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Dynamic behavior of axially functionally graded simply supported beams

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Abstract. This paper focuses on the free vibration analysis of axially functionally graded (FG) Euler-Bernoulli beams. The material properties of the beams are assumed to obey the linear law distribution. The complexities in solving differential equation of transverse vibration of composite beams which limit the analytical solution to some special cases are overcome using the Differential Transformation Method (DTM). Natural frequencies and corresponding normalized mode shapes are calculated. Validation targets are experimental data or finite element results. Different parameters such as reinforcement distribution, ratio of the reinforcement Young's modulus to the matrix Young's modulus and ratio of the reinforcement density to the matrix density are taken into investigation. The delivered results prove the capability and the robustness of the applied method. The studied parameters are demonstrated to be very crucial for the normalized natural frequencies and mode shapes.

Keywords: differential transformation method; functionally graded material; mode shape; natural frequency

1. Introduction

Functionally graded materials (FGMs) represent a new class of composites that consists of a graded pattern of material composition and/or microstructures. This relatively new advanced composite consists of different materials in order to achieve the desired properties according to the application where the FGM is used (Vatanabe *et al*. 2014, Beldjelili *et al*. 2016). FGMs have acquired a great attention of researchers in the past decade due to their graded properties at every single point in various dimensions (Fourn *et al*. 2018, Karami *et al*. 2018, Zaoui *et al*. 2019). FGMs have exceptional properties compared to traditional homogeneous materials due to the continuous transition of material properties.

FGMs have been developed as ultrahigh temperature resistant materials for aircraft, space vehicles. They have other engineering applications including mechanical, electronics, optics, chemical, biomedical, nuclear, and civil engineering. The increasing use of FG beams as structural components in various fields has necessitated the study of their dynamic behavior. Design of structures based on FGM to resist dynamic forces, such as wind and earthquakes, requires knowledge of their vibration properties particularly their natural frequencies and their mode shapes (Shabana *et al*. 2000, Bouafia *et al*. 2017).

Few analytical solutions are developed for arbitrary gradient change due to the difficulty of mathematical treatment of the problem (Aydogdu 2004).

Elishakoff and Guede (2004) used the semi-inverse method to treat a large class of problems involving graded beams of special forms and obtained explicit fundamental frequency. Li *et al*. (2013) and Tang *et al*. (2014) derived exact frequency equations of free vibration of uniform and non-uniform exponentially functionally graded beams using, respectively, the Euler-Bernoulli and Timoshenko theory. Alireza *et al*. (2017) investigated the dynamic response of functionally graded nanocomposite beams under the action of a moving load. The analysis is carried out by a mesh-free method using the two-dimensional theory of elasticity. Abdelaziz *et al*. (2017) developed and applied a simple hyperbolic shear deformation theory for the bending, vibration and buckling of FG sandwich plate with various boundary conditions. Equations of motion are obtained from Hamilton's principle. Numerical results for the natural frequencies, deflections and critical buckling loads of several types of FG graded sandwich plates under various boundary conditions were presented. Mouffoki *et al*. (2017) considered a novel simple trigonometric shear deformation theory to study the effects of moisture and temperature on free vibration characteristics of FG nanobeams resting on elastic foundation. Abualnour *et al*. (2018) investigated the free vibration analysis of FG plates resting on two-parameter elastic foundations using a hybrid quasi-3D higher-order shear deformation theory. The Governing equations of motion for FG plates were derived from Hamilton's principle and the closed form solutions were obtained by using Navier technique. Younsi *et al*. (2018) proposed higher shear deformation theories (HSDTs) for bending and free vibration of FG plates using hyperbolic shape function where the material properties vary continuously along the thickness direction. The governing equations which consider the effects of both transverse shear and thickness stretching were determined through the Hamilton's principle. The closed form solutions were deduced by employing Navier method. Meksi *et al*. (2019)

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improved a higher shear deformation theory to consider the influence of thickness stretching in FG plates. The kinematic of the improved HSDT was assumed by considering undetermined integral terms in in-plane displacements and a parabolic distribution of the vertical displacement within the thickness. Boutaleb *et al*. (2019) applied the theory of nonlocal elasticity based on the quasi 3D high shear deformation theory to determine the natural frequencies of the nanosize FG plate. The theory of nonlocal elasticity was utilized to examine the impact of the small scale on the natural frequency of the FG rectangular nanoplate.

With the aid of special functions, Huang and Li (2010) solved some free vibration and buckling problems of axially graded beams. Nevertheless, the assumption of non-homogeneity or non-uniformity still has special requirements and is not arbitrary.

Consequently, the application of numerical techniques seems to be inevitable. Various numerical methods are developed in order to calculate the frequencies and mode shapes of beams. Different variational techniques such as Rayleigh Ritz and Galerkin methods had been applied in the past. Due to advancement in computational techniques and availability of software, FE method is quite a less cumbersome than the conventional methods (Kahya and Turan 2017).

The dynamic analysis of through-thickness FG structural elements has been the subject of many researches. However, a number of further issues related to FG beams still need to be investigated. An important question to be addressed in this context consists of studying the vibrational behavior of axially FG beams. Axially FG beam is a special kind of nonhomogeneous FG material structure, whose material properties vary continuously along the axial direction of the beam. It is difficult to obtain precise solutions for the vibration of axially FG beams because of the variable coefficients of the governing equations. In order to solve the previously mentioned question, a semi-analytical method known as differential transformation method is adopted. This method can be applied to a wide range of vibration problems formulated as complex differential equations with different boundary conditions.

In this study, the free vibration of axially functionally graded Euler-Bernoulli beams is considered. Homogenous and linear FG beams are investigated. The differential transform method is used to determine the dimensionless frequencies and the mode shapes. The effect of the reinforcement distribution, the ratio of the reinforcement Young's modulus to the matrix Young's modulus and the ratio of the reinforcement density to the matrix density on the dynamic behavior of such beams is investigated.

This paper is organized as follows. Section 1 gives an introduction about FGM and its application. In the same section a literature review about the methods used previously for the vibration analysis of FG beams is presented. Section 2 provides basic idea of DTM. The formulation of governing differential equation of motion of linearly axially FG Euler-Bernoulli beam is discussed in Section 3. Section 4 presents the numerical results of free vibration of axially FG composite beams. The validation of obtained results and parametric study are discussed in the same section 4. Conclusions are exposed in section 5.

2. Differential transformation method

The DTM is a semi-analytical approach based on Taylor expansion series for solving linear and nonlinear differential equations. This method provides solutions in terms of convergent series with easily computable components (Mohammad *et al*. 2017). It was first used in structural dynamics by Malik and Dang (1998). Then, it was used in various domains like fluid flow, heat transfer problems and nonlinear oscillators' problems (Hassan 2002, Catal 2008, Shahba *et al*. 2013, Sepasgozar *et al*. 2017, Catal *et al*. 2017, Cherif *et al*. 2018).

DTM was used to solve forced vibration differential equations of motion of Euler-Bernoulli beams with different boundary conditions and various dynamic loads (Catal 2012). It was also used to solve the forth order differential equations for critical buckling load of partially embedded and semi-rigid connected pile with shear deformation (Catal 2014). It was reported that good agreement is found between DTM and analytical method. The effectiveness of DTM on free vibrations of axial-loaded Timoshenko beams resting on viscoelastic foundation was investigated by Bozyigit *et al*. (2018). The results of DTM were validated against dynamic stiffness method through several numerical examples an excellent agreement was observed.

Compared to power series solution method, which was adopted by Lin and Hsiao (2001), DTM predicts natural frequencies more accurately. DTM is directly used to solve governing equations and gives the solutions for whole domain. In this method, incorporation of boundary conditions is easily performed. On the contrary, finite element method (FEM) divides the domain into several elements and the accuracy of results depends on the number of elements. Also applying boundary conditions with FEM is not as easy as DTM. Moreover, DTM does not pose any restrictions on both the type of material gradation and the variation of the cross-section profile. Hence it could cover most of the engineering problems dealing with the mechanical behavior of nonuniform and nonhomogenous structures. These advantages have gained attention of several authors (see; Bert and Zeng 2004, Arikoglu and Ozkol 2005, Ozdemir and Kaya 2006).

The conceptual feature of the DTM is to transform the governing differential equations and related boundary conditions as well as continuity conditions into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution.

The differential transform of a sufficiently differentiable function u(x) is defined as Davit *et al*. (2018)

$$U(k) = \frac{1}{k!}\left(\frac{d^k u(x)}{dx^k}\right)_{x_0} \tag{1}$$

where k is a positive integer. In this paper, x_0 is set to zero.

The inverse differential transform is known as a presentation of u(x) by power series

$$u(x) = \sum_{k=0}^{\infty} x^k U(k) \tag{2}$$

From Eqs. (1)-(2), one can obtain

$$U(k) = \sum_{k=0}^{+\infty} \frac{x^k}{k!} \left(\frac{d^k u(x)}{dx^k} \right)_{x=0} \tag{3}$$

3. Formulation of the problem

Let's consider a rectangular beam of uniform cross-section A, width b, depth h, length L, and made of axially FGM. The rectangular Cartesian coordinate axes are used with the x-axis along the geometric centroidal axis. The volume fraction of the reinforcement is assumed to vary continuously along the length direction according to linear-law form

$$v_X = v_L + v_R \frac{X}{L} \tag{4}$$

where v_L is the volume fraction of the reinforcement at the left side of the beam. v_R is taken such that $(v_L+v_R)/2 = 0.2$.

According to the Euler-Bernoulli beam theory, the governing differential equation for free transverse vibration of axially FG beams reads as

$$\rho(X) A \frac{\partial^2 w(X,t)}{\partial t^2} + \frac{\partial^2}{\partial X^2} \left[E(X) I \frac{\partial w(X,t)}{\partial X^2} \right] = 0 \tag{5}$$

where w(X, t) represents the transverse displacement of the beam, E(X) is the modulus of elasticity, I is moment of inertia and ρ(X) is the density of the beam material. X is the distance from the left end of the beam.

Assuming the transverse displacement of the beam as follows, where ω is the circular natural frequency

$$w(X,t) = Y(X)\, exp(\, i\omega t) \tag{6}$$

Substituting the expression of the transverse displacement into the governing differential equation, Eq. (5) takes the form

$$-\rho(X) A Y(X) \omega^2 + \frac{d^2}{dX^2} \left[E(X) I \frac{d^2 Y(X)}{dX^2} \right] = 0 \tag{7}$$

which can be written

$$\frac{d^4 Y(X)}{dX^4} + 2I \frac{dE(X)}{dX} \frac{d^3 Y(X)}{dX^3} + \frac{d^2 E(X)}{dX^2} \frac{d^2 Y(X)}{dX^2} = \rho(X) A \omega^2 Y(X) \tag{8}$$

For simply supported beam, the corresponding boundary conditions are

$$w(0,t) = 0; \qquad \frac{\partial^2 w(0,t)}{\partial X^2} = 0; \tag{9}$$

$$w(L,t) = 0; \qquad \frac{\partial w^2(L,t)}{\partial X^2} = 0 \tag{9}$$

After substituting the expression of the transverse displacement into the boundary condition equations, Eq. (9) takes the form

$$Y(0) = 0; \quad \frac{\partial^2 Y(0)}{\partial X^2} = 0; \quad Y(L) = 0; \quad \frac{\partial^2 Y(L)}{\partial X^2} = 0 \tag{10}$$

For the convenience of deduction and calculation, the dimensionless parameters are defined

$$x = \frac{X}{L}, \quad u(x) = \frac{Y(X)}{L}, \quad S(X) = \frac{E(X)}{E_L}, \quad p(x) = \frac{\rho(X)}{\rho_L}$$

The Equation of motion (Eq. (8)) can be rewritten in terms of dimensionless variables as

$$p(x)\Omega^2 u(x) = S(x) \frac{d^4 u(x)}{dx^4} + 2I \frac{dS(x)}{dx} \frac{d^3 u(x)}{dx^3} + \frac{d^2 S(x)}{dx^2} \frac{d^2 u(x)}{dx^2} \tag{11}$$

where $\Omega^2 = \omega^2 \frac{\rho_0 A L^4}{E_0 I}$.

with E_0 is the matrix Young's modulus and p_0 is the matrix density.

The material properties of the beam at a distance X is

$$P_X = v_X P_r + (1 - v_X) P_m \tag{12}$$

where P stands for the Young's modulus or the density. Then

$$P(x) = (v_L + v_R x) P_r + (1 - (v_L + v_R x)) P_m \tag{13}$$

which follows

$$S(x) = fx + g \tag{14}$$

where

$$f = \frac{(E_r - E_m) v_R}{v_L E_r + (1 - v_L E_m)}$$

$$g = \frac{(E_r - E_m) v_L}{v_L E_r + (1 - v_L E_m)} + \frac{E_m}{v_L E_r + (1 - v_L E_m)}$$

and

$$p(x) = mx + e \tag{15}$$

where

$$m = \frac{(\rho_r - \rho_m) v_R}{v_L \rho_r + (1 - v_L \rho_m)}$$

$$e = \frac{(\rho_r - \rho_m) v_L}{v_L \rho_r + (1 - v_L \rho_m)} + \frac{\rho_m}{v_L \rho_r + (1 - v_L \rho_m)}$$

The dimensionless boundary conditions can be expressed as follows

$$u(0) = 0, \quad \frac{d^2 u}{\mathrm{dx}^2}(0) = 0, \tag{16}$$

$$u(1) = 0, \quad \frac{d^2u}{dx^2}(1) = 0 \tag{16}$$

Applying the principle of differential transformation method to the nondimensional governing equation (Eq. (11)), the following recurrence relations are obtained:

For uniform reinforcement distribution

$$U_{k+4} = \Omega^2 \frac{U_k}{(k+1)(k+2)(k+3)(k+4)} \tag{17}$$

For linear reinforcement distribution

$$U_{k+4} = \Omega^2 \frac{eU_k + mU_{k-1}}{g(k+1)(k+2)(k+3)(k+4)} - \frac{f}{g}\frac{k+2}{k+4}U_{k+3} \tag{18}$$

Applying the Differential Transform Method to the non-dimensional boundary conditions equations, Eq. (16) yields

$$\begin{aligned}
u(0) &= \sum_{k=0}^{+\infty} U(k)x^k \\
&= U(0)x^0 + U(1)x^1 + U(2)x^2 + \cdots = 0 \\
u''(0) &= \sum_{k=0}^{+\infty} (k+1)(k+2)U(k+2)x^k \\
&= (1)(2)U(2)x^0 + (2)(3)U(3)x^1 \\
&\quad +(3)(4)U(4)x^2 + (4)(5)U(5)x^3 + \cdots = 0 \\
u(1) &= \sum_{k=0}^{+\infty} U(k)x^k \\
&= U(0)x^0 + U(1)x^1 + U(2)x^2 + \cdots = 0 \\
u''(1) &= \sum_{k=0}^{+\infty} (k+1)(k+2)U(k+2)x^k \\
&= (1)(2)U(2)x^0 + (2)(3)U(3)x^1 \\
&\quad +(3)(4)U(4)x^2 + (4)(5)U(5)x^3 + \ldots = 0
\end{aligned} \tag{19}$$

which gives

$$\begin{aligned}
&U(0) = 0 \\
&U(2) = 0 \\
&\sum_{k=0}^{+\infty} U(k) = 0 \\
&\sum_{k=0}^{+\infty} k(k+1)U(k) = 0
\end{aligned} \tag{20}$$

U(1) and U(3) are set to unknown constants: U(1) = c and U(3) = d.

Both Eqs. (20c)-(20d) give nonlinear equation in terms of Ω and linear in terms of c and d.

Putting the boundary condition equations in matrix form, we get

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{21}$$

Where A and C are the coefficients of c, B and D are the coefficients of d.

Since c and d are different to zero, the determinant of matrix must be equal to zero. Hence

$$A.D - B.C = 0 \tag{22}$$

Depending upon the number of terms N taken, a higher degree polynomial in Ω can be gotten. The solution of the polynomial equation gives the dimensionless frequency Ω.

4. Numerical results

In this section, firstly the convergence and the validation of the DTM are examined. Afterwards, the first three frequency parameters for the free vibration of FG beam are investigated. The reinforcement volume fraction is kept constant equal to 20%. The effect of the reinforcement distribution, the ratio of the reinforcement Young's modulus to the matrix Young's modulus and the ratio of the reinforcement density to the matrix density are investigated.

4.1 Convergence and validation of the analysis

4.1.1 Convergence analysis

In order to check the convergence of the DTM, the case of 20% volume fraction homogenous beams is considered. For this case, the natural frequencies can be exactly calculated. Evaluated results of first three non-dimensional natural frequencies are tabulated in Table 1.

Table 1 shows that the numerical results have rapid convergence. It can be seen that the first non-dimensional frequency converges first (N = 25) then the second one (N = 35) and finally the third non-dimensional frequency which requires relatively high number of terms (N = 45). It is observed that increasing the number of terms, N, improves the accuracy of results and leads to convergent solutions at N = 45. Hence, N = 45 is used in the following numerical calculations.

4.1.2 Validation of the analysis

The validation of the analysis is done through direct comparison with exact solution for homogeneous beams and with finite element results obtained by using commercial finite element software package (ANSYS2013).

Table 1 The first three non-dimensional frequencies of simply supported FG beam for different number of terms N

N	Ω_1	Ω_2	Ω_3
19	9.8696044699	39.4169139196	-
20	9.8696044699	39.4169139196	-
25	9.8696044011	39.4784501712	87.8912222720
30	9.8696044011	39.4784176959	88.8107229014
35	9.8696044011	39.4784176044	88.8264496362
40	9.8696044011	39.4784176044	88.8264396509
45	9.8696044011	39.4784176044	88.8264396098

4.1.2.1 Comparison with available results

To check the above written formulation, the first three frequency parameters of homogeneous beams are compared with exact solution.

From Table 1, one can conclude that the estimated first three normalized frequency using DTM (Ω_1 = 9.8696044011, Ω_2 = 39.4784176044, Ω_3 = 88.8264396098) are identical to the exact results, which prove the efficiency of the present approach.

4.1.2.2 Modeling of axially FG beams using ANSYS

As already mentioned, the material properties of the FG beam vary linearly throughout its longitudinal axis. The ratio of the reinforcement Young's modulus to the matrix Young's modulus and the ratio of the reinforcement density to the matrix density are taken 50 and 0.5, respectively. The beam left side is reinforced with 15% whereas its right side is reinforced with 25%. In order to model the FG beam, the numerical model has been divided into several layers so that the changes in properties can be made. Each layer has the finite portion of the beam length and treated like isotropic material. Material properties of each layer have been calculated at its mid-plane by using linear distribution. To study the convergence of the analysis, various number of layers has been taken; 2, 4, 6, 8 and 10. The FE modeling has been performed using ANSYS (2013). Higher order 3-D, 10-node elements (SOLID187) has been used for modeling of FG beams. SOLID187 has a quadratic displacement behavior. This element has three degrees of freedom at each node: translations in the nodal x, y, and z directions. To simulate the pin support, the x, y and z bottom edge displacements are constrained while the roller support is free to move in the axial direction. 126064 elements with 196724 nodes are needed.

Table 2 reports the fundamental frequency parameters delivered by the FE analysis. r_E and r_ρ denote the ratio of the reinforcement Young's modulus to the matrix Young's modulus and the ratio of the reinforcement density to the matrix density; respectively.

Table 2 FE predictions of fundamental frequency parameters for v_L = 0.15, r_E = 50 and r_ρ = 0.5

N	2	4	6	8	10
Ω_1	33,92285	33,99842	34,00338	34,004	34,01081

Interpretation: It can be interpreted that the number of layers has a great influence on the fundamental frequencies of the modeled FG beam. From Table 2, one can consider that the convergence is reached for 10 layers. The FE results gotten for ten layers beams are confronted against those obtained using DTM. It is found that the fundamental frequency computed by DTM (34.05) is comparable with FE predictions. The satisfactory results concerning the frequency parameters give confidence in the predictions reported in next sections.

4.2 Effect of the reinforcement distribution

In this subsection, keeping the ratio of the reinforcement Young's modulus to the matrix Young's modulus and the ratio of the reinforcement density to the matrix density constant, three distribution cases are considered.

Case 1: The reinforcement volume fraction varies from 15% on the beam left side to 25% on its right side.

Case 2: The reinforcement volume fraction varies from 17.5% on the beam left side to 22.5% on its right side.

Case 3: Uniform distribution with 20% of the reinforcement.

For each case, the three first non-dimensional natural frequencies are calculated and listed in Tables 3-5.

It is observed from Tables 3-5 that by increasing the reinforcement volume fraction on the beam left side, the normalized fundamental frequency of simply supported beams increase whatever the values of the ratio r_E. On the contrary, the evolution of the second and the third normalized frequencies depends on the value of r_E. Ω_2 and Ω_3 decrease when increasing the reinforcement volume fraction for reinforcement having the same Young's modulus as the matrix (r_E = 1) and increase with the reinforcement volume fraction for any given value of (r_E > 1).

4.3 Effect of the ratio of the reinforcement density to the matrix density

In order to determine the effect of the ratio of the reinforcement density to the matrix density on the

Table 3 First three frequency parameters of simply supported FG beams with r_ρ = 0.5

r_E	v_L (%)	Ω_1	Ω_2	Ω_3
1	15	9,009607248560	36,041863758150	81,095212073456
	17.5	9,009658016957	36,039490739947	81,089108386785
	20	9,009674940214	36,038699760859	81,087074461928
5	15	12,079173477530	48,311976845915	108,704337886703
	17.5	12,085608672506	48,341264108489	108,758715681274
	20	12,087747372898	48,350989491592	108,789726356076
50	15	29,510957596060	117,938172657652	267,426592103038
	17.5	29,584569936774	118,312963025071	266,227830687296
	20	29,608813203268	118,435252813072	266,479318829399

Table 4 First three frequency parameters of simply supported FG beams with $r_\rho = 1$

r_E	v_L (%)	Ω_1	Ω_2	Ω_3
1	15	9,869604401089	39,478417604357	88,826439609799
	17.5	9,869604401089	39,478417604357	88,826439609799
	20	9,8696044010893	39,478417604357	88,826439609799
5	15	13,230042637945	52,906316206196	119,057598224092
	17.5	13,238614404079	52,951005182011	119,140129672886
	20	13,241463811120	52,965855244480	119,173174300074
50	15	32,317181860740	129,121254338507	294,384671746423
	17.5	32,405672540061	129,587319094395	291,598595055664
	20	32,434829784773	129,739319139093	291,913468062951

Table 5 First three frequency parameters of simply supported FG beams with $r_\rho = 2$

r_E	v_L (%)	Ω_1	Ω_2	Ω_3
1	15	9,009607248560	36,041863758150	81,095212073456
	17.5	9,009658016957	36,039490739947	81,089108386785
	20	9,009674940214	36,038699760859	81,087074461928
5	15	12,079173477530	48,311976845915	108,704337886703
	17.5	12,085608672506	48,341264108489	108,758715681274
	20	12,087747372898	48,350989491592	108,789726356076
50	15	29,510957596060	117,938172657652	267,426592103038
	17.5	29,584569936774	118,312963025071	266,227830687296
	20	29,608813203268	118,435252813072	266,479318829399

Table 6 First three frequency parameters of simply supported FG beams for $v_L = 15\%$

r_E	r_ρ	Ω_1	Ω_2	Ω_3
1	0.5	10,403441764192	41,615529749753	93,635464033999
	1	9,869604401089	39,478417604357	88,826439609799
	2	9,009607248560	36,041863758150	81,095212073456
5	0.5	13,440293434133	53,760558409946	120,976474353301
	1	13,230042637945	52,906316206196	119,057598224092
	2	12,079173477530	48,311976845915	108,704337886703
50	0.5	34,057676260109	136,067352933289	311,778015135180
	1	32,317181860740	129,121254338507	294,384671746423
	2	29,510957596060	117,938172657652	267,426592103038

normalized natural frequencies, different values of r_ρ are taken in consideration (Tables 6-8).

For different reinforcement distributions and Young's modulus ratios, one can see from Tables 6-8 that an increment in density ratio r_ρ leads to decrement in value of first thee normalized frequencies.

4.4 Effect of the ratio of the reinforcement Young's modulus to the matrix Young's modulus

The effect of the ratio of the reinforcement Young's modulus to the matrix Young's modulus on the normalized natural frequency is investigated in this section. r_ρ is taken equal to 0.5 and the reinforcement volume fraction incorporated in the beam is assumed to vary from 15% on the beam left side to 25% on its right side.

Table 9 gives results of the first three non-dimensional frequencies for different values of r_E.

It can be demonstrated from Table 9 that the first three normalized frequencies of the FG beam increase with increasing r_E. For FG beam comprising 15%, on its left side, of reinforcement 50 times stiffer than the matrix, Ω_1, Ω_2, and Ω_3 are increased by 3.27, 3.27 and 3.33; respectively.

Table 7 First three frequency parameters of simply supported FG beams for v_L = 17.5%

r_E	r_ρ	Ω_1	Ω_2	Ω_3
1	0.5	10,403467819035	41,614311945085	93,632332302055
	1	9,869604401089	39,478417604357	88,826439609799
	2	9,009658016957	36,039490739947	81,089108386785
5	0.5	13,954339779884	55,813672886482	125,592757579427
	1	13,238614404079	52,951005182011	119,140129672886
	2	12,085608672506	48,341264108489	108,758715681274
50	0.5	34,156679995006	136,587687248232	307,351477810235
	1	32,405672540061	129,587319094395	291,598595055664
	2	29,584569936774	118,312963025071	266,227830687296

Table 8 First three frequency parameters of simply supported FG beams for v_L = 20%

r_E	r_ρ	Ω_1	Ω_2	Ω_3
1	0.5	10,403476504088	41,613906016352	93,631288536787
	1	9,869604401089	39,478417604357	88,826439609799
	2	9,009674940214	36,038699760859	81,087074461928
5	0.5	13,957728399277	55,830913597111	125,619555593492
	1	13,241463811120	52,965855244480	119,173174300074
	2	12,087747372898	48,350989491592	108,789726356076
50	0.5	34,189312546584	136,757250186337	307,703812919244
	1	32,434829784773	129,739319139093	291,913468062951
	2	29,608813203268	118,435252813072	266,479318829399

Table 9 First three frequency parameters of simply supported FG beams with v_L = 15% and r_ρ = 0.5

r_E	Ω_1	Ω_2	Ω_3
1	10,403441764192	41,615529749753	93,635464033999
2	11,394563142540	45,577731083593	102,549672668279
5	13,440293434133	53,760558409946	120,976474353301
10	17,374077046874	69,456414178960	155,906064176076
20	22,725681062282	90,825568376213	200,675656388715
30	27,036424352765	108,049031072586	233,356149899951

4.5 Mode shapes

The effect of the reinforcement distribution, the density ratio and the Young's modulus ratio on the amplitude of vibration are calculated and shown in Figs. 1-5.

Fig. 1 shows that the effect of the volume fraction on the first, second and third mode shapes is nearly the same. The amplitudes of vibration of simply supported beams associated to the three mode shapes increase when increasing the reinforcement volume fraction.

Mode shapes for different density ratios are plotted in Fig. 2. It is seen that the effect of r_ρ on the amplitude of vibration for the three mode shapes is negligible. It can be also noted that the amplitudes of vibration of homogeneous beam corresponding to the three mode shapes are higher than that of axially FG beam with 15% of reinforcement having r_E = 50.

Figs. 3-5 show the three mode shapes of simply supported axially FG beams with different Young's modulus ratios for v_L = 15% and r_ρ = 0.5.

From Figs. 3-5, it can be seen that the value of r_E (1, 2, 5, 10, 20, 30, 40, and 50) has a significant effect on the mode shapes and the deflection values. The deflection of simply supported FG beams decreases as r_E increases.

5. Conclusions

The DTM approach has been demonstrated to be an effective technique to solve the free vibration of Euler-Bernoulli axially FG beams. The effectiveness of the method has been confirmed by comparing DTM predictions with existing results and performed FE data. Based on the numerical results, it is found that the dimensionless frequencies and mode shapes are highly sensitive to the reinforcement volume fraction and to the Young's modulus ratio. The effect of the density ratio on the free vibration of simply supported FG beam is found to be negligible. The normalized frequencies of FG beams increase with increasing the Young's modulus ratio. It is also observed that the deflection of the axially FG beam depends closely on the studied parameters.

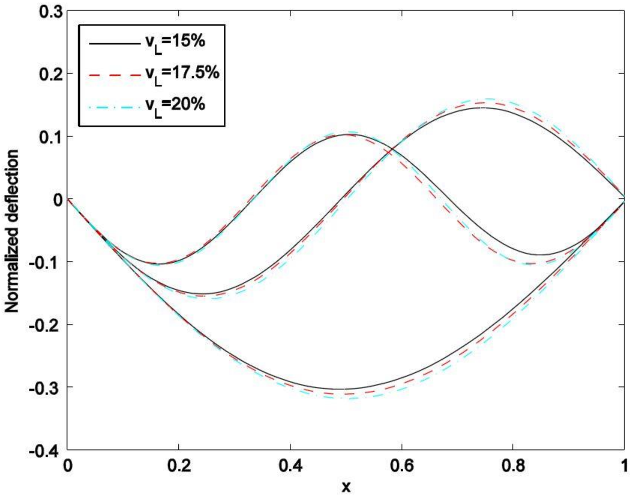


Fig. 1 The first three mode shapes for $r_E = 50$ and $r_\rho = 0.5$

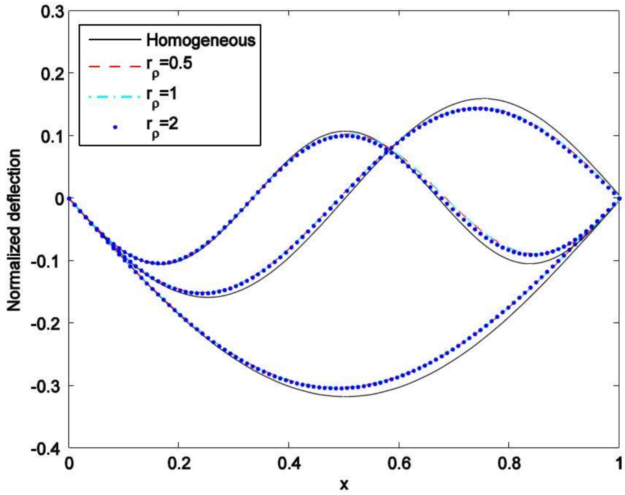
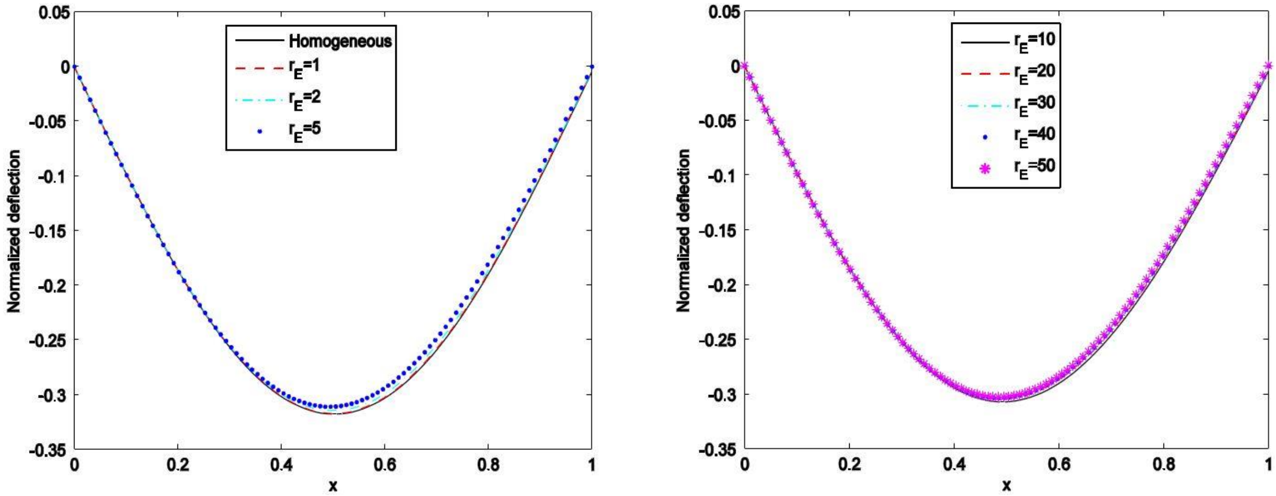


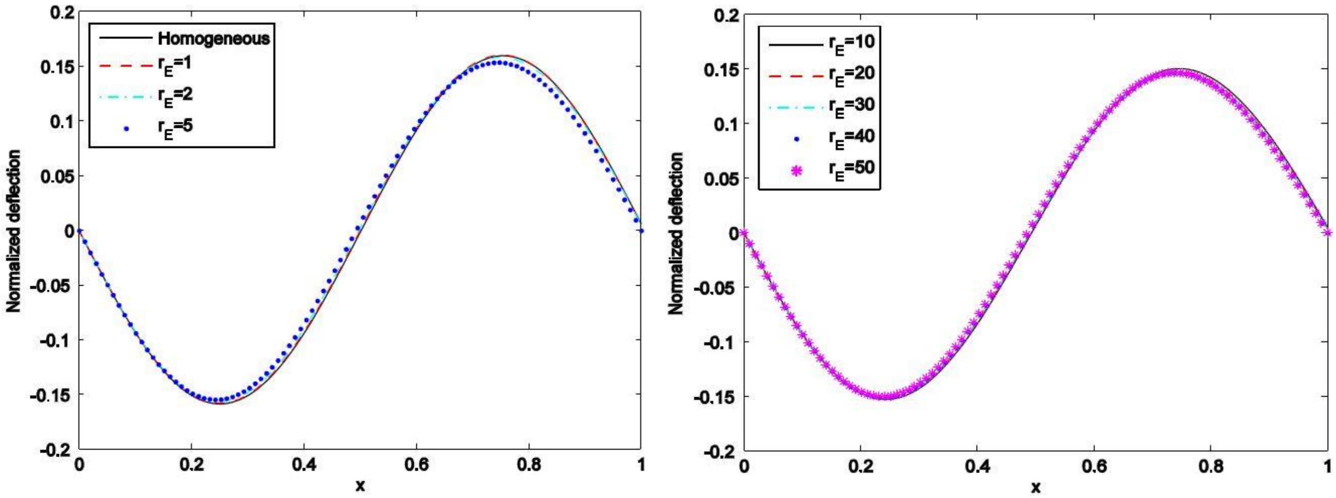
Fig. 2 The first three mode shapes for $v_L = 15\%$ and $r_E = 50$



(a) $r_E = (1, 2, 5)$

(b) $r_E = (10, 20, 30, 40, 50)$

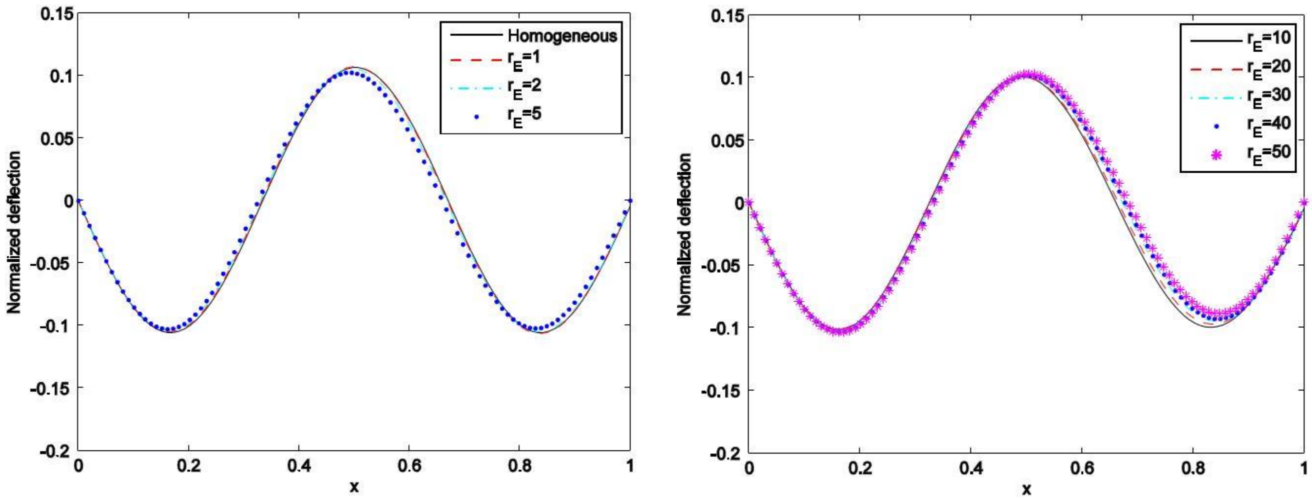
Fig. 3 The first mode shape for $v_L = 15$ and $r_\rho = 0.5$



(a) $r_E = (1, 2, 5)$

(b) $r_E = (10, 20, 30, 40, 50)$

Fig. 4 The second mode shape for $v_L = 15$ and $r_\rho = 0.5$



(a) r_E = (1, 2, 5)

(b) r_E = (10, 20, 30, 40, 50)

Fig. 5 The third mode shape for v_L = 15% and r_ρ = 0.5

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