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Modal parameter identification with compressed samples by sparse decomposition using the free vibration function as dictionary

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Abstract. Compressive sensing (CS) is a newly developed data acquisition and processing technique that takes advantage of the sparse structure in signals. Normally signals in their primitive space or format are reconstructed from their compressed measurements for further treatments, such as modal analysis for vibration data. This approach causes problems such as leakage, loss of fidelity, etc., and the computation of reconstruction itself is costly as well. Therefore, it is appealing to directly work on the compressed data without prior reconstruction of the original data. In this paper, a direct approach for modal analysis of damped systems is proposed by decomposing the compressed measurements with an appropriate dictionary. The damped free vibration function is adopted to form atoms in the dictionary for the following sparse decomposition. Compared with the normally used Fourier bases, the damped free vibration function spans a space with both the frequency and damping as the control variables. In order to efficiently search the enormous two-dimension dictionary with frequency and damping as variables, a two-step strategy is implemented combined with the Orthogonal Matching Pursuit (OMP) to determine the optimal atom in the dictionary, which greatly reduces the computation of the sparse decomposition. The performance of the proposed method is demonstrated by a numerical and an experimental example, and advantages of the method are revealed by comparison with another such kind method using POD technique.

Keywords: compressive sensing; sparse decomposition; redundant dictionary; orthogonal matching pursuit; modal parameter identification

1. Introduction

Continuous monitoring of key infrastructure (such as bridges, dams, offshore structures and building complexes) is increasing in demand for the integrity and safety of these structures. Structural health monitoring (SHM) technology emerged a few decades ago, and was essentially developed to serve this purpose. SHM integrates sensing technology, data acquisition and data processing to make meaningful assessment of the condition of structures. In fact, monitoring of bridges in real time has become a widely accepted practice worldwide today. An extensive review of the methods and applications of vibration-based SHM was given by Wei and Qiao (2011).

Cable wired system is a natural choice to realize SHM technology. However, the cost of cable installation and the overwhelming demand of maintenance to sustain a reliably wired SHM system are prohibitive to the widespread use of SHM technology. Alternatively, wireless sensor networks (WSN) offer a low cost, manageable and yet scalable solution for SHM (Lynch *et al*. 2002, 2003, Xu *et al*. 2004, Lynch 2007). There are several such platforms currently available for researchers, one of which is the wireless smart sensor (WSS) platform developed by the Smart Structure Technology Laboratory at the University of Illinois at Urbana-Champaign (Cho *et al*. 2010, Spencer *et al*. 2017). A Imote2 based WSS was deployed on the 2nd Jindo Bridge in South Korea, and claimed to be the world's largest wireless sensor network for civil structure monitoring (Spencer *et al*. 2017).

Due to the limited capacity of devices and resources on board of a smart sensor node, there are several issues that need to be better resolved before popularizing WSN-based SHM technology, among which are data synchronization for multi-channel sensing, data integrity due to packet loss in data transmission, data management of large data sets, and power management to minimize power consumption. These issues are especially true for the dynamic signal sensing.

In recent years, compressive sensing (CS) has brought great impact to traditional signal sampling and processing (Donoho 2006, Candès 2006, Candès *et al*. 2006, Fornasier and Rauhut 2011), which was built upon the Shannon-Nyquist sampling theory. The theory states that the number of samples required to reconstruct a signal without error is dictated by its bandwidth or the frequency range of interest. CS argues that it is possible to acquire far fewer samples than what is usually considered necessary, to reconstruct the signal accurately, and sometimes even exactly (Candès 2006). If the signal has a sparse representation, i.e., when it is decomposed on some basis, only a few coefficients are non zeros, and most are zeros or nearly zeros, we can sample the signal in a compressed form. The compressed measurements are linear inner products of the signal and an

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observation matrix, and the original signal can be recovered from the compressed measurements by pursuing the sparse solution using the l_0 norm (Fornasier and Rauhut 2011). The number of compressed samples needed for recovery is about 10-20% of those under the Shannon-Nyquist sampling framework (Duan and Kang 2014).

CS is believed to have great potential in structural health monitoring by addressing the above mentioned issues of WSN. The structural vibration signals are considered sparse when they are decomposed on the Fourier basis, or the wavelet basis. The sparsity reveals the vibration modes of the structure, which is the intrinsic dynamic property of the structure. Also, many problems in SHM, such as damage detection and modal updating, can be formulated as sparse recovery problems, as damages that exist in structures are sparse in nature. Applications of CS to vibration-based SHM have been presented in some literatures, and most research focus on data compression by CS. Bao *et al*. (2011) investigated the potential of CS for data compression of vibration data by reconstructing the signal both by wavelet and Fourier bases, and they compared the capacity of CS with tradition data compression methods. Mascarenas *et al*. (2013) designed and implemented a digital version of a compressed sensor. They collected compressed measurements by this sensor node and reconstructed original signal for detecting structural damage. Duan and Kang (2014) developed an improved recovery method by using the Polar interpolation in the Orthogonal Matching Pursuit (OMP) algorithm. This method improved the recovery accuracy of signals with increasing a little computation. Huang *et al*. (2016) proposed a Bayesian CS algorithm for the first time to reconstruct approximately sparse signals. Compared with other CS methods, the proposed algorithm showed superior performance in reconstruction robustness. Bao *et al*. (2017) developed a group sparse optimization algorithm to reconstruct the original data from incomplete measurements. They found that smaller reconstruction errors can be achieved using data from multiple sensors with the group sparse optimization method than using data from only single sensor. Pan *et al*. (2018) proposed a sparse recovery algorithm based on l_1-homotopy with a learned dictionary, which was more accurate and easily-implemented for streaming acoustic emission signals compared to alternative techniques/dictionaries.

Besides signal compression, CS has been introduced to deal with more problems in SHM. In application to WSN, CS is employed to recover lost data collected by WSN in the Shannon-Nyquist framework (Bao *et al*. 2013), and to reduce the volume of data and number of sensors in vibration monitoring (Ganesan *et al*. 2017). CS is further explored to reduce the power consumption of data acquisition in WSN (Bajwa *et al*. 2006, Feng *et al*. 2009, Ling and Tian 2010, O'Connor *et al*. 2014). The sparse recovery theory is also discussed for the structural damage detection and moving force identification (Zhou *et al*. 2013, 2015, Wang and Hao 2015, Zhang and Xu 2016, Hou *et al*. 2018).

In the application of CS in SHM, normally signals are compressively sampled under the framework of CS, then transmitted to a central data repository, where the original data is reconstructed from the compressed measurements. Subsequent data processing, such as modal parameter identification from structural vibration signals is performed in the same way as done with the Shannon-Nyquist sampling data (Bao *et al*. 2011, 2013, 2017, Mascarenas *et al*. 2013, Duan and Kang 2014, Huang *et al*. 2016, Ganesan *et al*. 2017, Bajwa *et al*. 2006, Feng *et al*. 2009). One problem of this approach is that distortion always exists in the reconstructed signals, and it will jeopardize the subsequent data processing and modal analysis. Moreover, the computation for the signal reconstruction is costly due to the nonlinearity of the sparsity recovery problem. Park *et al*. (2014) proposed a new approach to estimate the structural mode shapes from the compressed samples without requiring a reconstruction of the vibration signals. They proposed three measurement schemes and showed that mode shapes can be estimated directly from the compressed measurements with sufficient accuracy by just performing SVD to the compressive samples (Park *et al*. 2014). This procedure was interpreted as a POD technique (Park *et al*. 2015).

In this paper a new method for modal data identification without reconstruction of the original data will be proposed. The fundamental idea is to design an appropriate dictionary on which the compressed measurements are decomposed. The damped free vibration function is employed as the basis of the dictionary, by which mode shapes, modal frequencies and modal damping ratio can be extracted. The dictionary is designed to be redundant to ensure the sparse feature of the recovered variables. The proposed method can be applied to damped systems, and deliver the best possible estimation of modal information no matter what sampling scheme is adopted.

The rest of this paper is organized as follows: In section 2, following the structural modal analysis theory, the sparse recovery problem of modal parameter identification with compressed measurements is formulated. In section 3, design of the dictionary is introduced, and an algorithm to solve the sparse problem of modal identification is proposed. In section 4, synthetic compressed signals from a numerical and an experimental example respectively are used to support our analysis and discussion. Finally, Section 5 concludes this paper.

2. Theoretical development

The equation of motion of an unforced N-degree MDOF system is

$$[M]\{\ddot{u}(t)\}+[C]\{\dot{u}(t)\}+[K]\{u(t)\}=\{0(t)\} \tag{1}$$

where $[M]$, $[C]$ and $[K]$ are respectively the $N\times N$ symmetric mass, damping and stiffness matrices; $\{u(t)\}$ is the vector of system displacements. Note that $\{u(t)\}\{u_1(t),\ldots,u_N(t)\}$, and each $\{u(t)\}(i)=u_i(t),\ i\in 1,\ldots,N$ are displacement responses, which can be viewed as signals observed at the ith sensor node.

In case that $[C]$ is proportionally damped, the general solution to Eq. (1) is

$$\{u(t)\} = \sum_{n=1}^{N} \{\psi_n\} A_n e^{-\xi_n \omega_n t} sin(\omega_{d,n} t + \theta_n) \tag{2}$$

where $\{\psi_n\}$ is the normal mode shape, A_n is the amplitude of the mode signal, ξ_n is the modal damping ratio, ω_n and $\omega_{d,n}$ are the undamped and damped natural modal frequencies of the system, respectively; and θ_n is the phases of modal coordinate vector. $\{\psi_n\}$, ξ_n, ω_n and $\omega_{d,n}$ are determined by the characteristics of the system and unrelated to the system's initial conditions.

Firstly, reformulation of Eq. (2) by introducing the trigonometric function decomposition yields

$$\begin{aligned}\{u(t)\} = &\sum_{n=1}^{N} \{\psi_n\} A_n' e^{-\xi_n \omega_n t} sin(\omega_{d,n} t) \\ &+ \sum_{n=1}^{N} \{\psi_n\} A_n'' e^{-\xi_n \omega_n t} cos(\omega_{d,n} t)\end{aligned} \tag{3}$$

where $A_n' = A_n cos(\theta_n)$, $A_n'' = A_n sin(\theta_n)$ Then Eq. (3) can be written in a matrix-vector multiplication form as

$$\{u(t)\} = [\Psi][\Gamma]\{S\} \tag{4}$$

$$[\Gamma] = [[\Gamma'] \quad [\Gamma'']] \tag{5}$$

$$\{S\} = \{\{S'\} \quad \{S''\}\}^T \tag{6}$$

where

$$[\Gamma'] = \begin{bmatrix} A_1' & 0 & \cdots & 0 \\ 0 & A_2' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_N' \end{bmatrix}, \quad [\Gamma''] = \begin{bmatrix} A_1'' & 0 & \cdots & 0 \\ 0 & A_2'' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_N'' \end{bmatrix} \tag{7}$$

$$\{S'\} = \begin{Bmatrix} e^{-\xi_1 \omega_1 t} sin(\omega_{d,1} t) \\ e^{-\xi_2 \omega_2 t} sin(\omega_{d,2} t) \\ \vdots \\ e^{-\xi_N \omega_N t} sin(\omega_{d,N} t) \end{Bmatrix}, \quad \{S''\} = \begin{Bmatrix} e^{-\xi_1 \omega_1 t} cos(\omega_{d,1} t) \\ e^{-\xi_2 \omega_2 t} cos(\omega_{d,2} t) \\ \vdots \\ e^{-\xi_N \omega_N t} cos(\omega_{d,N} t) \end{Bmatrix} \tag{8}$$

$[\Psi] = [\{\psi_1\} \quad \{\psi_2\} \quad \cdots \quad \{\psi_N\}]$ denotes the mode shape matrix, and each column $\{\psi_n\}$ is a modal vector of *N* elements; $[\Gamma']$ and $[\Gamma'']$ are $N \times N$ diagonal matrices; $\{S'\}$ and $\{S''\}$ are $N \times 1$ modal coordinate vectors.

Assuming that *t* is sampled at *M* distinct instants in time, $t = t_1, \dots, t_M (M \gg N)$, then Eq. (4) with discrete times can be written in the following.

$$[U] = [\Psi][\Gamma][S] \tag{9}$$

$$[S] = [[S'] \quad [S'']]^T \tag{10}$$

$$[S'] = \begin{bmatrix} s_1'(t_1) & s_1'(t_2) & \cdots & s_1'(t_M) \\ s_2'(t_1) & s_2'(t_2) & \cdots & s_2'(t_M) \\ \vdots & \vdots & \ddots & \vdots \\ s_N'(t_1) & s_N'(t_2) & \cdots & s_N'(t_M) \end{bmatrix}, \quad [S''] = \begin{bmatrix} s_1''(t_1) & s_1''(t_2) & \cdots & s_1''(t_M) \\ s_2''(t_1) & s_2''(t_2) & \cdots & s_2''(t_M) \\ \vdots & \vdots & \ddots & \vdots \\ s_N''(t_1) & s_N''(t_2) & \cdots & s_N''(t_M) \end{bmatrix} \tag{11}$$

where $[S']$ and $[S'']$ are $N \times M$ matrices with $s_i'(t_j) = e^{-\xi_i \omega_i t} sin(\omega_{d,i} t_j)$, $s_i''(t_j) = e^{-\xi_i \omega_i t} cos(\omega_{d,i} t_j)$.

For a compressive sampling approach, where Eq. (9) is multiplied by an observation matrix$[\Phi]$, the compressed samples are obtained by

$$[Y] = [U][\Phi] = [\Psi][\Gamma][S][\Phi] \tag{12}$$

where $[\Phi]$ is a $M \times L$ random matrix ($M \gg L$, L is the length of compressed measurements); $[Y]$ is the compressed measurement matrix.

Park *et al*. (2014) have shown that with some sampling strategies that is different from the Shannon-Nyquist sampling, rows of the $[S]$ in Eq. (9) or $[S][\Phi]$ in Eq. (12) are nearly orthogonal, and thus modal vectors can be estimated by computing SVD of $[U]$ in Eq. (9) or $[Y]$ in Eq. (12).

As shown in Eqs. (2) and (9), the response of the system $[U]$ is a linear combination of the modal coordinate vectors in $[S]$. Inspired by the idea of sparse decomposition, if we can establish a dictionary $[D]$, which contain the modal coordinate vectors in $[S]$, then $[U]$ can be decomposed as

$$[U] = [\Upsilon][D] \tag{13}$$

Similarly, with the compressed measurements in Eq. (12), $[Y]$ can be decomposed as

$$[Y] = [\Upsilon][D][\Phi] \tag{14}$$

In Eqs. (13)-(14) , $[\Upsilon]$ is a $N \times J$ coefficient matrix, $[D]$ is a $J \times M$ matrix of dictionary, and each row of $[D]$ is an atom. If $[D]$ is designed to be redundant, i.e. $J \gg 2N$, and all rows of $[S]$ are contained in $[D]$, then $[\Upsilon]$ is sparse, which means that $[\Upsilon]$ will have at most 2*N* nonzero column vectors. Then the mode shapes, i.e., $[\Psi][\Gamma]$ as a whole, can be extracted from those nonzero column vectors of $[\Upsilon]$, and the at most 2*N* atoms in $[D]$, which correspond to the 2*N* nonzero column vectors in $[\Upsilon]$, are used to estimate the modal frequencies and damping in $[S]$. The remaining problem is how to design an appropriate redundant dictionary $[D]$.

3. Dictionary and sparse decomposition algorithm

3.1 Design of redundant dictionary

The modal coordinates in $\{S'\}$ and $\{S''\}$ are damped free vibration functions, and they are used as the bases for the design of the dictionary matrix $[D]$. Let's rephrase the modal coordinates in $\{S'\}$ and $\{S''\}$ as $e^{\frac{-abt}{\sqrt{1-a^2}}} sin(bt)$ and $e^{\frac{-abt}{\sqrt{1-a^2}}} cos(bt)$, respectively, where the parameters a, b represent the damping ratio and the damped natural frequency, respectively.

Discretization of the damped natural frequency, $b_l(l \in 1, \ldots, p)$ and the damping coefficient, $a_m(m \in 1, \ldots, q)$ gives atoms of the dictionary as follows

$$d'_{l,m} = e^{\frac{-a_m b_l t}{\sqrt{1-a_m^2}}} sin(b_l t) \quad \text{and} \quad d''_{l,m} = e^{\frac{-a_m b_l t}{\sqrt{1-a_m^2}}} cos(b_l t) \tag{15}$$

Therefore, the dictionary is formatted as $\{D\} = \{\{D'\} \quad \{D''\}\}^T$, where the two component vectors are

$$\{D'\} = \left\{ \begin{matrix} \left. \begin{matrix} d'_{l=1,m=1} \\ \vdots \\ d'_{l=1,m=q} \end{matrix} \right\} l = 1 \\ \vdots \\ \left. \begin{matrix} d'_{l=p,m=1} \\ \vdots \\ d'_{l=p,m=q} \end{matrix} \right\} l = p \end{matrix} \right\}, \quad \{D''\} = \left\{ \begin{matrix} \left. \begin{matrix} d''_{l=1,m=1} \\ \vdots \\ d''_{l=1,m=q} \end{matrix} \right\} l = 1 \\ \vdots \\ \left. \begin{matrix} d''_{l=p,m=1} \\ \vdots \\ d''_{l=p,m=q} \end{matrix} \right\} l = p \end{matrix} \right\} \tag{16}$$

There are altogether $2(p \times q)$ atoms in the dictionary. Note that the atoms in Eq. (16) are functions of time t. Time t is then sampled at M distinct points, $t = t_1, \ldots, t_M$, and a dictionary matrix $[D] = [[D'] \quad [D'']]^T$ is then obtained, in which $[D']$ and $[D'']$ are both of $(p \times q) \times M$ dimension.

In Eq. (15) $b_l(l \in 1, \ldots, p)$ is designed to sample from $[\omega_{dmin}, \omega_{dmax}]$, where ω_{dmin} and ω_{dmax} are the lowest and highest possible damped frequencies, respectively, and $a_m(m \in 1, \ldots, q)$ is usually sampled from $[0, \xi_{max}]$, where ξ_{max} is the maximum possible damping ratio. These bounds are determined by experience or range of interest.

By increasing the number of discretization points of b and$a$, the number of atoms in dictionary $[D']$ and $[D'']$ increases, and usually $p \times q \gg N$ is required, which in return enhances the sparsity of the variables to be sought, and reduces the leakage and improves accuracy of the identification of the modal parameters. But be aware of that increased redundancy of the dictionary also leads to increased coherence among the atoms in the dictionary (Rauhut *et al*. 2008), which will reduce the efficiency of the recovery algorithms employed to solve the sparse optimization problem.

3.2 Combination of OMP and two-step search strategy

With the redundant dictionary defined in Eq. (16), the compressed measurement matrix $[Y]$ is decomposed as shown in Eq. (14). $[Y]$ being column sparse, it has at most $2N$ nonzero column vectors. We then use matrix $[\hat{Y}]$ to approximate the nonzero columns in $[Y]$, and matrix $[\hat{D}]$ to approximate the at most $2N$ atoms in $[D]$corresponding to the nonzero columns in $[\hat{Y}]$. Thus, an approximation of $[Y]$ is made as

$$[Y] \approx [\hat{Y}][\hat{D}][\Phi] \tag{17}$$

To find the optimum $[\hat{Y}]$ and the corresponding $[\hat{D}]$, the following optimization problem is to be solved

$$\begin{aligned} & min\left\|[Y] - [\hat{Y}][\hat{D}][\Phi]\right\|_2 \\ & \text{subjct to } [\hat{D}]_{2N \times M} \in \begin{bmatrix} [D'] \\ [D''] \end{bmatrix}_{2(p \times q) \times M} \end{aligned} \tag{18}$$

Considering the $2(p \times q)$ number of atoms in the dictionary $[D]$, it needs tremendous computation to search for the optimal atoms in the dictionary. The Orthogonal Matching Pursuit (OMP), one of the most powerful algorithm to compute the sparse representations with the least complexity (Liu and Temlyakov 2012) is to be used to solve the above optimization problem. The OMP algorithm has an iteration scheme, and within each iteration, we propose a two-step strategy to determine the optimal atoms.

Firstly, fixing $m = 1$, the optimal atom with $b_{\hat{l}}(\hat{l} \in 1, \ldots, p)$ is chosen from the subset dictionary of $[D]$ with $d'_{l=\{1,\ldots,p\},m=1}$ and $d''_{l=\{1,\ldots,p\},m=1}$. This subset dictionary has all atoms with the same damping ratio $a_{m=1}$. This is actually to pick up the atom with the best matched undamped frequency if $a_{m=1} = 0$.

Secondly, fixing $l = \hat{l}$, the optimal atom with $a_{\hat{m}} = (\hat{m} \in 1, \ldots, q)$ is chosen from the subset dictionary of $[D]$ with $d'_{l=\hat{l},m=\{1,\ldots,q\}}$ and $d''_{l=\hat{l},m=\{1,\ldots,q\}}$. This subset dictionary has all atoms with the same frequency $b_{l=\hat{l}}$. This is to search the atom with the best matched damping ratio.

The two-step searching strategy is illustrated in Fig. 1, followed by the pseudo codes of the algorithm. This strategy greatly reduces the computation load in the sparse decomposition. Finally, within each loop, two optimal atoms $d'_{l=\hat{l},m=\hat{m}}$ and $d''_{l=\hat{l},m=\hat{m}}$ are searched out, and their parameters $b_{\hat{l}}$ and $a_{\hat{m}}$ are the optimal estimate of damped natural frequency and damping ratio respectively, according to the functions of atoms in Eq. (15).

The damping ratios of any order of modes always fall into a small positive range near zero, but the frequencies of structures may vary widely for different structures and order of modes. Therefore, in the above two-step searching strategy, we can conveniently fix the damping ratio at zero first, and search for the undamped frequency of the structures; and then given the optimal frequency, we further search for the optimal damping ratio. But if the order of searching is reversed, it would be hard for us to set an

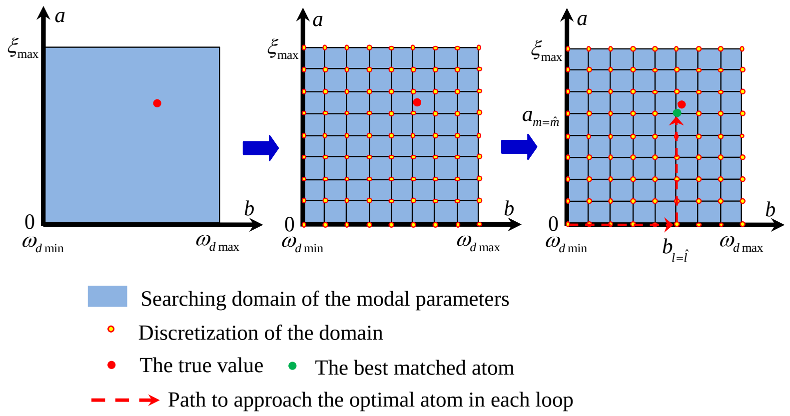

Fig. 1 Two-step searching strategy

appropriate initial value for the frequency, which may cause greater dependence of the searching results on the initially picked value.

Pseudo code of the algorithm

Input : $[Y]$, $[D']$, $[D'']$, and $[\Phi]$

Output: $[\hat{Y}]$ and $[\hat{D}]$

Initialization: $[\hat{Y}]=[\emptyset]$, $[\hat{D}]=[\emptyset]$, $\{Z\}=\emptyset$, $[R]=[Y]$

For i = 1 to N (N is the number of degree of the system)

(a) Form the subset dictionaries $[D']_{m=1}$ and $[D'']_{m=1}$ by picking up $d'_{l=\{1,\ldots,p\},m=1}$ and $d''_{l=\{1,\ldots,p\},m=1}$

(b) Search for the optimal atoms $d'_{l=\hat{l},m=1}$ and $d''_{l=\hat{l},m=1}$ by solving the following problem,

$$argmax\left(\left\|d'_{l,m=1}[\Phi][R]^T\right\|_2+\left\|d''_{l,m=1}[\Phi][R]^T\right\|_2\right)$$

subject to $d'_{l,m=1}\in[D']_{m=1}$, $d''_{l,m=1}\in[D'']_{m=1}$, $l\notin\{Z\}$

(c) Form the subset dictionaries $[D']_{l=\hat{l}}$ and $[D'']_{l=\hat{l}}$ by picking up $d'_{l=\hat{l},m=\{1,\ldots,q\}}$ and $d''_{l=\hat{l},m=\{1,\ldots,q\}}$

(d) Search for the optimal atoms $d'_{l=\hat{l},m=\hat{m}}$ and $d''_{l=\hat{l},m=\hat{m}}$ by solving the following problem,

$$argmax\left(\left\|d'_{l=\hat{l},m}[\Phi][R]^T\right\|_2+\left\|d''_{l=\hat{l},m}[\Phi][R]^T\right\|_2\right)$$

subject to $d'_{l=\hat{l},m}\in[D']_{l=\hat{l}}$, $d''_{l=\hat{l},m}\in[D'']_{l=\hat{l}}$

(e) Let $[\hat{D}]=\left[[\hat{D}]^T \quad d'^T{}_{l=\hat{l},m=\hat{m}} \quad d''^T{}_{l=\hat{l},m=\hat{m}}\right]^T$,$\{Z\}=\{Z,\hat{l}\}$

(f) $[\hat{Y}]=\frac{[R]}{([\hat{D}][\Phi])}$, $[R]=[R]-[\hat{Y}][\hat{D}]$

End

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4. Illustrative examples

4.1 Numerical experiment

In this section, the performance of the proposed method is tested by a numerical experiment. The synthetically compressed measurement data set is acquired from a simple 4-degree-of-freedom damped system under free vibration as shown in Fig. 2.

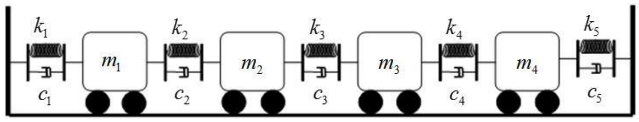

Fig. 2 A simple 4-degree-of-freedom damped system

The linear spring-damper elements are used to connect the adjacent pairs of masses. For simplicity's sake, the spring stiffness is set equal, $k_1=k_2=k_3=k_4=k_5=1000N/m$. The stiffness matrix $[K]$ of the above system is

$$[K]=\begin{bmatrix}k_1+k_2 & -k_2 & 0 & 0\\ -k_2 & k_2+k_3 & -k_3 & 0\\ 0 & -k_3 & k_3+k_4 & -k_4\\ 0 & 0 & -k_4 & k_4+k_5\end{bmatrix}$$
$$=\begin{bmatrix}2 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 2\end{bmatrix}\times 1000N/m.$$

All lumped masses of this system are set equal to $1kg$, and then the mass matrix $[M]$ is an identity matrix. Proportional damping is considered, $[C]=\beta[K]+0.1[M]$. Different damping levels are assumed by setting $\beta=0$, 0.001 and 0.002.

Matrices $[\Psi]$, $\{S'\}$ and $\{S''\}$ are acquired from the analytical model described above. The diagonal matrix $[\Gamma']$ and $[\Gamma'']$ are determined by the initial condition of a given system, which in this example are set as follows

$$[\Gamma']=[\Gamma'']=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0.8 & 0 & 0\\ 0 & 0 & 0.6 & 0\\ 0 & 0 & 0 & 0.4\end{bmatrix}$$

The free responses of the system are numerically generated by Eq. (4). The response matrix $[U]$ is formed by sampling $\{u(t)\}$ at 500 distinct points at time t = 0.00, 0.04, 0.08,…, 19.96s. The compressed measurements are synthesized by multiplying the data matrix $[U]$ with a random matrix $[\Phi]$. Here we use a Bernoulli random matrix, of which the element is either 1 or -1. The number of compressed measurements L defines the number of row

Table 1 Estimated modal parameters of the 4 DOFs system by SD and POD

β	[1]Compression Rate	Mode	MAC(SD)	MAC(POD)	Damped frequency by SD (Hz)		Damping ratio by SD(%)		Computing time (s)	
					[1]Exact	estimated	[2]Exact	estimated	SD	POD
0	2	1	0.9999	0.8760	3.11	3.11	0.26	0.21	79.1	1.1
		2	0.9999	0.6312	5.92	5.92	0.13	0.12		
		3	0.9999	0.7550	8.14	8.15	0.10	0.10		
		4	0.9996	0.9729	9.57	9.57	0.08	0.06		
	5	1	0.9948	0.9087		3.10		0.39	79.2	0.8
		2	0.9999	0.8477		5.92		0.15		
		3	0.9997	0.7781		8.14		0.11		
		4	1.0000	0.8594		9.56		0.09		
	10	1	0.9997	0.8256		3.10		0.24	79.3	0.7
		2	0.9997	0.6280		5.91		0.14		
		3	0.9999	0.9593		8.15		0.11		
		4	0.9994	0.7722		9.57		0.08		
0.001	2	1	1.0000	0.9982	3.11	3.12	1.23	1.19	77.1	1.1
		2	0.9998	0.9782	5.92	5.91	1.99	2.06		
		3	1.0000	0.8769	8.14	8.16	2.66	1.92		
		4	0.9998	0.8968	9.57	9.58	3.09	2.80		
	5	1	0.9999	0.9909		3.10		1.27	77.2	0.8
		2	1.0000	0.9263		5.89		2.22		
		3	0.9999	0.6817		8.12		1.76		
		4	0.9996	0.7246		9.49		3.00		
	10	1	0.9994	0.9909		3.10		1.27	76.3	0.7
		2	0.9976	0.9263		5.93		1.78		
		3	0.9371	0.6817		8.30		6.21		
		4	0.9938	0.7246		9.43		2.10		
0.002	2	1	0.9999	0.9968	3.11	3.12	2.21	1.99	77.9	1.1
		2	0.9968	0.9814	5.92	5.90	3.85	4.04		
		3	0.9908	0.8164	8.14	8.12	5.21	2.41		
		4	0.9960	0.8305	9.57	9.32	6.10	6.97		
	5	1	0.9996	0.9998		3.11		2.11	76.6	0.8
		2	0.9993	0.9721		5.85		4.10		
		3	0.9948	0.7724		8.10		2.35		
		4	0.9956	0.7924		9.37		5.23		
	10	1	0.9980	0.9970		3.11		2.42	77.9	0.7
		2	0.9700	0.7780		5.94		4.87		
		3	0.9835	0.6336		8.24		3.43		
		4	0.9761	0.7717		10.72		10.00		

of $[\Phi]$, and it is set to be 1/2, 1/5, and 1/10 respectively, of the number of the original discrete time data, which is 500 in this case, and compression rates of 2, 5 and 10 are thus achieved, respectively.

A redundant dictionary is needed to decompose the compressed measurements. According the compressive sensing theory, to ensure sparsity of the recovered parameter, a redundant dictionary is necessary. But more redundancy means more coherence among atoms in the dictionary, which weakens the restricted isometry property (RIP) of the compressed matrix, and reduces the effectiveness the sparse recovery algorithms (Eldar and Kutyniok 2012). On the contrary, less redundancy of the dictionary means coarse discretization of the control variables, which reduces the sparsity and increases leakage of the recovery variables. A tradeoff has to be made. Increased coherence between atoms with increasing number of discretization of frequency and damping ratio is shown in

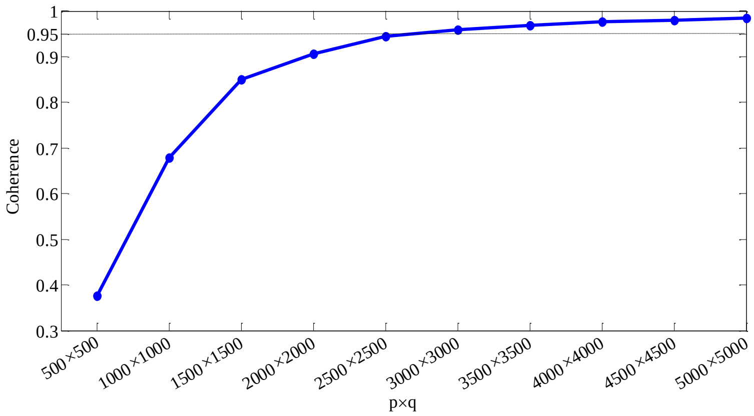

Fig. 3 Coherence of atoms in dictionary

Fig. 3. Discritization points of 2501 for both variables are chosen for the design of the dictionary in this example.

A redundant dictionary is established by the method developed in section 3.1. The damped frequency range is set to be [0-12.5] Hz, and this range is discretized uniformly at 2501 distinct points, $\{b_1 = 0, b_2 = 0.005, \ldots, b_{2501} = 12.5\}$ Hz. The damping ratio range is set to be [0-0.5], and the range is discretized at 2501 uniform distinct points, $\{a_1 = 0, a_2 = 0.0002, \ldots, a_{2501} = 0.5\}$. The quantization error is 0.005 Hz for the frequency and 0.0002 for the damping ratio, which are believed to be sufficient.

With the above discretization scheme, a dictionary with $2 \times 2501 \times 2501$ atoms is generated, which is used to decompose the compressed measurements by the OMP algorithm with the proposed two-step search strategy. It is worth emphasizing that, with the proposed searching strategy, the optimal atoms are selected from a sub dictionary with $2 \times (2501 + 2501)$ atoms in each loop, which remarkably reduces the computational time.

The modal frequencies, modal damping and mode shapes are then identified from the selected atoms. The sparse decomposition method developed in this paper is denoted as SD in the following part. For comparison purpose, the Proper Orthogonal Decomposition (POD) (Park *et al*. 2014) is also used to estimate the mode shapes. The accuracy of the mode shapes estimated by the SD and POD is evaluated by the modal assurance criterion (*MAC*)

$$MAC = \frac{\left(\{\hat{\psi}_i\}^T\{\psi_i\}\right)^2}{\left(\{\hat{\psi}_i\}^T\{\hat{\psi}_i\}\right)\left(\{\psi_i\}^T\{\psi_i\}\right)} \tag{19}$$

where $\{\hat{\psi}_i\}$ and $\{\psi_i\}$ denote the estimated and theoretical values of the *i*th mode shape vector, respectively. The comparison of the accuracy of the estimated mode shapes by SD and POD are shown in Table 1, and the analytical and identified frequencies and damping ratios are also given in the table.

The results in Table 1 show that, without reconstruction of the original signals, the sparse decomposition method accurately identifies the modal frequencies and modal shape vectors from the compressed measurements for the three compression ratio cases. Comparing with the POD technique, the SD technique attains a better estimate of model shape vectors in every compression ratio and damping case. However, damping identification is more challenging, especially for heavy damped systems. Computation time of the SD method is less than 80 seconds for each run of simulation on PC with CPU i3-4170 3.70 Ghz and memory 8 G bytes, showing that SD is more computational intensive than POD.

As we study a method that identify the modal parameters from the compressed measurements directly, without a prior reconstruction of the time history of signals, to be fair, we only compare the performance of the proposed SD method with the method of its kind, namely, the POD technique proposed by Park *et al*. (2014, 2015). The computational time of the POD technique and the SD method, shown in Table 1 indicates that a better recovery of modal data by SD over the POD is achieved at the cost of more computation.

On the other hand, if we choose to recover the time history signal first, and then do modal analysis using conventional methods in time domain, such as ITD, or methods in frequency domain, such as FDD, it is shown that distortion of the reconstructed signals jeopardizes the subsequent modal analysis, which renders the modal analysis results sensitive to the quality of the reconstructed signals and noises level. Moreover, the computation for the time history signal reconstruction itself is costly due to the nonlinear nature of the sparsity recovery problem.

In order to investigate the noise effect of the signal on the modal data recovery by the proposed method, two noise levels of 20% and 40% are considered in the simulated vibration signals. White noise is assumed, and the noise percentage is the RMS of the noise over that of the signal. The results of the case with $\beta = 0.001$ and compression rate 10 are shown in Table 2, which indicates that the performance of the proposed method is fairly good even in a very noisy environment.

4.2 Model experiment

A hammer-hit test of a three-story structure model in the laboratory was conducted to confirm the effectiveness of the proposed method. The structure was made up of aluminum with three rigid plates supported by four

Table 2 Performance of SD with different noise percentage

Noise percentage	Mode	MAC	Damped frequency(Hz)	Damping ratio(%)
0%	1	0.9994	3.10	1.27
	2	0.9976	5.93	1.78
	3	0.9371	8.30	6.21
	4	0.9938	9.43	2.10
20%	1	0.9964	3.11	1.82
	2	0.9894	5.93	1.30
	3	0.9723	8.12	2.11
	4	0.9935	9.71	2.30
40%	1	0.9977	3.11	2.02
	2	0.9971	5.95	1.98
	3	0.9816	8.13	2.66
	4	--	--	--

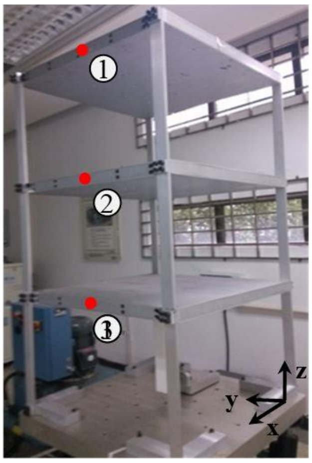

Fig. 4 The 3-story structure model and layout of sensors

rectangular columns at each corner, connected by bolts. The structure is based on a rigid plate, which is anchored to the ground by bolts. Three uniaxial accelerometers were deployed on each floor to acquire the X-direction accelerations, as shown in Fig. 4. The impact load of the hammer was imposed on the third floor along the X-direction, and the vibration signals in the X-direction were

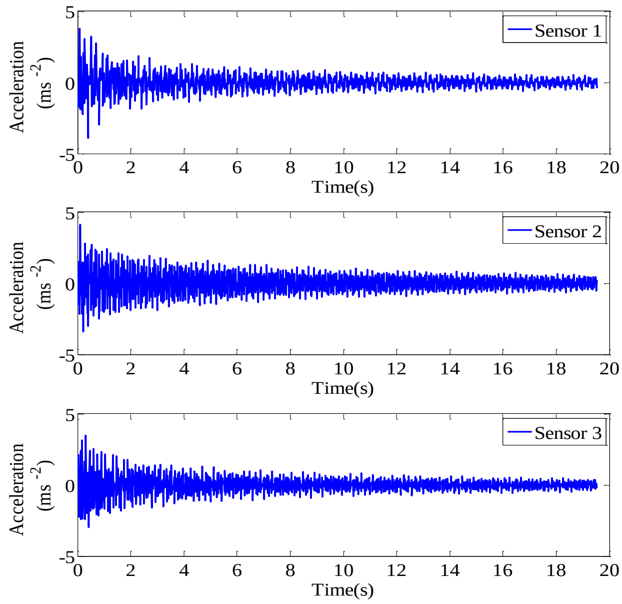

Fig. 5 X-axial accelerations collected from the structure model before compression

collected with a sampling frequency of 51.2 Hz. The sampling duration was 19.53 seconds for 1000 data points. Three examples of acceleration histories are shown in Fig. 5.

The experiment data is firstly used to estimate modal parameters of the structure by the Ibrahim Time Domain (ITD) method, results of which are taken as benchmarks shown in the Table 3. Then compressed measurements are obtained by multiplying the collected data with a random matrix. Compressed data length of 100 and 50 is considered, and the compression ratios are 10 and 20, respectively.

A redundant dictionary is created for the sparse recovery of modal parameters. The range of frequency is [0-25.6]Hz with an interval of 0.005 Hz, i.e., $\{b_1 = 0, b_2 = 0.005, \ldots, b_{5121} = 25.600\}$ Hz; and the range of damping ratio is [0-0.5] with an interval of 0.0001, i.e., $\{a_1 = 0, a_2 = 0.0001, \ldots, a_{5001} = 0.5\}$. Modal frequencies and damping ratios estimated by the SD using the compressed measurements are shown in Table 3. Table 3 also shows the MACs of the mode shapes estimated from the compressed measurements by the POD and SD, respectively, with

Table 3 Estimated modal parameters of the structure model by SD, POD and ITD

Compression rate	Mode	MAC(SD)	MAC(POD)	Damped frequency(Hz)		Damping ratio(%)	
				*ITD	SD	*ITD	SD
10	1	0.9967	0.9717	4.98	5.02	1.20	1.44
	2	0.9978	0.9657	13.95	13.99	0.76	0.96
	3	0.9999	0.9397	20.82	20.83	0.18	0.23
20	1	0.9957	0.9681		5.02		1.10
	2	0.9976	0.9408		14.00		1.30
	3	0.9898	0.9243		20.83		0.42

*ITD: "ITD" is obtained by the original time-domain data, and are regardless of the compression rate modal shape vectors identified by ITD as the benchmarks to compute MACs using Eq. (22).

The experiment results show that the identified modal frequencies from the compressed measurements match very well with those identified by ITD from the uncompressed signals. It also confirms that the SD technique performs better in identification of modal shape vectors over the POD technique. As for the damping identification, SD results are fairly consistent with those by ITD for this light damping structure.

5. Conclusions

A structural modal parameter identification method with compressed measurements is proposed in this research. Based on the formulation of free vibration of linear structures, a sparse decomposition problem is formulated to perform modal analysis directly on the compressively measured responses of structures without a prior reconstruction of the original data. By introducing the damped free vibration function as the basis for the design of the dictionary, a redundant dictionary with frequency and damping ratio as the control variables is designed. An efficient two-step search strategy to search the optimal atoms in the dictionary is put forward. Combined with the Orthogonal Matching Pursuit (OMP) algorithm, the sparse decomposition problem is solved, and modal parameters are extracted. Both the numerical and the experimental examples show that the proposed SD technique successfully recovers the modal frequencies, modal shape vectors and modal damping ratios with good accuracy. The SD technique outperforms the POD technique for the cases tested, and it extends the application of this category of methods from the undamped system to the damped one. However, it remains a problem to extract modal parameters from compressed measurements in the prevailing ambient vibration conditions.

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Nomenclature

N	number of degree of a system
$[M]$	mass matrix
$[C]$	damping matrix
$[K]$	stiffness matrices
$[\Psi]$	mode shape matrix
$\{\psi_i\}$	the ith mode shape vector
$\{\hat{\psi}_i\}$	the estimated$i$thmode shape vector
$\{u(t)\}$	vector of displacement
ξ_i	the ith modal damping ratio
ω_i	the ith undamped natural frequency
$\omega_{d,i}$	the ith damped natural frequency
θ_i	the phase of the ith modal coordinate vector
A_i	amplitude of the ith mode signal
A_i', A_i''	factorized amplitude of the ith mode signal
$[\Gamma]$	amplitude matrix
$[\Gamma']$, $[\Gamma'']$	subdivided amplitude matrices
$\{S\}$	modal coordinate matrix
$\{S'\}$, $\{S''\}$	subdivided modal coordinate matrix
$[U]$	matrix of system displacements
$[Y]$	compressed measurement matrix
$[\Phi]$	observation matrix
L	the length of compressed measurements
M	number of discrete time instant
$[D]$	dictionary for decomposition
$[\hat{D}]$	submatrix of $[D]$
$\{D\}$	dictionary designed based on damped free vibration functions
$\{D'\}$, $\{D''\}$	submatrices of $\{D\}$
$d_{l,m}'$, $d_{l,m}''$	atms in $\{D'\}$ and $\{D''\}$
$d_{l=\hat{l},m=\hat{m}}'$, $d_{l=\hat{l},m=\hat{m}}''$	atoms searched out by OMP
ω_{dmin}, ω_{dmax}	the lowest and highest possible damped frequencies
ξ_{max}	the maximum possible damping ratio
$[\Upsilon]$	coefficient matrix
$[\hat{\Upsilon}]$	submatrix of $[\Upsilon]$
$s_i'(t_j)$, $s_i''(t_j)$	the ith modal coordinate at time t_j
J	number of atom in $[D]$
a	parameter for representing damping ratio
a_m	discretized damping coefficient
b	parameter for representing damped natural frequency
b_l	discretized damped natural frequency
t	time