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Smart analysis of doubly curved piezoelectric nano shells: Electrical and mechanical buckling analysis

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Abstract. Stability analysis of three-layered piezoelectric doubly curved nano shell with accounting size dependency is performed in this paper based on first order shear deformation theory and curvilinear coordinate system relations. The elastic core is integrated with sensor and actuator layers subjected to applied electric potentials. The principle of virtual work is employed for derivation of governing equations of stability. The critical electrical and mechanical buckling loads are evaluated in terms of important parameters of the problem such as size-dependent parameter, two principle angle of doubly curved shell and two parameters of Pasternak's foundation. One can conclude that mechanical buckling loads are decreased with increase of nonlocal parameter while the electrical buckling loads are increased.

Keywords: stability analysis; electrical and mechanical buckling loads; three-layered nano shells; doubly curved piezoelectric; size dependent parameter; first-order shear deformation theory; applied electric potential

1. Introduction

Analysis of structures in very small scale (micro or nano scales) has enforced researchers to find new non-classical theories to cover prediction of behavior of those in various environments. These theories were presented to account size-dependency in the constitutive relations. Advances in development of non-classical theories leads to various theories in micro and nano scales for better prediction of behavior of small scale structures. Eringen nonlocal elasticity theory, modified couple stress theory, strain gradient theory and nonlocal strain gradient theory have been proposed for analysis of structures in nano and micro scales. Although application of above mentioned theories to the custom structures such as rods, beams and plates has been presented by various researchers, analysis of non-flat structures such as curved beam and doubly curved shell has not been performed comprehensively. Literature review on the subject of paper is presented to justify the novelties and necessities of this study.

Kapania and Yang (1986) studied post-buckling analysis of an imperfect doubly curved shell. The influence of various aspect ratio and imperfections was studied on the results. Fan and Zhang (1992) used curvilinear coordinate system to study static and dynamic analysis of the simply supported orthotropic doubly curved shells based on a unified analytical solution for thin, moderately thick, and thick laminated shells. Vibration analysis of geometrically imperfect single and multilayered composite double-curved shallow panels subjected to transverse loads and various in-lane boundary conditions was studied by Librescu and Chang (1993). They studied influence of transverse shear deformations, lamination and various in-plane boundary conditions on the responses of doubly curved shell. Wu and Liu (2007) presented three dimensional piezo-elasticity formulations of simply-supported doubly curved functionally graded elastic and piezoelectric shells based on state space approach. Using successive approximation method, the shell was divided into a multilayered shell with small thickness. Chandrashekhar (1989) presented free vibration analysis of laminated composite doubly curved shells. First order shear deformation theory was used for description of displacement field. Influence of in-plane and rotary inertia was accounted in the elements of mass matrix. The influence of shell geometry, orientation of layers, material parameters, and boundary conditions was studied on the free vibration responses of doubly curved shells. Arefi and Zenkour (2019b) studied the influence of thermo-magneto-electro-mechanical loads on the static analysis of a three-layered nanoplate. Sinusoidal shear-deformation plate theory was used for formulation of the problem and principle of virtual displacement was employed for derivation of the governing equations.

Qatu and Asadi (2012) presented free vibration analysis of a thin shallow shell with various boundary conditions based on Ritz method. The influence of various parameters such as different boundary conditions and radii of curvature was studied on the responses. Size-dependent free vibration analysis of orthotropic doubly-curved shallow shells with simply-supported boundary conditions was studied by Ghavanloo and Fazelzadeh (2013) based on strain gradient theory and Novozhilov’s linear shallow shell theory. The various length scale parameters were employed based on strain gradient theory for better prediction of behavior of small scale structure. Shooshtari and Razavi (2015) studied

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nonlinear and linear free vibration analyses of laminated magneto-electro-elastic doubly-curved thin shell with simply supported curved edges resting on an elastic foundation based on Donnell's shell theory. The numerical results were calculated based on Lindstedt-Poincare perturbation method. The influence of parameters of foundation, geometrical characteristics and electric and magnetic potentials was studied on the linear and nonlinear behavior of these smart shells. Nonlinear vibrations of doubly curved cross-ply shells with simply supported boundary conditions were studied by Yazdi (2013) based on von-Karman geometric nonlinear theory using Donnell's shell equations. The nonlinear governing equations of motion were reduced to a second order nonlinear ordinary differential equation using Galerkin approach and then solved using homotopy perturbation method. Zhang *et al*. (2001) studied the influence of impact load on the dynamic stability (buckling and post-buckling) of thin-walled doubly curved shells.

First order shear deformation theory was used by Sharma *et al*. (2013) to present governing equations of laminated composite doubly curved panels subjected to uniformly transverse loads. The effect of panel thickness, curvature, boundary conditions, lamination scheme and material property was studied on the static response of panel. Shen (2002) and Chen *et al*. (2017a) focused on the vibration and buckling analysis of composite structures in thermal environment with considering nonlinear strains. Thakur *et al*. (2017) studied analysis of a doubly curved composite shells based on higher order shear deformation theory. Pouresmaeeli and Fazelzadeh (2016) studied vibration analysis of doubly curved FG composite panels reinforced by carbon nanotube based on first-order shear deformation theory and Galerkin's method. The vibration responses were calculated in terms of important parameters such as volume fraction of carbon nanotubes, thickness ratio, aspect ratio, curvature ratio and shallowness ratio. Veysi *et al*. (2017) studied the nonlinear size dependent analysis of doubly curved micro shell based on modified couple stress theory, von-Karman geometric nonlinear relations and first-order shear deformation theory. They employed multiple scales method to solve governing equations of motion. Influence of various micro length scale parameters was studied on the responses. They mentioned that the effect of shell dimensions on the vibration characteristics of micro shell is strongly depending on the type of shell (this is important for spherical shell and not important for hyperbolic paraboloidal shells). Arefi and Zenkour (2019a) used sinusoidal shear deformation theory for thermo-magneto-electro-elastic analysis of a three layered curved nanobeam. Nonlocal elasticity relations and Hamilton's principle was employed for derivation of the governing equations of motion. They mentioned that applied electric and magnetic potential leads to important changes of responses. The sandwich structure was made from a nano core and two piezomagnetic face-sheets. Influence of nonlocal parameter, applied electric and magnetic potentials and two parameters of Pasternak's foundation was studied on the responses of the system. The numerical results indicate that increase of nonlocal parameters leads to decrease of stiffness of structure. Some important works on the stability analysis of structures are observed in Reference (Chen *et al*. 2017b). Hamdia *et al*. (2018) provided a sensitivity analysis for identification of key input parameters affecting energy conversion factor of flexoelectric materials. The numerical results indicated that the flexoelectric constants are the most dominant factors influencing the uncertainties in the energy conversion factor. Some related works to optimization and computational methods of flexoelectric and piezoelectric structures were studied by various researchers (Yeh 2014, Zehetner and Irschik 2008, Karami and Shahsavari 2019).

Some important numerical methods have been developed by researchers to cover wide range of engineering problems. Rabczuk *et al*. (2019) studied application of a novel nonlocal operator theory for solution of partial differential equations based on variational principle. The proposed formulation had capability to solve the differential electromagnetic vector wave equations based on electric fields. Guo *et al*. (2019) proposed a deep collocation method for thin plate bending problems. A loss function was built with the aim that the governing partial differential equations of Kirchhoff plate bending problems, and the boundary/initial conditions were minimized at those collocation points. Anitescu *et al*. (2019) presented application of artificial neural networks and an adaptive collocation strategy for solving partial differential equations. They showed capability of their solution method in classical problems such as Poisson and Helmholtz equations.

A comprehensive literature review on the various types of shells especially doubly curved shells and various size dependent theories has been completed in Introduction. Based on the best author's knowledge and complete review on the previous related works, it is confirmed that there is no published works on the calculation of mechanical and electrical buckling loads of doubly curved piezoelectric nano shells. The novelties of the present paper are application of nonlocal piezoelasticity relations and shear deformation theory to mechanical and electrical buckling loads of doubly curved piezoelectric nano shells. The principle of virtual work is applied to derive governing equations for a doubly curved piezoelectric nano shell. The mechanical and electrical buckling loads are calculated in terms of significant parameters of the problem such as nonlocal parameter, two angles of doubly curved nano shell and two parameters of Pasternak's foundation.

2. Stability formulation of piezoelectric doubly curved nano shells

The stability formulation of shear deformable doubly curved shell made of piezoelectric materials based on piezoelasticity relations, nonlocal elasticity theory and curvilinear coordinate system is derived in this section. The three-layered doubly curved nanoshell is composed of an elastic core and two piezoelectric nanoshell. Shown in Fig. 1 is a doubly curved piezoelectric nanoshell subjected to electrical loads. In this figure, α, β, z are employed

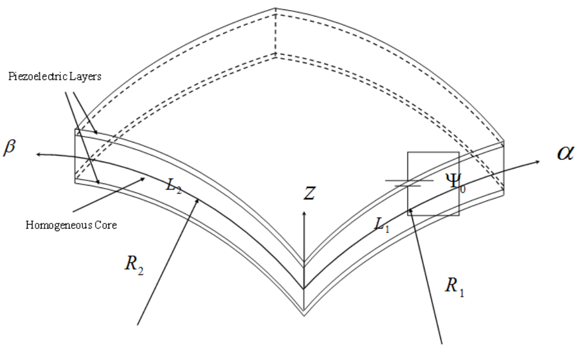

Fig. 1 The schematic of a doubly curved piezoelectric shell

coordinates along two planar and thickness directions. In addition, two principle radii of curvature are depicted with R_1, R_2 and the lengths of middle surfaces are defined with L_1, L_2.

First order shear deformation theory is developed for kinematic relations of deformations. Based on this theory the deformation at a point is linear function of z coordinate. The strain components based on first order shear deformation theory are defined as follows

$$\varepsilon_\alpha = \varepsilon_1^0 + zk_1, \tag{1a}$$

$$\varepsilon_\beta = \varepsilon_2^0 + zk_2, \tag{1b}$$

$$\gamma_{\beta z} = \varepsilon_4^0, \tag{1c}$$

$$\gamma_{\alpha z} = \varepsilon_5^0, \tag{1d}$$

$$\gamma_{\alpha\beta} = \varepsilon_6^0 + zk_6, \tag{1e}$$

in which the defined variables in Eq. (1) are expressed as (Arefi 2018a, b)

$$\varepsilon_1^0 = \frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\frac{\partial u}{\partial \alpha} + \frac{w}{R_1}, \tag{2a}$$

$$\varepsilon_2^0 = \frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\frac{\partial v}{\partial \beta} + \frac{w}{R_2}, \tag{2b}$$

$$\varepsilon_6^0 = \frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\frac{\partial v}{\partial \alpha} + \frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\frac{\partial u}{\partial \beta}, \tag{2c}$$

$$\varepsilon_4^0 = \phi_2 + \frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\frac{\partial w}{\partial \beta} - \frac{v}{R_2}, \tag{2d}$$

$$\varepsilon_5^0 = \phi_1 + \frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\frac{\partial w}{\partial \alpha} - \frac{u}{R_1}, \tag{2e}$$

$$k_1 = \frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\frac{\partial \phi_1}{\partial \alpha} \tag{2f}$$

$$k_2 = \frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\frac{\partial \phi_2}{\partial \beta} \tag{2g}$$

$$\begin{aligned} k_6 = &\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\frac{\partial \phi_2}{\partial \alpha} + \frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\frac{\partial \phi_1}{\partial \beta} \\ &+\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\frac{\partial v}{\partial \alpha} - \frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\frac{\partial u}{\partial \beta}\right), \end{aligned} \tag{2h}$$

in which u, v, w are displacements of middle surface of piezoelectric doubly curved nano shell and ϕ_1, ϕ_2 are rotation functions about β, α directions, respectively.

Based on the assumed strain field (Eq. (1)), the displacement field is continuous and no discontinuity is occurred between core and integrated layers.

The piezoelectric layers are subjected to applied electric potential. Electric potential distribution is assumed as follows (Arefi *et al*. 2018, Arefi and Zenkour 2017a-c)

$$\breve{\Psi} = \frac{2z}{h}\Psi_0 - \Psi(\alpha,\beta)cos\frac{\pi z}{h}, \tag{3}$$

In which Ψ_0 is applied electric potential and $\Psi(\alpha,\beta)$ is two-dimensional distribution of electric potential along α,β directions. Based on Eq. (3), the first term is represented the applied electric potential and the second term applies for homogeneous conditions at four boundaries and also top and bottom. Electric field components are derived using electric potential distribution as follows

$$E_\alpha = \frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\frac{\partial \Psi}{\partial \alpha}\cos\frac{\pi z}{h}, \tag{4a}$$

$$E_\beta = \frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\frac{\partial \Psi}{\partial \beta}\cos\frac{\pi z}{h}, \tag{4b}$$

$$E_z = -\frac{2}{h}\Psi_0 - \frac{\pi}{h}\Psi sin\frac{\pi z}{h}, \tag{4c}$$

The nonlocal constitutive relations for elastic nano core are expressed as (Arefi and Zenkour 2017a ,b)

$$(1-(e_0a)^2\nabla^2)\begin{Bmatrix}\sigma_\alpha\\ \sigma_\beta\\ \sigma_{\beta z}\\ \sigma_{\alpha z}\\ \sigma_{\alpha\beta}\end{Bmatrix} = \begin{bmatrix} {C_{11}}^c & {C_{12}}^c & 0 & 0 & 0\\ {C_{12}}^c & {C_{22}}^c & 0 & 0 & 0\\ 0 & 0 & {C_{44}}^c & 0 & 0\\ 0 & 0 & 0 & {C_{55}}^c & 0\\ 0 & 0 & 0 & 0 & {C_{66}}^c\end{bmatrix}\begin{Bmatrix}\varepsilon_\alpha\\ \varepsilon_\beta\\ \gamma_{\beta z}\\ \gamma_{\alpha z}\\ \gamma_{\alpha\beta}\end{Bmatrix}, \tag{5}$$

in which e_0a is nonlocal parameter, ${C_{ij}}^c$ stiffness coefficients of core and σ_i, ε_i are stress and strain components (superscript c indicates that this property is related to core). In addition, the nonlocal stress-strain relations based on nonlocal piezo-elasticity relations for doubly curved piezoelectric layers are expressed as

$$(1-(e_0a)^2\nabla^2)\begin{Bmatrix}\sigma_\alpha\\ \sigma_\beta\\ \sigma_{\beta z}\\ \sigma_{\alpha z}\\ \sigma_{\alpha\beta}\end{Bmatrix} = \begin{bmatrix} {C_{11}}^p & {C_{12}}^p & 0 & 0 & 0\\ {C_{12}}^p & {C_{22}}^p & 0 & 0 & 0\\ 0 & 0 & {C_{44}}^p & 0 & 0\\ 0 & 0 & 0 & {C_{55}}^p & 0\\ 0 & 0 & 0 & 0 & {C_{66}}^p\end{bmatrix}\begin{Bmatrix}\varepsilon_\alpha\\ \varepsilon_\beta\\ \gamma_{\beta z}\\ \gamma_{\alpha z}\\ \gamma_{\alpha\beta}\end{Bmatrix} \tag{6}$$

$$-\begin{bmatrix} 0 & 0 & e_{13} \\ 0 & 0 & e_{23} \\ 0 & e_{42} & 0 \\ e_{51} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_\alpha \\ E_\beta \\ E_z \end{Bmatrix} \quad (6)$$

In which e_{ij} are piezoelectric coefficients, and E_i electric field components (superscript p indicates that this property is related to face-sheets). The electric displacement relations for piezoelectric layers are expressed as

$$\begin{Bmatrix} D_\alpha \\ D_\beta \\ D_z \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & e_{51} & 0 \\ 0 & 0 & e_{42} & 0 & 0 \\ e_{13} & e_{23} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \gamma_{\beta z} \\ \gamma_{\alpha z} \\ \gamma_{\alpha\beta} \end{Bmatrix}, + \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix} \begin{Bmatrix} E_\alpha \\ E_\beta \\ E_z \end{Bmatrix} \quad (7)$$

In which k_{ij} are dielectric coefficients. In addition, E_i is electric field components derived using divergence of electric potential.

The principle of virtual work is used to derive governing equations of stability problem. Strain energy U is defined as follows

$$U = \frac{1}{2}\int_\alpha \int_\beta \int_z \begin{bmatrix} \sigma_\alpha \varepsilon_\alpha + \sigma_\beta \varepsilon_\beta + \sigma_{yz}\gamma_{yz} + \sigma_{xz}\gamma_{xz} \\ +\sigma_{xy}\gamma_{xy} - D_\alpha E_\alpha - D_\beta E_\beta - D_z E_z \end{bmatrix} dV \quad (8)$$

Substitution of strain and electric displacement in variation form yields variation of strain energy as follows

$$\delta U = \int_\alpha \int_\beta \begin{bmatrix} \{N_\alpha \frac{1}{R_1}\frac{\partial \delta u}{\partial \alpha} + N_\alpha^1 \frac{\delta w}{R_1} + M_\alpha \frac{1}{R_1}\frac{\partial \delta \phi_1}{\partial \alpha}\} \\ +\{N_\beta \frac{1}{R_2}\frac{\partial \delta v}{\partial \beta} + N_\beta^1 \frac{\delta w}{R_2} + M_\beta \frac{1}{R_2}\frac{\partial \delta \phi_2}{\partial \beta}\} \\ +\left\{N_{\beta z}^1 \delta\phi_2 + N_{\beta z}\frac{1}{R_2}\frac{\partial \delta w}{\partial \beta} - N_{\beta z}^1 \frac{\delta v}{R_2}\right\} \\ +\left\{N_{\alpha z}^1 \delta\phi_1 + N_{\alpha z}\frac{1}{R_1}\frac{\partial \delta w}{\partial \alpha} - N_{\alpha z}^1 \frac{\delta u}{R_1}\right\} \\ +N_{\alpha\beta}\frac{1}{R_1}\frac{\partial \delta v}{\partial \alpha} + N_{\beta\alpha}\frac{1}{R_2}\frac{\partial \delta u}{\partial \beta} \\ +M_{\alpha\beta}\frac{1}{R_1}\frac{\partial \delta \phi_2}{\partial \alpha} + M_{\beta\alpha}\frac{1}{R_2}\frac{\partial \delta \phi_1}{\partial \beta} \\ +\frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{1}{R_1}M_{\alpha\beta}\frac{\partial \delta v}{\partial \alpha} - \frac{1}{R_2}M_{\beta\alpha}\frac{\partial \delta u}{\partial \beta}\right) \\ -\bar{D}_\alpha \frac{1}{R_1}\frac{\partial \delta \Psi}{\partial \alpha} - \bar{D}_\beta \frac{1}{R_2}\frac{\partial \delta\Psi}{\partial \beta} + \bar{D}_z \delta\Psi \end{bmatrix} d\beta d\alpha. \quad (9)$$

in which the resultant components are defined as

$$\begin{aligned} \{N_\alpha, N_\alpha^1, M_\alpha\} &= \int_{-h/2-h_p}^{-h/2} \sigma_\alpha{}^p \left[1 + \frac{z}{R_2}\right]\left\{1, \left[1 + \frac{z}{R_1}\right], z\right\} dz \\ &+ \int_{-h/2}^{+h/2} \sigma_\alpha \left[1 + \frac{z}{R_2}\right]\left\{1, \left[1 + \frac{z}{R_1}\right], z\right\} dz \\ &+ \int_{+h/2}^{+h/2+h_p} \sigma_\alpha{}^p \left[1 + \frac{z}{R_2}\right]\left\{1, \left[1 + \frac{z}{R_1}\right], z\right\} dz. \end{aligned} \quad (10a)$$

$$\begin{aligned} \{N_\beta, N_\beta^1, M_\beta\} &= \int_{-h/2-h_p}^{-h/2} \sigma_\beta{}^p \left[1 + \frac{z}{R_1}\right]\left\{1, \left[1 + \frac{z}{R_2}\right], z\right\} dz \\ &+ \int_{-h/2}^{+h/2} \sigma_\beta \left[1 + \frac{z}{R_1}\right]\left\{1, \left[1 + \frac{z}{R_2}\right], z\right\} dz \\ &+ \int_{+h/2}^{+h/2+h_p} \sigma_\beta{}^p \left[1 + \frac{z}{R_1}\right]\left\{1, \left[1 + \frac{z}{R_2}\right], z\right\} dz. \end{aligned} \quad (10b)$$

$$\begin{aligned} \{N_{\beta z}, N_{\beta z}^1\} &= \int_{-h/2-h_p}^{-h/2} \tau_{\beta z}{}^p \left[1 + \frac{z}{R_1}\right]\left\{1, \left[1 + \frac{z}{R_2}\right]\right\} dz \\ &+ \int_{-h/2}^{+h/2} \tau_{\beta z} \left[1 + \frac{z}{R_1}\right]\left\{1, \left[1 + \frac{z}{R_2}\right]\right\} dz \\ &+ \int_{+h/2}^{+h/2+h_p} \tau_{\beta z}{}^p \left[1 + \frac{z}{R_1}\right]\left\{1, \left[1 + \frac{z}{R_2}\right]\right\} dz. \end{aligned} \quad (10c)$$

$$\begin{aligned} \{N_{\alpha z}, N_{\alpha z}^1\} &= \int_{-h/2-h_p}^{-h/2} \tau_{\alpha z}{}^p \left[1 + \frac{z}{R_2}\right]\left\{1, \left[1 + \frac{z}{R_1}\right]\right\} dz \\ &+ \int_{-h/2}^{+h/2} \tau_{\alpha z} \left[1 + \frac{z}{R_2}\right]\left\{1, \left[1 + \frac{z}{R_1}\right]\right\} dz \\ &+ \int_{+h/2}^{+h/2+h_p} \tau_{\alpha z}{}^p \left[1 + \frac{z}{R_2}\right]\left\{1, \left[1 + \frac{z}{R_1}\right]\right\} dz. \end{aligned} \quad (10d)$$

$$\begin{aligned} \{N_{\alpha\beta}, M_{\alpha\beta}\} &= \int_{-h/2-h_p}^{-h/2} \tau_{\alpha\beta}{}^p \left[1 + \frac{z}{R_2}\right]\{1, z\} dz \\ &+ \int_{-h/2}^{+h/2} \tau_{\alpha\beta} \left[1 + \frac{z}{R_2}\right]\{1, z\} dz \\ &+ \int_{+h/2}^{+h/2+h_p} \tau_{\alpha\beta}{}^p \left[1 + \frac{z}{R_2}\right]\{1, z\} dz. \end{aligned} \quad (10e)$$

$$\begin{aligned} \{N_{\beta\alpha}, M_{\beta\alpha}\} &= \int_{-h/2-h_p}^{-h/2} \tau_{\alpha\beta}{}^p \left[1 + \frac{z}{R_1}\right]\{1, z\} dz \\ &+ \int_{-h/2}^{+h/2} \tau_{\alpha\beta} \left[1 + \frac{z}{R_1}\right]\{1, z\} dz \\ &+ \int_{+h/2}^{+h/2+h_p} \tau_{\alpha\beta}{}^p \left[1 + \frac{z}{R_1}\right]\{1, z\} dz. \end{aligned} \quad (10f)$$

$$\begin{aligned} \{\bar{D}_\alpha, \bar{D}_\beta\} &= \int_{-h/2-h_p}^{-h/2} \left\{D_\alpha \left[1 + \frac{z}{R_2}\right], D_\beta \left[1 + \frac{z}{R_1}\right]\right\} \cos\frac{\pi z}{h} dz \\ &+ \int_{+h/2}^{+h/2+h_p} \left\{D_\alpha \left[1 + \frac{z}{R_2}\right], D_\beta \left[1 + \frac{z}{R_1}\right]\right\} \cos\frac{\pi z}{h} dz. \end{aligned} \quad (10g)$$

$$\begin{aligned}\bar{D}_z = &\int_{-h/2-h_p}^{-h/2} \frac{\pi}{h} sin\frac{\pi z}{h} D_z[1+\frac{z}{R_1}][1+\frac{z}{R_2}]\,dz \\ &+ \int_{+h/2}^{+h/2+h_p} \frac{\pi}{h} sin\frac{\pi z}{h} D_z[1+\frac{z}{R_1}][1+\frac{z}{R_2}]\,dz\end{aligned} \tag{10h}$$

The work done by external forces including uniform transverse loads and reaction of Pasternak's foundation is calculated as

$$\begin{aligned}\delta W_{Ext1} = \int \Big\{ &-q\left[1+\frac{h}{2R_1}\right]\left[1+\frac{h}{2R_2}\right] \\ &+R_f\left[1-\frac{h}{2R_1}\right]\left[1-\frac{h}{2R_2}\right]\Big\} R_1R_2\delta w d\beta d\alpha\end{aligned} \tag{11}$$

in which the reaction of the Pasternak's foundation is defined as $R_f = K_1 w - K_2\nabla^2 w$. The work done by in-plane external forces is calculated as

$$\begin{aligned}\delta W_{Ext2} = -\int_\beta\int_\alpha \Big[&\left(N_{0\alpha}+N_{E\alpha}\right)\frac{1}{{R_1}^2}\frac{\partial^2 w}{\partial\alpha^2} \\ &+\left(N_{0\beta}+N_{E\beta}\right)\frac{1}{{R_2}^2}\frac{\partial^2 w}{\partial\beta^2}\Big] R_1R_2\delta w d\alpha d\beta\end{aligned} \tag{12}$$

In which $(N_{0\alpha}, N_{0\beta})$ and $(N_{E\alpha}, N_{E\beta})$ are pre-mechanical and pre-electrical loads. Substitution of strain energy and work due to external forces into principle of virtual work $\delta U - \delta W_{Ext} = 0$ yields the stability governing equations as follows

$$\begin{aligned}\delta u:\quad &\frac{\partial}{\partial\alpha}\left(\frac{N_\alpha}{R_1}\right)+\frac{\partial}{\partial\beta}\left(\frac{N_{\beta\alpha}}{R_2}\right)+\frac{N_{\alpha z}^1}{R_1} \\ &-\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{\partial}{\partial\beta}\left(\frac{M_{\beta\alpha}}{R_2}\right)=0,\end{aligned} \tag{13a}$$

$$\delta\phi_1:\quad \frac{\partial}{\partial\alpha}\left(\frac{M_\alpha}{R_1}\right)-N_{\alpha z}^1+\frac{\partial}{\partial\beta}\left(\frac{M_{\beta\alpha}}{R_2}\right)=0, \tag{13b}$$

$$\begin{aligned}\delta v:\quad &+\frac{\partial}{\partial\beta}\left(\frac{N_\beta}{R_2}\right)+\frac{\partial}{\partial\alpha}\left(\frac{N_{\alpha\beta}}{R_1}\right) \\ &+\frac{N_{\beta z}^1}{R_2}+\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{\partial}{\partial\alpha}\left(\frac{M_{\alpha\beta}}{R_1}\right)=0,\end{aligned} \tag{13c}$$

$$\delta\phi_2:\quad \frac{\partial}{\partial\beta}\left(\frac{M_\beta}{R_2}\right)-N_{\beta z}^1+\frac{\partial}{\partial\alpha}\left(\frac{M_{\alpha\beta}}{R_1}\right)=0, \tag{13d}$$

$$\begin{aligned}\delta w:\quad &-\frac{N_\alpha^1}{R_1}-\frac{N_\beta^1}{R_2}+\frac{\partial}{\partial\beta}\left(\frac{N_{\beta z}}{R_2}\right)+\frac{\partial}{\partial\alpha}\left(\frac{N_{\alpha z}}{R_1}\right) \\ &+\left(N_{0\alpha}+N_{E\alpha}+N_{M\alpha}\right)\frac{1}{{R_1}^2}\frac{\partial^2 w}{\partial\alpha^2} \\ &+\left(N_{0\beta}+N_{E\beta}+N_{M\beta}\right)\frac{1}{{R_2}^2}\frac{\partial^2 w}{\partial\beta^2} \\ &= q\left[1+\frac{h}{2R_1}\right]\left[1+\frac{h}{2R_2}\right] \\ &-R_f\left[1-\frac{h}{2R_1}\right]\left[1-\frac{h}{2R_2}\right],\end{aligned} \tag{13e}$$

$$\delta\Psi:\quad +\frac{\partial}{\partial\alpha}\left(\frac{\bar{D}_\alpha}{R_1}\right)+\frac{\partial}{\partial\beta}\left(\frac{\bar{D}_\beta}{R_2}\right)+\bar{D}_z=0, \tag{13f}$$

In which the resultant components are defined in Appendix A.

Substitution of resultant components from Appendix A into governing equations leads to final governing equations in terms of primary displacement field as follows

$$\begin{aligned}\delta u:\quad &\frac{A_1}{R_1}\frac{\partial^2 u}{\partial\alpha^2}+\left(\frac{\chi[A_{70}-A_{66}]+A_{54}-A_{58}}{R_2}\right)\frac{\partial^2 u}{\partial\beta^2} \\ &-\frac{A_{45}}{R_1}u+\frac{A_3}{R_1}\frac{\partial^2\phi_1}{\partial\alpha^2}+\left(\frac{A_{56}-\chi A_{68}}{R_2}\right)\frac{\partial^2\phi_1}{\partial\beta^2} \\ &+\frac{A_{43}}{R_1}\phi_1+\frac{A_4}{R_1}\frac{\partial^2 v}{\partial\alpha\partial\beta} \\ &+\left(\frac{A_{57}+A_{53}-\chi[A_{65}+A_{69}]}{R_2}\right)\frac{\partial^2 v}{\partial\alpha\partial\beta} \\ &+\left(\frac{A_6}{R_1}+\frac{A_{55}-\chi A_{67}}{R_2}\right)\frac{\partial^2\phi_2}{\partial\alpha\partial\beta} \\ &+\left(\frac{A_5+A_2+A_{44}}{R_1}\right)\frac{\partial w}{\partial\alpha}+\left(\frac{A_7-A_{46}}{R_1}\right)\frac{\partial\Psi}{\partial\alpha} \\ &=-\frac{\partial}{\partial\alpha}\left(\frac{{N_\alpha}^\Psi}{R_1}\right) \\ \delta\phi_1:\quad &\frac{A_{15}}{R_1}\frac{\partial^2 u}{\partial\alpha^2}+\frac{(A_{66}-A_{70})}{R_2}\frac{\partial^2 u}{\partial\beta^2}+\frac{A_{17}}{R_1}\frac{\partial^2\phi_1}{\partial\alpha^2} \\ &+\frac{A_{68}}{R_2}\frac{\partial^2\phi_1}{\partial\beta^2}-A_{43}\phi_1 \\ &+\left(\frac{A_{65}+A_{69}}{R_2}+\frac{A_{18}}{R_1}\right)\frac{\partial^2 v}{\partial\alpha\partial\beta}+A_{45}u \\ &+\left(\frac{A_{20}}{R_1}+\frac{A_{67}}{R_2}\right)\frac{\partial\phi_2}{\partial\alpha\partial\beta}+\left(\frac{A_{19}+A_{16}}{R_1}-A_{44}\right)\frac{\partial w}{\partial\alpha} \\ &+\left(\frac{A_{21}}{R_1}+A_{46}\right)\frac{\partial\Psi}{\partial\alpha}=-\frac{\partial}{\partial\alpha}\left(\frac{{M_\alpha}^\Psi}{R_1}\right) \\ \delta v:\quad &\left(\frac{A_4}{R_2}+\frac{\chi[A_{60}-A_{64}]+A_{48}-A_{52}}{R_1}\right)\frac{\partial^2 u}{\partial\alpha\partial\beta} \\ &+\left(\frac{A_{50}+\chi A_{62}}{R_1}+\frac{A_6}{R_2}\right)\frac{\partial^2\phi_1}{\partial\alpha\partial\beta} \\ &+\left(\frac{A_{47}+A_{51}+\chi[A_{59}+A_{63}]}{R_1}\right)\frac{\partial^2 v}{\partial\alpha^2}+\frac{A_{22}}{R_2}\frac{\partial^2 v}{\partial\beta^2} \\ &-\frac{A_{41}}{R_2}v+\left(\frac{A_{49}+\chi A_{61}}{R_1}\right)\frac{\partial^2\phi_2}{\partial\alpha^2}+\frac{A_{24}}{R_2}\frac{\partial^2\phi_2}{\partial\beta^2} \\ &+\frac{A_{39}}{R_2}\phi_2+\left(\frac{A_{40}+A_5+A_{23}}{R_2}\right)\frac{\partial w}{\partial\beta} \\ &+\left(\frac{A_7-A_{42}}{R_2}\right)\frac{\partial\Psi}{\partial\beta}=-\frac{\partial}{\partial\beta}\left(\frac{{N_\beta}^\Psi}{R_2}\right) \\ \delta\phi_2:\quad &\left(\frac{A_{18}}{R_2}+\frac{A_{60}-A_{64}}{R_1}\right)\frac{\partial^2 u}{\partial\alpha\partial\beta}+A_{41}v \\ &+\left(\frac{A_{62}}{R_1}+\frac{A_{20}}{R_2}\right)\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+\left(\frac{A_{59}+A_{63}}{R_1}\right)\frac{\partial^2 v}{\partial\alpha^2}\end{aligned} \tag{14}$$

$$\begin{aligned}\delta\phi_2:\quad &\left(\frac{A_{18}}{R_2}+\frac{A_{60}-A_{64}}{R_1}\right)\frac{\partial^2 u}{\partial\alpha\partial\beta}+A_{41}v \\ &+\left(\frac{A_{62}}{R_1}+\frac{A_{20}}{R_2}\right)\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+\left(\frac{A_{59}+A_{63}}{R_1}\right)\frac{\partial^2 v}{\partial\alpha^2}\end{aligned}$$

$$+\frac{A_{28}}{R_2}\frac{\partial^2 v}{\partial\beta^2}+\left(\frac{A_{30}}{R_2}+\frac{A_{61}}{R_1}\right)\frac{\partial^2\phi_2}{\partial\alpha^2}-A_{39}\phi_2+\left(\frac{A_{29}+A_{19}}{R_2}-A_{40}\right)\frac{\partial w}{\partial\beta}+\left(\frac{A_{21}}{R_2}+A_{42}\right)\frac{\partial\Psi}{\partial\beta}=-\frac{\partial}{\partial\beta}\left(\frac{{M_\alpha}^{\Psi}}{R_2}\right)$$

$$\begin{aligned}\delta w:\quad &\left(-\frac{A_{11}}{R_2}-\frac{A_8+A_{37}}{R_1}\right)\frac{\partial u}{\partial\alpha}+\left(\frac{A_{35}-A_{10}}{R_1}-\frac{A_{13}}{R_2}\right)\frac{\partial\phi_1}{\partial\alpha}\\&+\left(-\frac{A_{33}+A_{25}}{R_2}-\frac{A_{11}}{R_1}\right)\frac{\partial v}{\partial\beta}-\frac{A_{27}}{R_2}\frac{\partial\phi_2}{\partial\beta}+\left(-\frac{A_9+A_{12}}{R_1}-\frac{A_{12}+A_{26}}{R_2}\right)w\\&+\left(-\frac{A_{13}}{R_1}-\frac{A_{31}}{R_2}\right)\frac{\partial\phi_2}{\partial\beta}+\frac{A_{36}}{R_1}\frac{\partial^2 w}{\partial\alpha^2}+\left(\frac{A_{32}-A_{34}}{R_2}\right)\frac{\partial^2\Psi}{\partial\beta^2}-\frac{A_{38}}{R_1}\frac{\partial^2\Psi}{\partial\alpha^2}\\&+\left(-\frac{A_{14}}{R_2}-\frac{A_{14}}{R_1}\right)\Psi+\left(N_{0\alpha}+N_{E\alpha}\right)\frac{1}{{R_1}^2}\frac{\partial^2 w}{\partial\alpha^2}+\left(N_{0\beta}+N_{E\beta}\right)\frac{1}{{R_2}^2}\frac{\partial^2 w}{\partial\beta^2}\\&=+\frac{{N_\alpha}^{1\Psi}}{R_1}+\frac{{N_\beta}^{1\Psi}}{R_2}+(1-(e_0a)^2\nabla^2)\left\{\begin{matrix}q\left[1+\frac{h}{2R_1}\right]\left[1+\frac{h}{2R_2}\right]\\-R_f\left[1-\frac{h}{2R_1}\right]\left[1-\frac{h}{2R_2}\right]\end{matrix}\right\}\\\delta\Psi:\quad&\left(-\frac{A_{73}}{R_1}+A_{79}\right)\frac{\partial u}{\partial\alpha}+\left(\frac{A_{71}}{R_1}+A_{81}\right)\frac{\partial\phi_1}{\partial\alpha}+\left(A_{82}-\frac{A_{77}}{R_2}\right)\frac{\partial v}{\partial\beta}+\left(\frac{A_{75}}{R_2}+A_{84}\right)\frac{\partial\phi_2}{\partial\beta}\\&+\frac{A_{72}}{R_1}\frac{\partial^2 w}{\partial\alpha^2}+\frac{A_{76}}{R_2}\frac{\partial^2 w}{\partial\beta^2}+(A_{80}+A_{83})w-\frac{A_{78}}{R_2}\frac{\partial^2\Psi}{\partial\beta^2}-\frac{A_{74}}{R_1}\frac{\partial^2\Psi}{\partial\alpha^2}+A_{85}\Psi=-{D_z}^{\Psi}\end{aligned}\tag{14}$$

In which the integration constants A_i and other undefined variables are expressed in Appendix B.

3. Solution procedure

Solution procedure is illustrated in this section based on double trigonometric solution for the simply supported boundary conditions. The four boundary conditions are assumed simply-supported. In addition, the homogeneous boundary conditions are assumed for electric potentials.

Based on this procedure the solution is expressed as follows

$$\begin{Bmatrix}u\\\phi_1\\v\\\phi_2\\w\\\Psi\end{Bmatrix}=\begin{Bmatrix}U cos\lambda_m\alpha sin\mu_n\beta\\\Phi_1 cos\lambda_m\alpha sin\mu_n\beta\\V sin\lambda_m\alpha cos\mu_n\beta\\\Phi_2 sin\lambda_m\alpha cos\mu_n\beta\\W sin\lambda_m\alpha sin\mu_n\beta\\\Psi sin\lambda_m\alpha sin\mu_n\beta\end{Bmatrix}\tag{15}$$

In which the $\{X\}=\{U,\Phi_1,V,\Phi_2,W,\Psi\}^T$ are unknown amplitudes and $\lambda_m=\frac{m R_1}{L_1}$, $\mu_n=\frac{n R_2}{L_2}$. Substitution of proposed solution from Eq. (15) into governing equations leads to following well-known format as follows

$$[K]\{X\}=\{F\}\tag{16}$$

Elements of stiffness matrix $[K]$ and force matrix $\{F\}$ are defined in Appendix C.

4. Numerical results and discussion

The material properties of piezoelectric doubly curved nano shell are presented for the core and piezoelectric layers as

Core:

$$E=169GPa, v=0.3$$

Piezoelectric:

$$\begin{aligned}&{C_{11}}^P=138.499GPa,\\&{C_{22}}^P=138.499GPa,\\&{C_{33}}^P=114.745GPa\\&{C_{12}}^P=77.371GPa,\\&{C_{13}}^P=73.643GPa,\\&{C_{23}}^P=73.643GPa\\&{C_{44}}^P=25.6GPa,\\&{C_{55}}^P=25.6GPa,\\&{C_{66}}^P=30.6GPa\\&e_{13}=e_{31}=-5.2\,{C}/{m^2},\\&e_{23}=e_{32}=-5.2\,{C}/{m^2},\\&e_{33}=15.8\,{C}/{m^2},\\&e_{15}=12.72\,{C}/{m^2},\\&e_{24}=12.72\,{C}/{m^2},\\&k_{11}=1.306\times10^{-8}\,{F}/{m},\\&k_{22}=1.306\times10^{-8}\,{F}/{m},\\&k_{33}=1.151\times10^{-8}\,{F}/{m}\end{aligned}$$

In addition, the dimensions of doubly curved piezoelectric nano shell for Figs. 2-17 are summarized as follows

$$\begin{aligned}&L_1=L_2=20\,nm,\\&R_1=R_2=20\,nm,\\&h_e=1\,nm,\qquad h_p=0.1\,nm\end{aligned}$$

4.1 Critical applied electrical loads

In this section, the distribution of critical applied electrical loads $\Psi_{0,cr}$ is presented in term of two principle angles of doubly curved shells θ_1,θ_2. One can see that the trend of critical applied electrical loads $\Psi_{0,cr}$ is not uniform for all values of two principle angles θ_1,θ_2. For example; for $\theta_2=0.5$, the critical applied electrical loads $\Psi_{0,cr}$ is decreased with increase of θ_1, while they are increased for $\theta_2=1$. Shown in Fig. 3 is distribution of critical applied electrical loads $\Psi_{0,cr}$ in terms of first angle

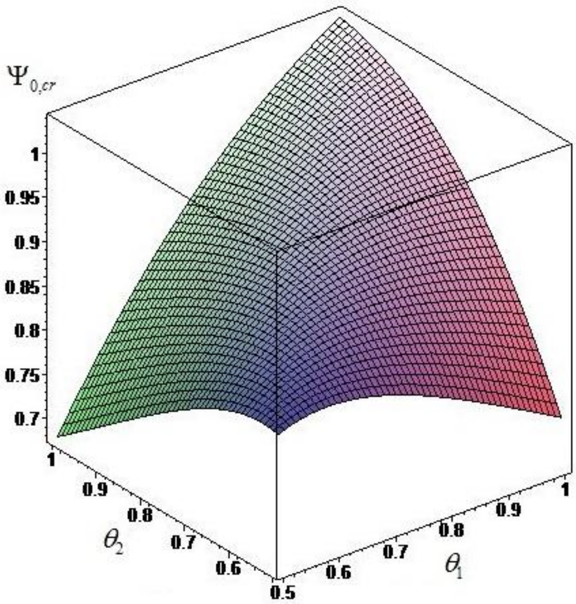

Fig. 2 Distribution of critical applied electrical loads $\Psi_{0,cr}$ in term of two principle angles of doubly curved shells θ_1, θ_2

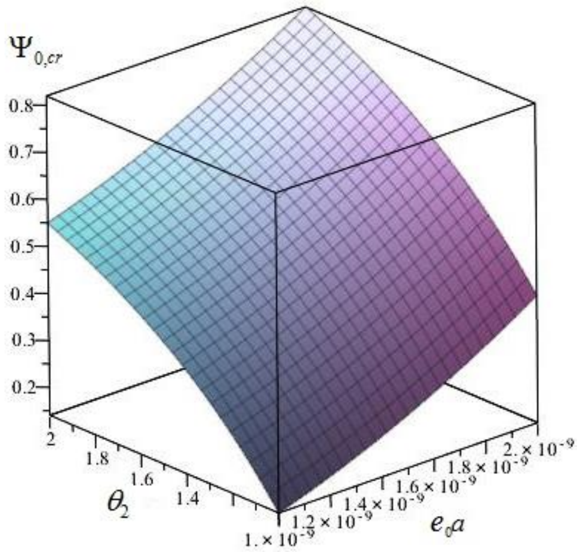

Fig. 4 Distribution of critical applied electrical loads $\Psi_{0,cr}$ in term of second angle of shells θ_2 and small scale parameter $\mathrm{e}_0 a$

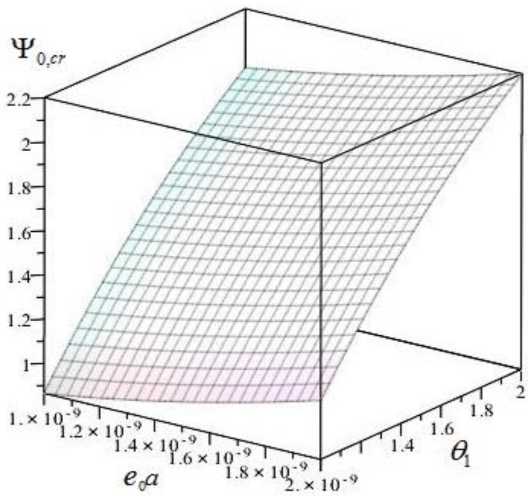

Fig. 3 Distribution of critical applied electrical loads $\Psi_{0,cr}$ in term of first angle of shells $\theta_1$and small scale parameter $\mathrm{e}_0 a$

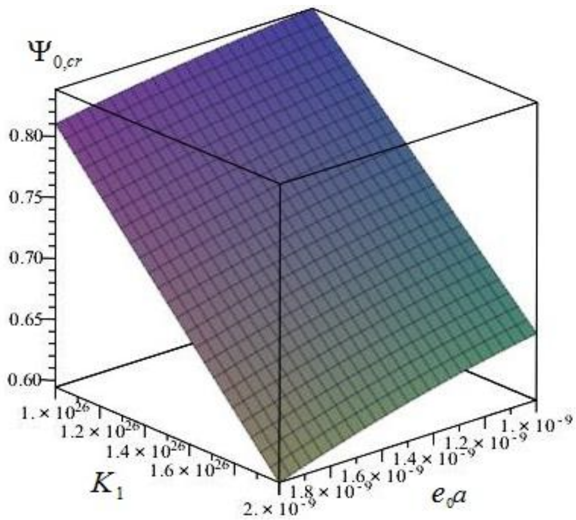

Fig. 5 Distribution of critical applied electrical loads $\Psi_{0,cr}$ in term of winker parameter of foundation $\mathrm{K}_1$and small scale parameter $\mathrm{e}_0 a$

of shells θ_1 and small scale parameter $\mathrm{e}_0 a$. One can observe that with increase of nonlocal parameter, the critical applied electrical loads $\Psi_{0,cr}$ are increased significantly. In addition the increase behavior can be observed for variation of critical applied electrical loads $\Psi_{0,cr}$ in terms of variation of first angle of shells θ_1.

Fig. 4 shows variation of critical applied electrical loads $\Psi_{0,cr}$ in terms of second angle of shells θ_2 and small scale parameter $\mathrm{e}_0 a$. It is concluded that critical applied electrical loads $\Psi_{0,cr}$ are increased significantly with increase of both parameters (second angle of shells θ_2 and small scale parameter $\mathrm{e}_0 a$).

Shown in Fig. 5 is the effect of winker parameter of foundation K_1 and small scale parameter $\mathrm{e}_0 a$ on the critical applied electrical loads $\Psi_{0,cr}$. The numerical results indicate that with increase of both Winkler parameter of foundation and nonlocal parameter, the critical applied electrical loads $\Psi_{0,cr}$ are decreased significantly.

Shown in Fig. 6 is the effect of shear parameter of foundation and small scale parameter $\mathrm{e}_0 a$ on the critical applied electrical loads $\Psi_{0,cr}$. One can conclude that with increase of both shear parameter of foundation and nonlocal parameter, the critical applied electrical loads $\Psi_{0,cr}$ are decreased significantly.

The effect of two principle angles of doubly curved shell on the mechanical buckling loads of doubly curved nano shell is presented in Fig. 7. One can see that with increase of first angle of shells θ_1 and decrease of second angle of shells θ_2, the mechanical buckling loads $\mathrm{N}_{0,cr}$ are

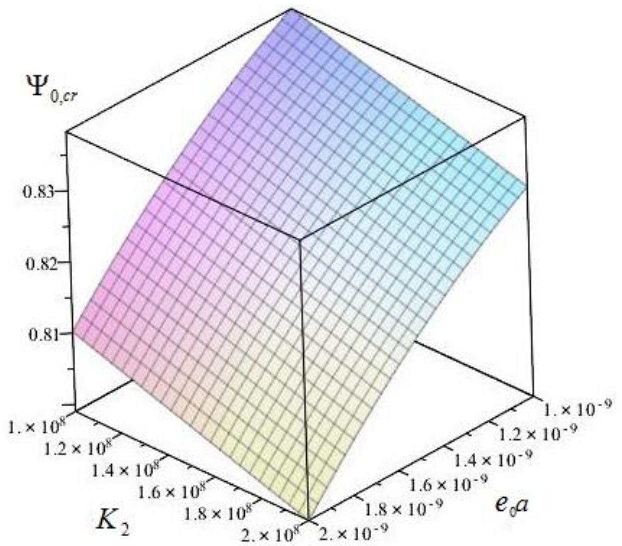

Fig. 6 Distribution of critical applied electrical loads $\Psi_{0,cr}$ in term of shear parameter of foundation K_2 and small scale parameter $\mathrm{e}_0 a$

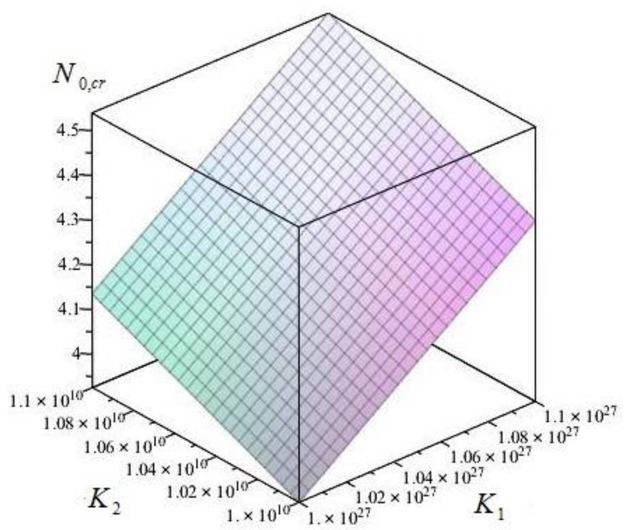

Fig. 8 Distribution of mechanical buckling loads $\mathrm{N}_{0,cr}$ in term of two parameters of Pasternak foundation K_1, K_2.

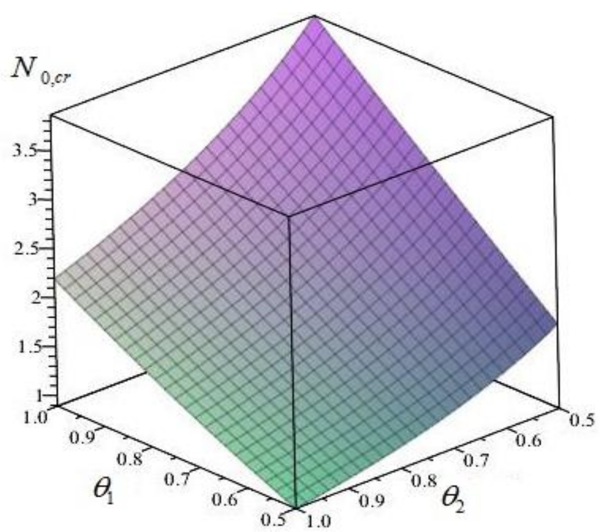

Fig. 7 Distribution of mechanical buckling loads $\mathrm{N}_{0,cr}$ in term of two principle angles of doubly curved shell θ_1, θ_2

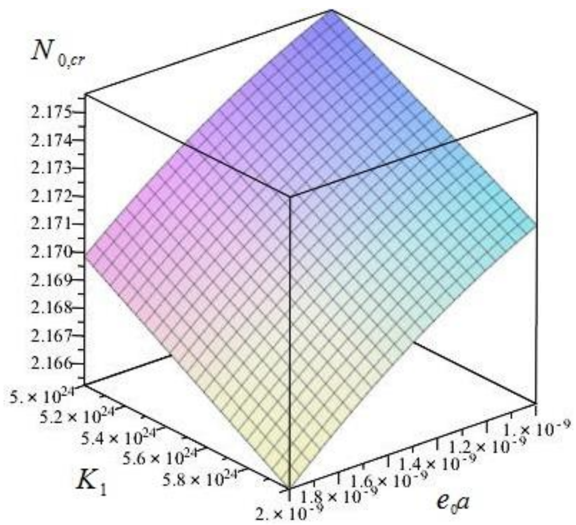

Fig. 9 Distribution of mechanical buckling loads $\mathrm{N}_{0,cr}$ in term of direct parameter of foundation K_1 and small scale parameter $\mathrm{e}_0 a$

increased significantly.

Fig. 8 depicts variation of mechanical buckling loads in terms of direct and shear parameters of Pasternak's foundation $K_{1,} K_2$. One can conclude that with increase of direct and shear parameters of foundation, the stiffness of structure is increased and consequently the mechanical buckling loads are increased.

Shown in Figs. 9, 10 are the effect of two parameters of foundation $(\mathrm{K}_1, \mathrm{K}_2)$ and small scale parameter $\mathrm{e}_0 a$ on the mechanical buckling loads $\mathrm{N}_{0,cr}$. The numerical results indicate that with increase of Winkler parameter of foundation, the mechanical buckling loads are increased while with increase of nonlocal parameter they are decreased significantly. It is concluded that increase of nonlocal parameter leads to significant decrease of stiffness of nano shell and consequently decrease of its mechanical buckling load.

5. Conclusions

Stability analysis of shear deformable doubly curved nano shell including a nano core and two piezoelectric nanoshells was studied in this paper based on size dependent constitutive relations and first order shear deformation theory. Principle of virtual work was used in this work to derive governing equations of stability analysis. The piezoelectric layers have been subjected to initial electric potential. The governing equations of stability have been solved based on double trigonometric functions for simply supported boundary conditions. The

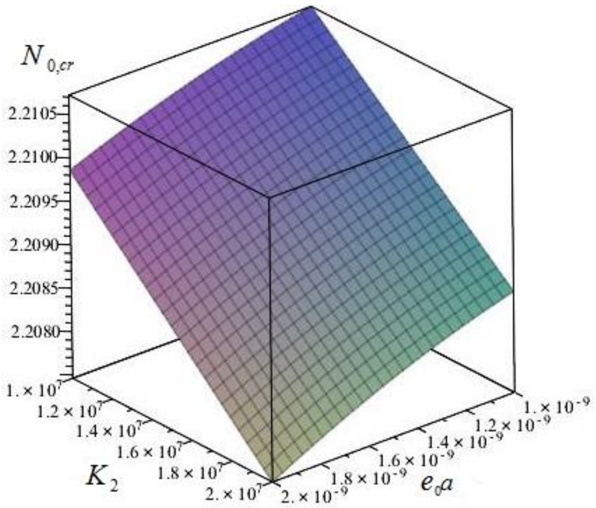

Fig. 10 Distribution of mechanical buckling loads $\mathrm{N}_{0,cr}$ in term of shear parameter of foundation K_2 and and small scale parameter $\mathrm{e}_0 a$

critical applied electric potentials and critical mechanical loads were evaluated in terms of significant inputs of the problem such as nonlocal parameter, two parameters of Pasternak's foundation and two principle angles of doubly curved nano shell. The main significant conclusions of this work are classified as follows:

The critical applied electrical loads $\Psi_{0,cr}$ were evaluated in terms of significant parameters of the problem. One can conclude that these outputs are increased with increase of small scale parameter. In addition, investigation on the effect of two principle angles of doubly curved nano shell indicates that for $\theta_2 = 0.5$, the critical applied electrical loads $\Psi_{0,cr}$ is decreased with increase of θ_1, while they are increased for $\theta_2 = 1$. It is confirmed that changes of critical applied electrical loads $\Psi_{0,cr}$ are strongly depending on the two two principle angles and consequently a quantitative presentation is not possible. Furthermore, it is concluded that increase of two parameters of Pasternak's foundation leads to decrease of critical applied electrical loads.

The mechanical buckling loads have been evaluated in terms of important parameters of the problem. This investigation indicates that the mechanical buckling loads are decreased with increase of nonlocal parameter. In addition, it is concluded that increase of direct and shear parameters of foundation leads to increase of mechanical buckling loads. In addition, one can see that with increase of first angle of shells $\theta_1$and decrease of second angle of shells θ_2, the critical applied electrical loads $\Psi_{0,cr}$ are increased significantly.

References

Anitescu, C., Atroshchenko, E., Alajlan, N. and Rabczuk, T. (2019), "Artificial neural network methods for the solution of second order boundary value problems", *Comput. Mater. Cont.*, **59**(1), 345-359.

Arefi, M. (2018a), "Nonlocal free vibration analysis of a doubly curved piezoelectric nano shell", *Steel Compos. Struct., Int. J.*, **27**(4), 479-493. https://doi.org/10.12989/scs.2018.27.4.479

Arefi, M. (2018b), "Analysis of a doubly curved piezoelectric nano shell: nonlocal electro-elastic bending solution", *Eur. J. Mech.-A/Solids.*, **70**, 226-237. https://doi.org/10.1016/j.euromechsol.2018.02.012

Arefi, M. and Zenkour, A.M. (2017a), "Size-dependent free vibration and dynamic analyses of piezo-electro-magnetic sandwich nanoplates resting on viscoelastic foundation", *Phys. B: Cond. Matt.*, **521**, 188-197. https://doi.org/10.1016/j.physb.2017.06.066

Arefi, M. and Zenkour, A.M. (2017b), "Transient sinusoidal shear deformation formulation of a size-dependent three-layer piezo-magnetic curved nanobeam", *Acta. Mech.*, **228** (10), 3657-3674. https://doi.org/10.1007/s00707-017-1892-6

Arefi, M. and Zenkour, A.M. (2017c), "Transient analysis of a three-layer microbeam subjected to electric potential", *Int. J. Smart. Nano. Mater.*, **8**(1), 20-40. https://doi.org/10.1080/19475411.2017.1292967

Arefi, M. and Zenkour, A.M. (2019a), "Influence of magneto-electric environments on size-dependent bending results of three-layer piezomagnetic curved nanobeam based on sinusoidal shear deformation theory", *J. Sandw. Struct. Mater.*, **21**(8), 2751-2778. https://doi.org/10.1177/1099636217723186

Arefi, M. and Zenkour, A.M. (2019b), "Effect of thermo-magneto-electro-mechanical fields on the bending behaviors of a three-layered nanoplate based on sinusoidal shear-deformation plate theory", *J. Sandw. Struct. Mater.*, **21**(2), 639-669. https://doi.org/10.1177/1099636217697497

Arefi, M., Zamani, M.H. and Kiani, M. (2018), "Size-dependent free vibration analysis of three-layered exponentially graded nanoplate with piezomagnetic face-sheets resting on Pasternak's foundation", *J. Intel. Mater. Syst. Struct.*, **29**(5), 774-786. https://doi.org/10.1177/1045389X17721039

Chandrashekhara, K. (1989), "Free vibrations of anisotropic laminated doubly curved shells", *Comput. Struct.*, **33**(2), 435-440. https://doi.org/10.1016/0045-7949(89)90015-1

Chen, D., Yang, J. and Kitipornchai, S. (2017a), "Nonlinear vibration and postbuckling of functionally graded graphene reinforced porous nanocomposite beams", *Compos. Sci. Tech.*, **142**, 235-245. https://doi.org/10.1016/j.compscitech.2017.02.008

Chen, C.S., Liu, F.H. and Chen, W.R. (2017b), "Vibration and stability of initially stressed sandwich plates with FGM face sheets in thermal environments", *Steel Compos. Struct., Int. J.*, **23**(3), 251-261. https://doi.org/10.12989/scs.2017.23.3.251

Fan, J. and Zhang, J. (1992), "Analytical solutions for thick, doubly curved, laminated shells", *J. Eng. Mech.*, **118**(7), 1338-1356. https://doi.org/10.1061/(ASCE)0733-9399(1992)118:7(1338)

Ghavanloo, E. and Fazelzadeh, S.A. (2013), "Free vibration analysis of orthotropic doubly-curved shallow shells based on the gradient elasticity", *Compos. Part B: Eng.*, **45**(1), 1448-1457. https://doi.org/10.1016/j.compositesb.2012.09.054

Guo, H., Zhuang, X. and Rabczuk, T. (2019), "A deep collocation method for the bending analysis of Kirchhoff plate", *Comput. Mater. Continua*, **59**(2), 433-456.

Hamdia, K.M., Ghasemi, H., Zhuang, X., Alajlan, N. and Rabczuk, T. (2018), "Sensitivity and uncertainty analysis for flexoelectric nanostructures", *Comput. Meth. Appl. Mech. Eng.*, **337**, 95-109. https://doi.org/10.1016/j.cma.2018.03.016

Kapania, R.K. and Yang, T.Y. (1986), "Formulation of an imperfect quadrilateral doubly curved shell element for postbuckling analysis", *AIAA Journal*, **24**(2), 310-311. https://doi.org/10.2514/3.9261

Karami, B. and Shahsavari, D. (2019), "Nonlocal strain gradient model for thermal stability of FG nanoplates integrated with piezoelectric layers", *Smart Struct. Syst., Int. J.*, **23**(3), 27-36. https://doi.org/10.12989/sss.2019.23.3.215

Librescu, L. and Chang, M.Y. (1993), "Effects of geometric imperfections on vibration of compressed shear deformable laminated composite curved panels", *Acta Mech.*, **96**, 203-224. https://doi.org/10.1007/BF01340710

Pouresmaeeli, S. and Fazelzadeh, S.A. (2016), "Frequency analysis of doubly curved functionally graded carbon nanotube-reinforced composite panels", *Acta. Mech.*, **227**(10), 2765-2794. https://doi.org/10.1007/s00707-016-1647-9

Qatu, M.S. and Asadi, E. (2012), "Vibration of doubly curved shallow shells with arbitrary boundaries", *Appl. Acoust.*, **73**(1), 21-27. https://doi.org/10.1016/j.apacoust.2011.06.013

Rabczuk, T., Ren, H. and Zhuang, X. (2019), "A nonlocal operator method for partial differential equations with application to electromagnetic waveguide problem", *Comput. Mater. Continua*, **59**(1), 31-55.https://doi.org/10.32604/cmc.2019.04567

Sharma, A., Upadhyay, A.K. and Shukla, K.K. (2013), "Flexural response of doubly curved laminated composite shells", *Sci. China Phys. Mech. Astr.*, **56**(4), 812-817. https://doi.org/10.1007/s11433-013-5020-x

Shen, H-S. (2002), "Postbuckling analysis of axially-loaded functionally graded cylindrical shells in thermal environments", *Compos. Sci. Tech.*, **62**, 977-987. https://doi.org/10.1016/S0266-3538(02)00029-5

Shooshtari, A. and Razavi, S. (2015), "Linear and nonlinear free vibration of a multilayered magneto-electro-elastic doubly-curved shell on elastic foundation", *Compos. Part B: Eng.*, **78**(1), 95-108. https://doi.org/10.1016/j.compositesb.2015.03.070

Thakur, S.N., Ray, C. and Chakraborty, S. (2017), "A new efficient higher-order shear deformation theory for a doubly curved laminated composite shell", *Acta Mech.*, **228**(1), 69-87. https://doi.org/10.1007/s00707-016-1693-3

Veysi, A., Shabani, R. and Rezazadeh, Gh. (2017), "Nonlinear vibrations of micro-doubly curved shallow shells based on the modified couple stress theory", *Nonlinear. Dyn.*, **87**(3), 2051-2065. https://doi.org/10.1007/s11071-016-3175-5

Wu, C.P. and Liu, K.Y. (2007), "A state space approach for the analysis of doubly curved functionally graded elastic and piezoelectric shells", *Tech Sci Press: Comput. Mater. Continua*, **6**(3), 177-199.

Yazdi, A.A. (2013), "Applicability of homotopy perturbation method to study the nonlinear vibration of doubly curved cross-ply shells", *Compos. Struct.*, **96**, 526-531. https://doi.org/10.1016/j.compstruct.2012.09.040

Yeh, J-Y. (2014), "Axisymmetric dynamic instability of polar orthotropic sandwich annular plate with ER damping treatment", *Smart Struct. Syst., Int. J.*, **13**(1), 124-136. https://doi.org/10.12989/sss.2014.13.1.025

Zehetner, C. and Irschik, H. (2008), "On the static and dynamic stability of beams with an axial piezoelectric actuation", *Smart Struct. Syst., Int. J.*, **4**(1), 67-84. https://doi.org/10.12989/sss.2008.4.1.067

Zhang, J., Van Campen, D.H., Zhang, G.Q. and Bouwman, V. (2001), "Dynamic stability of doubly curved orthotropic shallow shells under impact", *AIAA Journal*, **39**(5), 956-961. https://doi.org/10.2514/2.1401

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Appendix A

$$(1-(e_0a)^2\nabla^2)N_\alpha = A_1\frac{\partial u}{\partial\alpha} + A_2 w + A_3\frac{\partial\phi_1}{\partial\alpha} + A_4\frac{\partial v}{\partial\beta} + A_5 w + A_6\frac{\partial\phi_2}{\partial\beta} + A_7\Psi + N_\alpha{}^\Psi$$

$$(1-(e_0a)^2\nabla^2)N_\alpha^1 = A_8\frac{\partial u}{\partial\alpha} + A_9 w + A_{10}\frac{\partial\phi_1}{\partial\alpha} + A_{11}\frac{\partial v}{\partial\beta} + A_{12} w + A_{13}\frac{\partial\phi_2}{\partial\beta} + A_{14}\Psi + N_\alpha{}^{1\Psi}$$

$$(1-(e_0a)^2\nabla^2)M_\alpha = A_{15}\frac{\partial u}{\partial\alpha} + A_{16} w + A_{17}\frac{\partial\phi_1}{\partial\alpha} + A_{18}\frac{\partial v}{\partial\beta} + A_{19} w + A_{20}\frac{\partial\phi_2}{\partial\beta} + A_{21}\Psi + M_\alpha{}^\Psi$$

$$(1-(e_0a)^2\nabla^2)N_\beta = A_4\frac{\partial u}{\partial\alpha} + A_5 w + A_6\frac{\partial\phi_1}{\partial\alpha} + A_{22}\frac{\partial v}{\partial\beta} + A_{23} w + A_{24}\frac{\partial\phi_2}{\partial\beta} + A_7\Psi + N_\beta{}^\Psi$$

$$(1-(e_0a)^2\nabla^2)N_\beta^1 = A_{11}\frac{\partial u}{\partial\alpha} + A_{12} w + A_{13}\frac{\partial\phi_1}{\partial\alpha} + A_{25}\frac{\partial v}{\partial\beta} + A_{26} w + A_{27}\frac{\partial\phi_2}{\partial\beta} + A_{14}\Psi + N_\beta{}^{1\Psi}$$

$$(1-(e_0a)^2\nabla^2)M_\beta = A_{18}\frac{\partial u}{\partial\alpha} + A_{19} w + A_{20}\frac{\partial\phi_1}{\partial\alpha} + A_{28}\frac{\partial v}{\partial\beta} + A_{29} w + A_{30}\frac{\partial\phi_2}{\partial\beta} + A_{21}\Psi + M_\alpha{}^\Psi$$

$$(1-(e_0a)^2\nabla^2)N_{\beta z} = A_{31}\phi_2 + A_{32}\frac{\partial w}{\partial\beta} - A_{33} v - A_{34}\frac{\partial\Psi}{\partial\beta}$$

$$(1-(e_0a)^2\nabla^2)N_{\beta z}{}^1 = A_{39}\phi_2 + A_{40}\frac{\partial w}{\partial\beta} - A_{41} v - A_{42}\frac{\partial\Psi}{\partial\beta}$$

$$(1-(e_0a)^2\nabla^2)N_{\alpha z} = A_{35}\phi_1 + A_{36}\frac{\partial w}{\partial\alpha} - A_{37} u - A_{38}\frac{\partial\Psi}{\partial\alpha}$$

$$(1-(e_0a)^2\nabla^2)N_{\alpha z}{}^1 = A_{43}\phi_1 + A_{44}\frac{\partial w}{\partial\alpha} - A_{45} u - A_{46}\frac{\partial\Psi}{\partial\alpha}$$

$$(1-(e_0a)^2\nabla^2)N_{\alpha\beta} = A_{47}\frac{\partial v}{\partial\alpha} + A_{48}\frac{\partial u}{\partial\beta} + A_{49}\frac{\partial\phi_2}{\partial\alpha} + A_{50}\frac{\partial\phi_1}{\partial\beta} + A_{51}\frac{\partial v}{\partial\alpha} - A_{52}\frac{\partial u}{\partial\beta}$$

$$(1-(e_0a)^2\nabla^2)N_{\beta\alpha} = A_{53}\frac{\partial v}{\partial\alpha} + A_{54}\frac{\partial u}{\partial\beta} + A_{55}\frac{\partial\phi_2}{\partial\alpha} + A_{56}\frac{\partial\phi_1}{\partial\beta} + A_{57}\frac{\partial v}{\partial\alpha} - A_{58}\frac{\partial u}{\partial\beta}$$

$$(1-(e_0a)^2\nabla^2)M_{\alpha\beta} = A_{59}\frac{\partial v}{\partial\alpha} + A_{60}\frac{\partial u}{\partial\beta} + A_{61}\frac{\partial\phi_2}{\partial\alpha} + A_{62}\frac{\partial\phi_1}{\partial\beta} + A_{63}\frac{\partial v}{\partial\alpha} - A_{64}\frac{\partial u}{\partial\beta}$$

$$(1-(e_0a)^2\nabla^2)M_{\beta\alpha} = A_{65}\frac{\partial v}{\partial\alpha} + A_{66}\frac{\partial u}{\partial\beta} + A_{67}\frac{\partial\phi_2}{\partial\alpha} + A_{68}\frac{\partial\phi_1}{\partial\beta} + A_{69}\frac{\partial v}{\partial\alpha} - A_{70}\frac{\partial u}{\partial\beta}$$

$$\overline{D}_\alpha = A_{71}\phi_1 + A_{72}\frac{\partial w}{\partial \alpha} - A_{73}u - A_{74}\frac{\partial \Psi}{\partial \alpha}$$

$$\overline{D}_\beta = A_{75}\phi_2 + A_{76}\frac{\partial w}{\partial \beta} - A_{77}v - A_{78}\frac{\partial \Psi}{\partial \beta}$$

$$D_z = A_{79}\frac{\partial u}{\partial \alpha} + A_{80}w + A_{81}\frac{\partial \phi_1}{\partial \alpha} + A_{82}\frac{\partial v}{\partial \beta} + A_{83}w + A_{84}\frac{\partial \phi_2}{\partial \beta} + D_z{}^{\Psi} + A_{85}\Psi$$

Appendix B

$$\{A_1, A_2, A_3\} = \int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]C_{11}{}^{p}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}, \frac{1}{R_1}, \frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz + \int_{-h/2}^{+h/2}\left[1+\frac{z}{R_2}\right]C_{11}{}^{c}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}, \frac{1}{R_1}, \frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz + \int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]C_{11}{}^{p}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}, \frac{1}{R_1}, \frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz$$

$$\{A_4, A_5, A_6\} = \int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]C_{12}{}^{p}\left\{\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}, \frac{1}{R_2}, \frac{z}{R_2\left(1+\frac{z}{R_2}\right)}\right\}dz + \int_{-h/2}^{+h/2}\left[1+\frac{z}{R_2}\right]C_{12}{}^{c}\left\{\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}, \frac{1}{R_2}, \frac{z}{R_2\left(1+\frac{z}{R_2}\right)}\right\}dz + \int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]C_{12}{}^{p}\left\{\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}, \frac{1}{R_2}, \frac{z}{R_2\left(1+\frac{z}{R_2}\right)}\right\}dz$$

$$\{A_7\} = \int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]e_{13}\frac{\pi}{h}sin\frac{\pi z}{h}dz + \int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]e_{13}\frac{\pi}{h}sin\frac{\pi z}{h}dz,$$

$$N_\alpha{}^{\Psi} = \int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]\frac{2}{h}\Psi_0 e_{13}\,dz + \int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]\frac{2}{h}\Psi_0 e_{13}\,dz$$

$$\{A_8, A_9, A_{10}\} = \int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]\left[1+\frac{z}{R_1}\right]C_{11}{}^{p}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}, \frac{1}{R_1}, \frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz$$

$$+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_2}\right]\left[1+\frac{z}{R_1}\right]C_{11}{}^{c}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_1},\frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]\left[1+\frac{z}{R_1}\right]C_{11}{}^{p}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_1},\frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz$$

$$\{A_{11},A_{12},A_{13}\}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]\left[1+\frac{z}{R_1}\right]C_{12}{}^{p}\left\{\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_2},\frac{z}{R_2\left(1+\frac{z}{R_2}\right)}\right\}dz+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_2}\right]\left[1+\frac{z}{R_1}\right]C_{12}{}^{c}\left\{\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_2},\frac{z}{R_2\left(1+\frac{z}{R_2}\right)}\right\}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]\left[1+\frac{z}{R_1}\right]C_{12}{}^{p}\left\{\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_2},\frac{z}{R_2\left(1+\frac{z}{R_2}\right)}\right\}dz$$

$$\{A_{14}\}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]\left[1+\frac{z}{R_1}\right]e_{13}\frac{\pi}{h}sin\frac{\pi z}{h}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]\left[1+\frac{z}{R_1}\right]e_{13}\frac{\pi}{h}sin\frac{\pi z}{h}dz,$$

$$N_{\beta}{}^{\Psi}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_1}\right]\frac{2}{h}\Psi_0e_{13}\,dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_1}\right]\frac{2}{h}\Psi_0e_{13}\,dz$$

$$N_{\alpha}{}^{1\Psi}=N_{\beta}{}^{1\Psi}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]\left[1+\frac{z}{R_1}\right]\frac{2}{h}\Psi_0e_{13}\,dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]\left[1+\frac{z}{R_1}\right]\frac{2}{h}\Psi_0e_{13}\,dz$$

$$\{A_{15},A_{16},A_{17}\}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]zC_{11}{}^{p}\left\{\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_2},\frac{z}{R_2\left(1+\frac{z}{R_2}\right)}\right\}dz+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_2}\right]zC_{11}{}^{c}\left\{\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_2},\frac{z}{R_2\left(1+\frac{z}{R_2}\right)}\right\}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]zC_{11}{}^{p}\left\{\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_2},\frac{z}{R_2\left(1+\frac{z}{R_2}\right)}\right\}dz$$

$$\{A_{18},A_{19},A_{20}\}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]zC_{12}{}^{p}\left\{\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_2},\frac{z}{R_2\left(1+\frac{z}{R_2}\right)}\right\}dz+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_2}\right]zC_{12}{}^{c}\left\{\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_2},\frac{z}{R_2\left(1+\frac{z}{R_2}\right)}\right\}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]zC_{12}{}^{p}\left\{\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_2},\frac{z}{R_2\left(1+\frac{z}{R_2}\right)}\right\}dz$$

$$\{A_{21}\}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]ze_{13}\frac{\pi}{h}sin\frac{\pi z}{h}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]ze_{13}\frac{\pi}{h}sin\frac{\pi z}{h}dz$$

$$M_{\alpha}{}^{\Psi}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]\frac{2}{h}z\Psi_0e_{13}\,dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]\frac{2}{h}z\Psi_0e_{13}\,dz$$

$$\{A_{22},A_{23},A_{24}\}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]C_{22}{}^{p}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_1},\frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_2}\right]C_{22}{}^{c}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_1},\frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]C_{22}{}^{p}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_1},\frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz$$

$$\{A_{25},A_{26},A_{27}\}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]\left[1+\frac{z}{R_1}\right]C_{22}{}^{p}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_1},\frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_2}\right]\left[1+\frac{z}{R_1}\right]C_{22}{}^{c}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_1},\frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]\left[1+\frac{z}{R_1}\right]C_{22}{}^{p}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_1},\frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz$$

$$\{A_{28},A_{29},A_{30}\}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]zC_{22}{}^{p}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_1},\frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_2}\right]zC_{22}{}^{c}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_1},\frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]zC_{22}{}^{p}\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_1},\frac{z}{R_1\left(1+\frac{z}{R_1}\right)}\right\}dz$$

$$\{A_{31},A_{32},A_{33}\}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_1}\right]C_{44}{}^{p}\left\{1,\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_1}\right\}dz+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_1}\right]C_{44}{}^{c}\left\{1,\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_1}\right\}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_1}\right]C_{44}{}^{p}\left\{1,\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_1}\right\}dz$$

$$A_{34}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_1}\right]e_{42}\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\cos\frac{\pi z}{h}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_1}\right]e_{42}\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\cos\frac{\pi z}{h}dz$$

$$\{A_{35},A_{36},A_{37}\}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_1}\right]C_{55}{}^{p}\left\{1,\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2}\right\}dz+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_1}\right]C_{55}{}^{c}\left\{1,\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2}\right\}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_1}\right]C_{55}{}^{p}\left\{1,\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2}\right\}dz$$

$$A_{38}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_1}\right]e_{51}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}cos\frac{\pi z}{h}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_1}\right]e_{51}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}cos\frac{\pi z}{h}dz$$

$$\{A_{39},A_{40},A_{41}\}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]C_{44}{}^{p}\left\{1,\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_1}\right\}dz+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]C_{44}{}^{c}\left\{1,\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_1}\right\}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]C_{44}{}^{p}\left\{1,\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_1}\right\}dz$$

$$A_{42}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]e_{42}\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\cos\frac{\pi z}{h}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]e_{42}\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\cos\frac{\pi z}{h}dz$$

$$\{A_{43},A_{44},A_{45}\}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]C_{55}{}^{p}\left\{1,\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2}\right\}dz+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]C_{55}{}^{c}\left\{1,\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2}\right\}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]C_{55}{}^{p}\left\{1,\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2}\right\}dz$$

$$A_{46}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]e_{51}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}cos\frac{\pi z}{h}dz+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]e_{51}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}cos\frac{\pi z}{h}dz$$

$$\{A_{47},A_{48},A_{49},A_{50},A_{51},A_{52}\}=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]C_{66}{}^{p}\left\{\begin{matrix}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\\ z\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)(\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\\ z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\end{matrix}\right\}dz+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_2}\right]C_{66}{}^{c}$$

$$\left\{\begin{array}{c}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\\ z\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\right),\\ z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\end{array}\right\}dz$$

$$+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]C_{66}{}^{p}$$

$$\left\{\begin{array}{c}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\\ z\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\right),\\ z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\end{array}\right\}dz$$

$$\{A_{53},A_{54},A_{55},A_{56},A_{57},A_{58}\}$$

$$=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_1}\right]C_{66}{}^{p}$$

$$\left\{\begin{array}{c}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\\ z\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\right),\\ z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\end{array}\right\}dz$$

$$+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_1}\right]C_{66}{}^{c}$$

$$\left\{\begin{array}{c}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\\ z\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\right),\\ z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\end{array}\right\}dz$$

$$+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_1}\right]C_{66}{}^{p}$$

$$\left\{\begin{array}{c}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\\ z\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\right),\\ z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\end{array}\right\}dz$$

$$\{A_{59},A_{60},A_{61},A_{62},A_{63},A_{64}\}$$

$$=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]zC_{66}{}^{p}$$

$$\left\{\begin{array}{c}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\\ z\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\right),\\ z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\end{array}\right\}dz$$

$$+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_2}\right]zC_{66}{}^{c}$$

$$\left\{\begin{array}{c}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\\ z\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\right),\\ z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\end{array}\right\}dz$$

$$+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]zC_{66}{}^{p}$$

$$\left\{\begin{array}{c}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\\ z\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\right),\\ z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\end{array}\right\}dz$$

$$\{A_{65},A_{66},A_{67},A_{68},A_{69},A_{70}\}$$

$$=\int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_1}\right]zC_{66}{}^{p}$$

$$\left\{\begin{array}{c}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\\ z\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\right),\\ z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\end{array}\right\}dz$$

$$+\int_{-h/2}^{+h/2}\left[1+\frac{z}{R_1}\right]zC_{66}{}^{c}$$

$$\left\{\begin{matrix} \frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}, \\ z\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\right), \\ z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{R_2\left(1+\frac{z}{R_2}\right)} \end{matrix}\right\}dz$$

$$+\int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_1}\right]zC_{66}{}^{p}$$

$$\left\{\begin{matrix} \frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}, \\ z\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\right) \\ ,z\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{R_2\left(1+\frac{z}{R_2}\right)} \end{matrix}\right\}dz$$

$$\{A_{71},A_{72},A_{73}\} = \int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]e_{15}\left\{1,\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2}\right\}\cos\frac{\pi z}{h}\,dz + \int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]e_{15}\left\{1,\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2}\right\}\cos\frac{\pi z}{h}\,dz$$

$$\{A_{74}\} = \int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_2}\right]k_{11}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\frac{\partial\Psi}{\partial\alpha}\cos^2\frac{\pi z}{h}dz + \int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_2}\right]k_{11}\frac{1}{R_1\left(1+\frac{z}{R_1}\right)}\frac{\partial\Psi}{\partial\alpha}\cos^2\frac{\pi z}{h}dz$$

$$\{A_{75},A_{76},A_{77}\} = \int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_1}\right]e_{24}\left\{1,\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_1}\right\}\cos\frac{\pi z}{h}\,dz + \int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_1}\right]e_{24}\left\{1,\frac{1}{R_2\left(1+\frac{z}{R_2}\right)},\frac{1}{R_1}\right\}\cos\frac{\pi z}{h}\,dz$$

$$\{A_{78}\} = \int_{-h/2-h_p}^{-h/2}\left[1+\frac{z}{R_1}\right]k_{22}\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\frac{\partial\Psi}{\partial\alpha}\cos^2\frac{\pi z}{h}\,dz + \int_{+h/2}^{+h/2+h_p}\left[1+\frac{z}{R_1}\right]k_{22}\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\frac{\partial\Psi}{\partial\alpha}\cos^2\frac{\pi z}{h}\,dz$$

$$\{A_{79},A_{80},A_{81},A_{82},A_{83},A_{84}\} = \int_{-h/2}^{+h/2}\frac{\pi}{h}sin\frac{\pi z}{h}\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]e_{31}$$

$$\left\{\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_1},z\frac{1}{R_1\left(1+\frac{z}{R_1}\right)},\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}, \frac{1}{R_2},z\frac{1}{R_2\left(1+\frac{z}{R_2}\right)}\right\}0dz$$

$$A_{85} = \int_{-h/2-h_p}^{-h/2}\left(\frac{\pi}{h}sin\frac{\pi z}{h}\right)^2\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]k_{33}\,dz + \int_{+h/2}^{+h/2+h_p}\left(\frac{\pi}{h}sin\frac{\pi z}{h}\right)^2\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]k_{33}\,dz$$

$$D_z{}^{\Psi} = \int_{-h/2-h_p}^{-h/2}\frac{\pi}{h}sin\frac{\pi z}{h}\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]\frac{2}{h}\Psi_0 k_{33}\,dz + \int_{+h/2}^{+h/2+h_p}\frac{\pi}{h}sin\frac{\pi z}{h}\left[1+\frac{z}{R_1}\right]\left[1+\frac{z}{R_2}\right]\frac{2}{h}\Psi_0 k_{33}\,dz$$

Appendix C

$$K_{11} = -\frac{A_1}{R_1}\lambda_m{}^2 - \frac{\chi[A_{70} - A_{66}] + A_{54} - A_{58}}{R_2}\mu_n{}^2 - \frac{A_{45}}{R_1},$$

$$K_{12} = -\frac{A_3}{R_1}\lambda_m{}^2 - \left(\frac{A_{56} - \chi A_{68}}{R_2}\right)\mu_n{}^2 + \frac{A_{43}}{R_1}$$

$$K_{13} = -\left(\frac{A_4}{R_1} + \frac{A_{57} + A_{53} - \chi[A_{65} + A_{69}]}{R_2}\right)\lambda_m\mu_n,$$

$$K_{14} = -\left(\frac{A_6}{R_1} + \frac{A_{55} - \chi A_{67}}{R_2}\right)\lambda_m\mu_n,$$

$$K_{15} = \left(\frac{A_5 + A_2 + A_{44}}{R_1}\right)\lambda_m,$$

$$K_{16} = \left(\frac{A_7 - A_{46}}{R_1}\right)\lambda_m,$$

$$F_1 = -\frac{\partial}{\partial\alpha}\left(\frac{N_\alpha{}^\Psi}{R_1}\right)$$

$$K_{21} = -\frac{A_{15}}{R_1}\lambda_m{}^2 - \left(\frac{A_{66} - A_{70}}{R_2}\right)\mu_n{}^2 + A_{45},$$

$$K_{22} = -\frac{A_{17}}{R_1}\lambda_m{}^2 - \frac{A_{68}}{R_2}\mu_n{}^2 - A_{43},$$

$$K_{23} = -\left(\frac{A_{65} + A_{69}}{R_2} + \frac{A_{18}}{R_1}\right)\lambda_m\mu_n,$$

$$K_{24} = -\left(\frac{A_{20}}{R_1} + \frac{A_{67}}{R_2}\right)\lambda_m\mu_n,$$

$$K_{25} = +\left(\frac{A_{19} + A_{16}}{R_1} - A_{44}\right)\lambda_m$$

$$K_{26} = \left(\frac{A_{21}}{R_1} + A_{46}\right)\lambda_m,$$

$$F_2 = -\frac{\partial}{\partial\alpha}\left(\frac{M_\alpha{}^\Psi}{R_1}\right)$$

$$K_{31} = -\left(\frac{A_4}{R_2} + \frac{\chi[A_{60} - A_{64}] + A_{48} - A_{52}}{R_1}\right)\lambda_m\mu_n,$$

$$K_{23} = -\left(\frac{A_{50} + \chi A_{62}}{R_1} + \frac{A_6}{R_2}\right)\lambda_m\mu_n,$$

$$K_{33} = -\left(\frac{A_{47} + A_{51} + \chi[A_{59} + A_{63}]}{R_1}\right)\lambda_m{}^2 - \frac{A_{22}}{R_2}\mu_n{}^2 - \frac{A_{41}}{R_2},$$

$$K_{34} = -\left(\frac{A_{49} + \chi A_{61}}{R_1}\right)\lambda_m{}^2 - \frac{A_{24}}{R_2}\mu_n{}^2 + \frac{A_{39}}{R_1}$$

$$K_{35} = +\left(\frac{A_5 + A_{23} + A_{40}}{R_2}\right)\mu_n,$$

$$K_{36} = +\left(\frac{A_7 - A_{42}}{R_2}\right)\mu_n,$$

$$F_3 = -\frac{\partial}{\partial\beta}\left(\frac{N_\beta{}^\Psi}{R_2}\right)$$

$$K_{41} = -\left(\frac{A_{18}}{R_2} + \frac{A_{60} - A_{64}}{R_1}\right)\lambda_m\mu_n,$$

$$K_{42} = -\left(\frac{A_{62}}{R_1} + \frac{A_{20}}{R_2}\right)\lambda_m\mu_n$$

$$K_{43} = -\left(\frac{A_{59} + A_{63}}{R_1}\right)\lambda_m{}^2 - \frac{A_{28}}{R_2}\mu_n{}^2 + A_{41},$$

$$K_{44} = -\frac{A_{61}}{R_1}\lambda_m{}^2 - \frac{A_{30}}{R_2}\mu_n{}^2 - A_{39}$$

$$K_{45} = \left(\frac{A_{29} + A_{19}}{R_2} - A_{40}\right)\mu_n,$$

$$K_{46} = \left(\frac{A_{21}}{R_2} + A_{42}\right)\mu_n,$$

$$F_4 = -\frac{\partial}{\partial\beta}\left(\frac{M_\alpha{}^\Psi}{R_2}\right)$$

$$K_{51} = \left(\frac{A_{11}}{R_2} + \frac{A_8 + A_{37}}{R_1}\right)\lambda_m,$$

$$K_{52} = -\left(\frac{A_{35} - A_{10}}{R_1} - \frac{A_{13}}{R_2}\right)\lambda_m,$$

$$K_{53} = \left(\frac{A_{33} + A_{25}}{R_2} + \frac{A_{11}}{R_1}\right)\mu_n,$$

$$K_{54} = \left(\frac{A_{27} - A_{31}}{R_2} + \frac{A_{13}}{R_1}\right)\mu_n$$

$$\begin{aligned} K_{55} = &-\frac{A_{36}}{R_1}\lambda_m{}^2 - \frac{A_{32}}{R_2}\mu_n{}^2 - \left(\frac{A_9 + A_{12}}{R_1} + \frac{A_{12} + A_{26}}{R_2}\right) \\ &-\left(N_{0\alpha} + N_{E\alpha} + N_{M\alpha}\right)\frac{1}{R_1{}^2}\lambda_m{}^2 \\ &-\left(N_{0\beta} + N_{E\beta} + N_{M\beta}\right)\frac{1}{R_2{}^2}\mu_n{}^2 \\ &+\left(1 + (e_0a)^2\{\lambda_m{}^2 + \mu_n{}^2\}\right)\left[1 - \frac{h}{2R_1}\right]\left[1 - \frac{h}{2R_2}\right] \\ &\left\{K_w + K_G\left(\frac{\lambda_m{}^2}{\left(R_1 - \frac{h}{2}\right)^2} + \frac{\mu_n{}^2}{\left(R_2 - \frac{h}{2}\right)^2}\right)\right\}, \end{aligned}$$

$$K_{56} = \frac{A_{38}}{R_1}\lambda_m{}^2 + \frac{A_{34}}{R_2}\mu_n{}^2 - \left(\frac{A_{14}}{R_2} + \frac{A_{14}}{R_1}\right),$$

$$\begin{aligned} F_5 = &+\frac{N_\alpha{}^{1\Psi}}{R_1} + \frac{N_\beta{}^{1\Psi}}{R_2} \\ &+\left(1 + (e_0a)^2\{\lambda_m{}^2 + \mu_n{}^2\}\right)q\left[1 + \frac{h}{2R_1}\right]\left[1 + \frac{h}{2R_2}\right] \end{aligned}$$

$$K_{61} = \left(\frac{A_{73}}{R_1} - A_{79}\right)\lambda_m,$$

$$K_{62} = -\left(\frac{A_{71}}{R_1} + A_{81}\right)\lambda_m,$$

$$K_{63} = \left(\frac{A_{77}}{R_2} - A_{82}\right)\mu_n,$$

$$K_{64} = -\left(\frac{A_{75}}{R_2} + A_{84}\right)\mu_n,$$

$$K_{65} = -\frac{A_{72}}{R_1}\lambda_m{}^2 - \frac{A_{76}}{R_2}\mu_n{}^2 + (A_{80} + A_{83})$$

$$K_{65} = +\frac{A_{78}}{R_2}\mu_n{}^2 + \frac{A_{74}}{R_1}\lambda_m{}^2 + A_{85}$$

$$F_6 = -D_z{}^\Psi$$