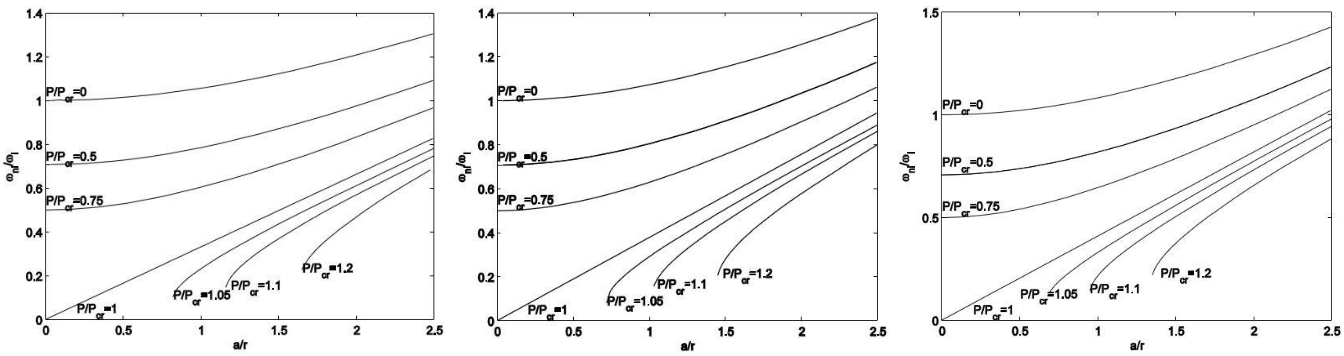
It can be noted from these figures that there is a clear dependence of the dimensionless nonlinear frequency on dimensionless amplitude. All beams exhibit typical hardening behavior, i.e., the nonlinear frequency ratio increases as the vibration amplitude is increased. Here, only positive vibration amplitudes are investigated because the quadratic nonlinear term representing the coupling effect vanishes ($\chi_2 = 0$).

The plots (Figs. 3(a)-(c)) of pre-buckled state (P < P_{cr}) and post-buckled state (P > P_{cr}) are separated by a straight



(a) r_E = 1 (b) r_E = 10 (c) r_E = 20

Fig. 2 Dimensionless nonlinear frequency versus dimensionless axial load for L/h = 20



(a) r_E = 1 (b) r_E = 10 (c) r_E = 20

Fig. 3 Dimensionless nonlinear frequency versus dimensionless amplitude for L/h = 20

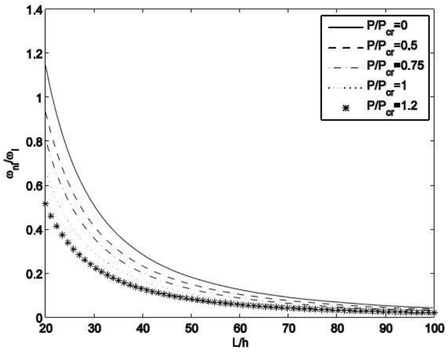


Fig. 4 Dimensionless nonlinear frequency versus beam slenderness ratio for rE = 2 and a/r = 2

line that represents the case where the axial load is equal to the buckling load. It is clearly seen that the variation of the dimensionless nonlinear frequency as a function of the dimensionless amplitude is parabolic. The parabolic branches change the concavity before and after the straight line. It is also shown that the branches look similar to those of linear vibration.

4.3 Influence of the beam slenderness ratio

In this subsection, we consider a Young's modulus ratio r_E = 10 and study the dimensionless frequency for dimensionless amplitude a/r = 2.

Fig. 4 plots the evolution of the dimensionless nonlinear frequency as a function of the beam slenderness ratio for various values of dimensionless axial loads.

In linear vibration analysis, one can expect that the natural frequency of the beam increases with increasing the slenderness ratio L/h (Selmi 2019). This proportionality is because the increase of the slenderness ratio results in the decrease of the shear deformation effect of the beam, which positively correlates to the flexibility of the beam. However, the relationship of the nonlinear frequency and the beam slenderness ratio does not follow the same pattern. It is demonstrated (Fig. 4) that the dimensionless nonlinear frequency drops as the increase of slenderness ratio, and presents a horizontal asymptote from L/h = 90. It is worth noting that the curves plotting the variation of dimensionless nonlinear frequency as a function of the beam slenderness ratio for a given value of dimensionless axial load converge to the same value.

4.4 Influence of Young's modulus ratio

The effect of the Young's modulus ratio on the dimensionless nonlinear frequencies (the nonlinear frequency normalized by the linear natural frequency of homogeneous beam made of epoxy) of FG beams in the pre-buckled and post-buckled stages are investigated in this subsection. The beam slenderness ratio and the dimensionless amplitude are taken equal to 20 and 2, respectively.

It can be demonstrated from Figs. 5(a)-(c) that there is a clear dependence of dimensionless nonlinear frequency on the Young's modulus ratio. For pre-buckled state, the normalized nonlinear frequency increases rapidly then

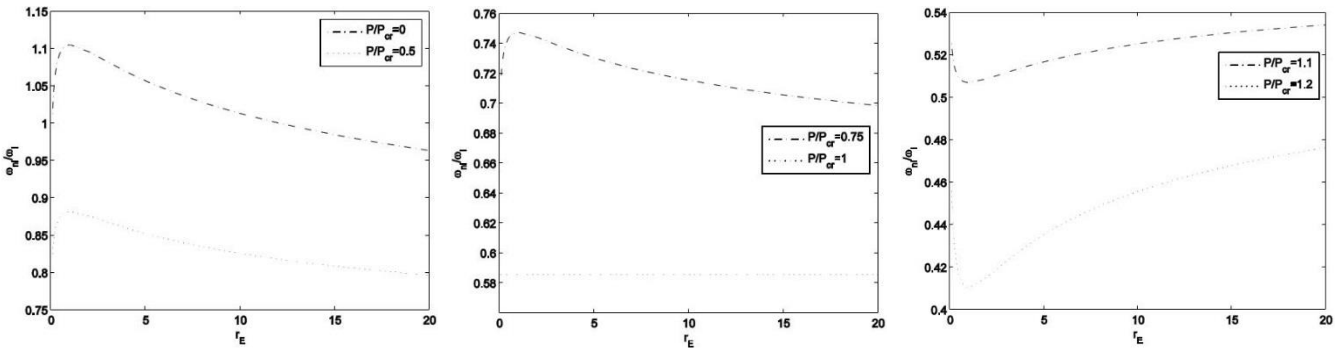
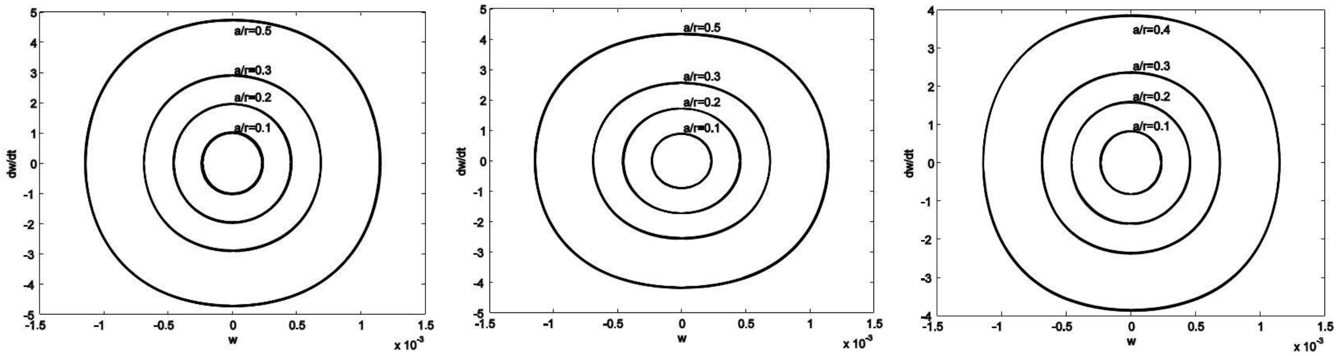
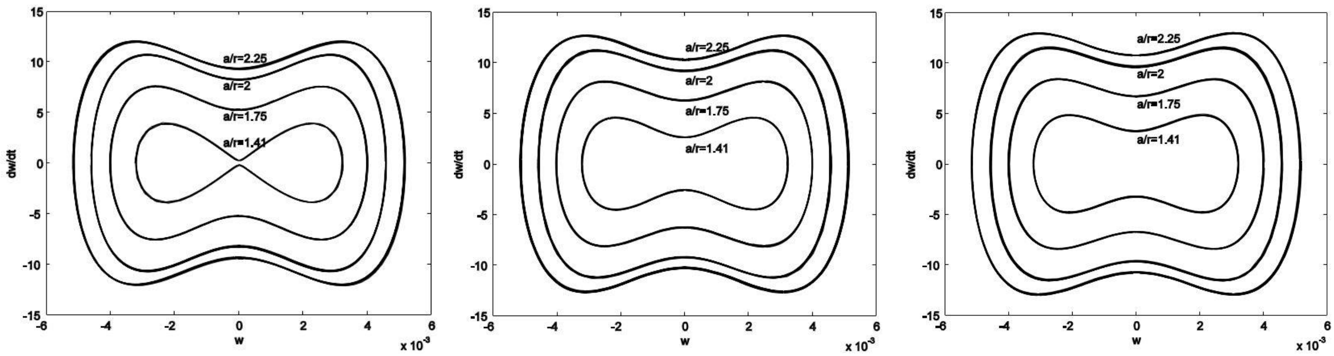


Fig. 5 Dimensionless nonlinear frequency versus Young's modulus ratio for a/r = 2 and L/h = 20



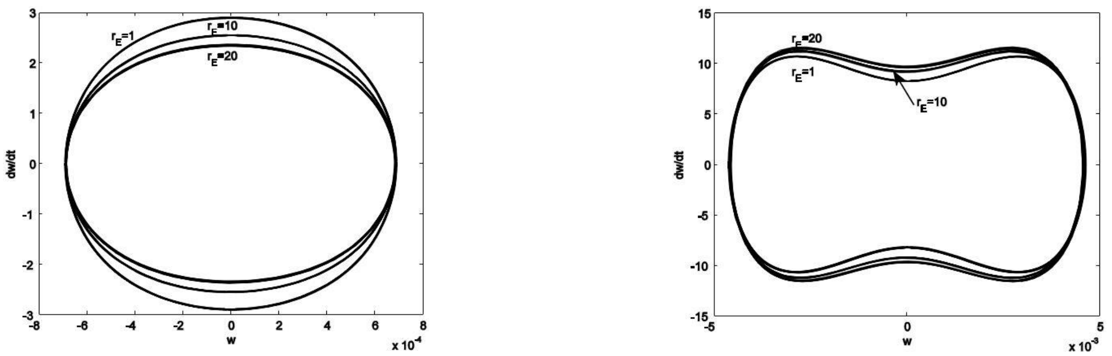
(a) r_E = 1 (b) r_E = 10 (c) r_E = 20

Fig. 6 Phase trajectory plot for pre-buckling stage (P/P_{cr} = 0.5) for L/h = 20



(a) $r_E = 1$ (b) $r_E = 10$ (c) $r_E = 20$

Fig. 7 Phase trajectory plot for post-buckling stage ($P/P_{cr} = 1.15$) for L/h = 20



(a) Pre-buckling stage ($P/P_{cr} = 0.5$) (b) Post-buckling stage ($P/P_{cr} = 1.15$)

Fig. 8 Variation of phase trajectory plot with Young's modulus ratio for L/h = 20

decreases with the Young's modulus ratio while a rapid decrease then an increase is seen for post-buckled state. When the axial load is equal to the buckling load, the normalized nonlinear frequency is independent on the Young's modulus ratio.

It is worth noting that in the pre-buckled state, the curves corresponding to high values of dimensionless axial load are below those corresponding to low values of dimensionless axial load. The opposite trend is observed in the post-buckled state.

5. Phase trajectory

The phase trajectory plots expose the velocity with respect to the displacement at a selected location. For different values of dimensionless amplitudes, the phase trajectory plots for pre-buckling stage ($P/P_{cr} = 0.5$) corresponding to the point located at x = 0.5 L, are shown in Figs. 6(a)-(c). From one figure to another, it is the Young's modulus ratio which varies from 1 to 20. Figs. 7(a)-(c) show the phase trajectory plots for post-buckling stage (P/Pcr = 1.15) corresponding to the same location and same material constituents as that taken for pre-buckling stage. In this section, the beam slenderness ratio is L/h = 20. For r_E = 1, 10 and 20, typical phase trajectory plots for L/h = 20 and a/r = 2 are depicted in Fig. 8(a) for pre-buckling stage and in Fig. 8(b) for the post-buckling one.

In pre-buckled stage, the phase trajectory plot is a closed symmetric curve which is more like an ellipse. In the post-buckled stage, the phase diagram has also a close trajectory which tends to be elliptic for high Young's modulus ratio. From Figs. 7(a)-(c), 8(a)-(c) and 8(a)-(b), it is noted that the phase trajectory plots of the studied FG beams are strongly dependent on the vibration amplitude. In the pre- and post-buckling stages, for the considered Young's modulus ratios, increasing vibration amplitude leads to swelling of phase trajectory plot. For the pre-buckling stage, the swelling is important for homogeneous beam and increases significantly when the Young's modulus ratio gets small (Fig. 8(a)). On the opposite, the swelling increase with increasing Young's modulus ratio for post-buckling stage (Fig. 8(b)).

6. Conclusions

Exact solution for nonlinear behavior of FG buckled beams with fixed-fixed boundary conditions are obtained using the Euler-Bernoulli beam theory and von Kármán's geometric nonlinearity. The nonlinear solution for buckled beams was derived through the elliptic function. The effects of the Young's modulus ratio, the beam slenderness ratio, the vibration amplitude and the magnitude of axial load on the nonlinear behavior are examined. Numerical results prove that (i) the increase of the vibration amplitude leads to swelling of the curve showing the variation of the dimensionless nonlinear frequency as a function of the dimensionless axial load; (ii) the nonlinear frequency ratio increases with an increase in the vibration amplitude and a decrease in the beam slenderness ratio; (iii) the monotony of the normalized nonlinear frequency depends on the Young's modulus ratio.

Acknowledgments

This project was supported by the Deanship of Scientific Research at Prince Sattam Bin Abdulaziz University under the research project No: 11267/01/ 2019.

References

Abdelghany, S.M., Ewis, K.M., Mahmoud, A.A. and Nassar, M.M. (2015), "Dynamic response of non-uniform beam subjected to moving load and resting on nonlinear viscoelastic foundation", *Aust. J. Basic Appl. Sci.*, **4**(3), 192-199. https://doi.org/10.1016/j.bjbas.2015.05.007

Addou, F.Y. Meradjah, M., Bousahla, A.A., Benachour, A., Bourada, F., Tounsi, A. and Mahmoud, S.R. (2019), "Influences of porosity on dynamic response of FG plates resting on Winkler/Pasternak/Kerr foundation using quasi 3D HSDT", *Comput. Concrete*, *Int. J.*, **24**(4), 347-367. https://doi.org/10.12989/cac.2019.24.4.347

Ali, G.A, Mahmoud, P. and Mohammad, A. (2018), "Non-linear free and forced vibration analysis of sandwich nano-beam with FG-CNTRC face-sheets based on nonlocal strain gradient theory", *Smart Struct. Syst.*, *Int. J.*, **22**(11), 105-120. https://doi.org/10.12989/sss.2018.22.1.105

Ansari, M., Esmailzadeh, E. and Younesian, D. (2011), "Frequency analysis of finite beams on nonlinear Kelvin-Voight foundation under moving loads", *J. Sound Vib.*, **330**(7), 1455-1471. https://doi.org/10.1016/j.jsv.2010.10.005

Ansari, R., Pourashraf, T. and Gholami, R. (2015), "An exact solution for the nonlinear forced vibration of functionally graded nanobeams in thermal environment based on surfaceelasticity theory", *Thin Wall. Struct.*, **93**, 169-176. https://doi.org/10.1016/j.tws.2015.03.013

Aydogdu, M. and Taskin, V. (2007), "Free vibration analysis of functionally graded beams with simply-supported edges", *Mater. Des.*, **28**, 1651-1656. https://doi.org/10.1016/j. matdes.2006.02.007

Balubaid, M., Tounsi, A., Dakhel, B. and Mahmoud, S.R. (2019), "Free vibration investigation of FG nanoscale plate using nonlocal two variables integral refined plate theory", *Comput. Concrete*, *Int. J.*, **24**(6), 579-586. https://doi.org/10.12989/cac .2019.24.6.579

Barari, A., Omidvar, M., Abdoul, R.G. and Ganji, D.D. (2008), "Application of homotopy perturbation method and variational iteration method to nonlinear oscillator differential equations", *Acta. Appl. Math.*, **104**, 161-171. https://doi.org/10.1007/s10440-008-9248-9

Belabed, Z., Bousahla, A.A., Houari, M.S.A., Tounsi, A. and Mahmoud, S.R. (2018), "A new 3-unknown hyperbolic shear deformation theory for vibration of functionally graded sandwich plate", *Earthq. Struct.*, *Int. J.*, **14**(2), 103-115. https://doi.org/10.12989/eas.2018.14.2.103

Belbachir, N., Draich, K., Bousahla, A., Bourada, M., Tounsi, A. and Mohammadimehr, M. (2019), "Bending analysis of anti-symmetric cross-ply laminated plates under nonlinear thermal and mechanical loadings", *Steel Compos. Struct.*, *Int. J.*, **33**(1), 913-924. https://doi.org/10.12989/scs.2019.33.1.081

Beldjelili, Y., Tounsi, A. and Mahmoud, S.R. (2016), "Hygro-thermo-mechanical bending of S-FGM plates resting on variable elastic foundations using a four-variable trigonometric plate theory", *Smart Struct. Syst.*, *Int. J.*, **18**(4), 755-786. https://doi.org/10.12989/sss.2016.18.4.755

Berghouti, H., Bedia, E.A.A., Benkhedda, A. and Tounsi, A. (2019), "Vibration analysis of nonlocal porous nanobeams made of functionally graded material", *Adv. Nano Res.*, *Int. J.*, **7**(5), 351-364. https://doi.org/10.12989/anr.2019.7.5.351

Bodaghi, M., Damanpack, A.R., Aghdam, M.M. and Shakeri, M. (2012), "Non-linear active control of FG beams in thermal environments subjected to blast loads with integrated FGP sensor/actuator layers", *Compos. Struct.*, **94**(12), 3612-3623. https://doi.org/10.1016/j.compstruct.2012.06.001

Bodaghi, M., Damanpack, A.R., Aghdam, M.M. and Shakeri, M. (2014), "Geometrically non-linear transient thermo-elastic response of FG beams integrated with a pair of FG piezoelectric sensors", *Compos. Struct.*, **107**, 48-59. https://doi.org/10.1016/j.compstruct.2013.07.045

Bouafia, K., Kaci, A., Houari, M.S.A., Benzair, A. and Tounsi, A. (2017), "A nonlocal quasi-3D theory for bending and free flexural vibration behaviors of functionally graded nanobeams", *Smart Struct. Syst.*, *Int. J.*, **19**(2), 115-126. https://doi.org/10.12989/sss.2017.19.2.115

Bourada, F., Bousahla, A.A., Bourada, M., Azzaz, A., Zinata, A. and Tounsi, A. (2019), "Dynamic investigation of porous functionally graded beam using a sinusoidal shear deformation theory", *Wind Struct.*, *Int. J.*, **28**(1), 19-30. https://doi.org/10.12989/was.2019.28.1.019

Boutaleb, S., Benrahou, K.H., Bakora, A., Algarni, A., Bousahla, A.A., Tounsi, A., Tounsi, A. and Mahmoud, S.R. (2019), "Dynamic Analysis of nanosize FG rectangular plates based on simple nonlocal quasi 3D HSDT", *Adv. Nano Res.*, *Int. J.*, **7**(3), 189-206. https://doi.org/10.12989/anr.2019.7.3.189

Byrd, M.D. and Friedman, P.F. (1971), *Handbook of elliptic integrals for engineers and physicists*, Springer, Berlin.

Damanpack, A.R., Bodaghi, M., Ghassemi, H. and Sayehbani, M., (2013a), "²Boundary element method applied to the Bending Analysis of Thin Functionally Graded Plates", *Lat. Am. J. Solids Stru.*, **10**(3), 549-570. http://dx.doi.org/10.1590/S1679-78252013000300006

Damanpack, A.R., Bodaghi, M., Aghdam, M.M. and Shakeri, M. (2013b), "Active control of geometrically non-linear transient response of sandwich beams with a flexible core using piezoelectric patches", *Compos. Struct.*, **100**, 517-531. https://doi.org/10.1016/j.compstruct.2012.12.029

Daouadji, T.H. and Tounsi, A. (2013), "A new higher order shear deformation model for static behavior of functionally graded plates", *Adv. Appl. Math Mech.*, **5**(3), 351-364. https://doi.org/10.1017/S2070073300002721

Ding, J., Chu, L., Xin, L. and Dui, G. (2018), "Nonlinear Vibration Analysis of Functionally Graded Beams Considering the Influences of the Rotary Inertia of the Cross Section and Neutral Surface Position", *Mech. Based Des. Struct.*, **46**(2), 225-237. https://doi.org/10.1080/15397734. 2017.1329020

Duc, N.D., Thang, P.T., Dao, N.T. and Tac, H.V. (2015), "Nonlinear buckling of higher deformable S-FGM thick circular cylindrical shells with metal-ceramic-metal layers surrounded on elastic foundations in thermal environment", *Compos. Struct.*, **121**, 134-141. https://doi.org/10.1016/j. compstruct.2014.11.009

Fereidoon, A., Ghadimi, M., Barari, A., Kaliji, H.D. and Domairry, G. (2011), "Nonlinear vibration of oscillation systems using frequency-amplitude formulation", *Shock Vib.*, **19**(3), 323-332. https://doi.org/10.3233/SAV-2011-0633

Fouladi, F., Hosseinzadeh, E., Barari, A. and Domairry, G. (2010), "Highly Nonlinear Temperature-Dependent Fin Analysis by Variational Iteration Method", *Heat Transf. Res.*, **41**(2), 155-165. https://doi.org/10.1615/HeatTransRes. v41.i2.40

Fourn, H., Ait Atmane, H., Bourada, M., Bousahla, A.A., Tounsi, A. and Mahmoud, S.R. (2018), "A novel four variable refined plate theory for wave propagation in functionally graded material plates", *Steel Compos. Struct., Int. J.*, **27**(1), 109-122. https://doi.org/10.12989/scs.2018.27.1.109

Ganapathi, M. and Prakash, T. (2006), "Thermal buckling of simply supported functionally graded skew plates", *Compos. Struct.*, **74**(2), 247-250. https://doi.org/10.1016/j.compstruct. 2005. 04.004

Ganji, S.S., Barari, B. and Ganji, D.D. (2011), "Approximate analyses of two mass-spring systems and buckling of a column", *Comput. Math. Appl.*, **61**(4), 1088-1095. https://doi.org/10.1016/j.camwa.2010.12.059

Ganji, S.S., Barari, A., Ibsen, L.B. and Domairry, G. (2012), "Differential transform method for mathematical modeling of jamming transition problem in traffic congestion flow", *Cent. Eur. J. Oper. Res.*, **20**, 87-100. https://doi.org/10.1007/s10100 -010-0154-7

He, J.H. (2006), "Some asymptotic methods for strongly nonlinear equations", *Int. J. Mod. Phys. B*, **20**(10), 1141-1199. https://doi.org/10.1142/S0217979206033796

He, J.H. and Wu, X.H. (2007), "Variational iteration method: new development and applications", *Comput. Math. Appl.*, **54**, 881-894. https://doi.org/10.1016/j.camwa.2006.12.083

Hosseini, H.S., Fadaee, M. and Atashipour, S.R. (2011), "A new exact analytical approach for free vibration of reissner-mindlin functionally graded rectangular plates", *Int. J. Mech. Sci.*, **53**(1), 11-22. https://doi:10.1016/j.ijmecsci.2010.10.002

Houari, M.S.A., Tounsi, A., Bessaim, A. and Mahmoud, S.R. (2016), "A new simple three-unknown sinusoidal shear deformation theory for functionally graded plates", *Steel Compos. Struct., Int. J.*, **22**(2), 257-276. https://doi.org/10.12989/ scs.2016.22.2.257

Huang, J.L., Su, R.K.L., Lee, Y.Y. and Chen, S.H. (2011), "Nonlinear vibration of a curved beam under uniform base harmonic excitation with quadratic and cubic nonlinearities", *J. Sound Vib.*, **330**, 5151-5164. https://doi.org/10.1016/j.jsv. 2011.05.023

Hussain, M. and Naeem, M. (2019a), "Rotating response on the vibrations of functionally graded zigzag and chiral single walled carbon nanotubes", *Appl. Math. Model.*, **75**, 506-520. https://doi.org/10.1016/j.apm.2019.05.039

Hussain, M. and Naeem, M.N. (2019b), "Vibration characteristics of zigzag and chiral FGM rotating carbon nanotubes sandwich with ring supports", *J. Mech. Eng. Sci., Part C.*, **233**(16), 5763-5780. https://doi.org/10.1177/095440 6218802320

Hussain, M., Naeem, M.N., and Isvandzibaei, M. (2018a), "Effect of Winkler and Pasternak elastic foundation on the vibration of rotating functionally graded material cylindrical shell", *Proceedings of the Institution of Mechanical Engineers, Part C: J. Mech. Eng. Sci.*, **232**(24), 4564-4577. https://doi.org/10.1177/0954406217753459

Hussain, M., Naeem, M.N., Shahzad, A., He, M. and Habib, S. (2018b), "Vibrations of rotating cylindrical shells with FGM using wave propagation approach", *IMechE Part C: J. Mech. Eng. Sci.*, **232**(23), 4342-4356. https://doi.org/10.1177/09544 06218802320

Ibsen, L.B., Barari, A. and Kimiaeifar, A.S. (2010), "Analysis of highly nonlinear oscillation systems using He's max-min method and comparison with homotopy analysis and energy balance methods", *Sadhana*, **35**(4), 433-448. https://doi.org/ 10.1007/s12046-010-0024-y

Karami, B., Janghorban, M., Shahsavari, D. and Tounsi, A. (2018), "A size-dependent quasi-3D model for wave dispersion analysis of FG nanoplates", *Steel Compos. Struct., Int. J.*, **28**(1), 99-110. https://doi.org/10.12989/scs.2018.28.1.099

Kamarian, S., Bodaghi, M., Pourasghar, A. and Talebi, S. (2016), "Vibrational behavior of non-uniform piezoelectric sandwich beams made of CNT-reinforced polymer nanocomposite by considering the agglomeration effect of CNTs", *Polym. Compos.*, **38**, E553-E562. https://doi.org/10. 1002/pc.23933

Ke, L.L., Yang, J. and Kitipornchai, S. (2010), "An analytical study on the nonlinear vibration of functionally graded beams", *Meccanica*, **45**(6), 743-752. https://doi.org/10. 1007/s11012-009-9276-1

Khoa, N.D., Thiem, H.T. and Duc, N.D. (2019), "Nonlinear buckling and postbuckling of imperfect piezoelectric S-FGM circular cylindrical shells withmetal-ceramic-metal layers in thermal environment using Reddy's third-order shear deformation shell theory", *Mech. Adv. Mater. Struct.*, **26**(3), 248-259. https://doi.org/10.1080/15376494.2017.1341583

Lee, Y.Y., Li, Q.S., Leung, A.Y.T. and Su, R.K.L. (2012), "The jump phenomenon effect on the sound absorption of a nonlinear panel absorber and sound transmission loss of a nonlinear panel backed by a cavity", *Nonlinear Dyn.*, **69**, 99-116. https://doi.org/10. 1007/ s11071-011-0249-2

Leung, A.Y.T., Guo, Z.J. and Fung, T.C. (2010), "The multi parameter homotopy harmonic balance method for steady state problems", *Int. J. Comput. Math.*, **87**(5), 1158-1177. https://doi.org/10.1080/00207160903229899

Mahmoudi, A., Benyoucef, S., Tounsi, A., Benachour, A., Bedia, E.A. and Mahmoud, S.R. (2019), "A refined quasi-3D shear deformation theory for thermo-mechanical behavior of functionally graded sandwich plates on elastic foundations", *J. Sandw. Struct. Mater.*, **21**(6), 1906-1926. https://doi.org/10.1177%2F1099636217727577

Moeenfard, H., Mojahedi, M. and Ahmadian, M.T. (2011), "A homotopy perturbation analysis of nonlinear free vibration of Timoshenko microbeams", *J. Mech. Sci. Technol.*, **25**, 557-565. https://doi.org/10.1007/s12206-011-0130-8

Pirbodaghi, T., Ahmadian, M.T. and Fesanghary, M. (2009), "On the homotopy analysis method for non-linear vibration of beams", *Mech. Res. Commun.*, **36**, 143-148. https://doi.org/10.1016/ j.mechrescom.2008.08.001

Praveen, G.N. and Reddy, J.N. (1998), "Nonlinear transient thermoelastic analysis of functionally graded ceramic-metal plates", *Int. J. Solids Struct.*, **35**(3), 4457-4476. https://doi.org/10.1016/S0020-7683(97)00253-9

Rafei, M., Ganji, D., Daniali, H. and Pashaei, H. (2007), "Thevariational iteration method for nonlinear oscillators with discontinuities", *J. Sound Vib.*, **305**, 614-620. https://doi.org/10.1016/j.jsv.2007.04.020

Selmi, A. (2019), "Buckling capacity of functionally graded beams: A comparative study of different shear deformation beam theories", *Int. J. Computat. Mater. Sci. Eng.*, **8**(2), 1950003. https:// doi.org/10. 1142/ S2047684119500039

Şimşek, M. (2010), "Fundamental frequency analysis of functionally graded beams by using different higher-order beam theories", *Nucl. Eng. Des.*, **240**(4), 697-705. https://doi.org/10.1016/j.nucengdes.2009.12.013

Şimşek, M. (2015), "Size dependent nonlinear free vibration of an axially functionally graded (AFG) microbeam using He's variational method", *Compos. Struct.*, **131**, 207-214. https://doi.org/10.1016/j.compstruct.2015.05.004

Sofiyev, A.H., Hui, D. and Valiyev, A.A. (2016), "Effects of shear stresses and rotary inertia on the stability and vibration of sandwich cylindrical shells with FGM core surrounded by elastic medium", *Mech. Based Des. Struct.*, **44**(4), 384-404. https://doi.org/10.1080/15397734.2015.1083870

Sundararajan, N., Prakash, T. and Ganapathi, M. (2005), "Nonlinear free flexural vibrations of functionally graded rectangular and skew plates under thermal environments", *Finite Elem. Anal. Des.*, **42**(2), 152-168. https://doi.org/10.1016/j.finel.2005.06.001

Tounsi, A., Houari, M.S.A. and Bessaim, A. (2016), "A new 3-unknowns non-polynomial plate theory for buckling and vibration of functionally graded sandwich plate", *Struct. Eng. Mech., Int. J.*, **60**(4), 547-565. https://doi.org/10.12989/sem.2016. 60.4.547

Tufekci, E., Eroglu, U. and Aya, S.A. (2016), "Exact solution for in-plane static problems of circular beams made of functionally graded materials", *Mech. Based Des. Struct.*, **44**(4), 476-494. https://doi.org/10.1080/15397734. 2015.1121398

Udupa, G., Rao, S.S. and Gangadharan, K.V. (2014), "Functionally graded composite materials: an overview", *Procedia Mater. Sci.*, **5**, 1291-1299. https://doi.org/10.1016/ j.mspro.2014.07.442

Yaghoobi, H. and Yaghoobi, P. (2013), "Buckling analysis of sandwich plates with FGM face sheets resting on elastic foundation with various boundary conditions: an analytical approach", *Meccanica*, **48**, 2019-2035. https://doi.org/10.10 07/s11012-013-9720-0

Yang, J. and Chen, Y. (2008), "Free vibration and buckling analysesof functionally graded beams with edge cracks", *Compos. Struct.*, **83**(1), 48-60. https://doi.org/10.1016/j. compstruct.2007.03.006

Yazdi, A.A. (2013), "Homotopy perturbation method for nonlinear vibration analysis of functionally graded plate", *J. Vib. Acoust.*, **135**(2), 021012 (6 pages). https://doi.org/10.1115/1.4023252

Younesian, D., Sadri, M. and Esmailzadeh, E. (2014), "Primary and secondary resonance analyses of clamped-clamped micro-beams", *Nonlinear Dyn.*, **76**, 1867-1884. https://doi.org/10.1007/s11071-014-1254-z

Zaoui, F.Z., Ouinas, D. and Tounsi, A. (2019), "New 2D and quasi-3D shear deformation theories for free vibration of functionally graded plates on elastic foundations", *Compos. Part B*, **159**, 231-247. https://doi.org/10.1016/j.compositesb. 2018.09.051.https://doi.org/10.1007/s11071-010-9790-7

CC