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Application of nonlocal elasticity theory on the wave propagation of flexoelectric functionally graded (FG) timoshenko nano-beams considering surface effects and residual surface stress

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Abstract. This research deals with wave propagation of the functionally graded (FG) nano-beams based on the nonlocal elasticity theory considering surface and flexoelectric effects. The FG nano-beam is resting in Winkler-Pasternak foundation. It is assumed that the material properties of the nano-beam changes continuously along the thickness direction according to simple power-law form. In order to include coupling of strain gradients and electrical polarizations in governing equations of motion, the nonlocal non-classical nano-beam model containg flexoelectric effect is used. Also, the effects of surface elasticity, di-electricity and piezoelectricity as well as bulk flexoelectricity are all taken into consideration. The governing equations of motion are derived using Hamilton principle based on first shear deformation beam theory (FSDBT) and also considering residual surface stresses. The analytical method is used to calculate phase velocity of wave propagation in FG nano-beam as well as cut-off frequency. After verification with validated reference, comprehensive numerical results are presented to investigate the influence of important parameters such as flexoelectric coefficients of the surface, bulk and residual surface stresses, Winkler and shear coefficients of foundation, power gradient index of FG material, and geometric dimensions on the wave propagation characteristics of FG nano-beam. The numerical results indicate that considering surface effects/flexoelectric property caused phase velocity increases/decreases in low wave number range, respectively. The influences of aforementioned parameters on the occurrence cut-off frequency point are very small.

Keywords: flexoelectric; surface effects; residual surface stresses; functionally graded beam; flexoelectricity

1. Introduction

The beams are one of the main components of engineering structures applicable in the mechanical and electrical instruments. They can be used in macro, micro or nano scales. Various analyses of the beam structures in mentioned scales need more consideration and theories. The various environments such as thermal, electrical and different types of foundation can significantly change the responses of the beams. In addition manufacturing the beams with piezoelectric materials leads to intelligent systems that can be used as elements of nano-electro-mechanical-systems. Combination of above mentioned aspects of beams leads to an interesting problems in scope of mechanical engineering and nano-electro-mechanical-systems (Arani *et al*. 2014, Arefi and Zenkour 2017 a, b, c, d, Arefi *et al.* 2017). In recent years, advances in materials engineering has led to the emergence of new materials known as functionally graded materials (FGMs). Smooth and continuous variation of thermo-mechanical properties in a specific direction is the important feature of FGMs. For this reason, in structures made of FGMs, there are smaller stresses concentrations and the avoidance of cracking and delamination phenomenon (Kanani *et al.* 2014). Therefore, the aforementioned materials have gained much attention in engineering applications. Due to vast variety of FGM applications, examine behavior of FG structures under different mechanical load is significant. Much efforts have been done to develop FGMs in the areas of manufacturing, characterization, design, testing, modeling and simulation. With the development of new methods for construction of FGMs, the researchers were interested to investigate structures made of FG materials. Many researchers have studied static and dynamic behaviors of FG structures (Pompe *et al.* 2003, Watari *et al*. 2004, Marin 2005).

The behaviors of the beams in various subjects were studied by the researchers in the form of articles and books (Reddy 2011, Reddy and El-Borgi 2014). One of the main these subjects is wave propagation analysis of the beams based on various beam theories (Arvin *et al.* 2010, Asghari *et al.* 2010). Various beam theories such as the Euler–Bernoulli, Timoshenko or first order shear deformation, Reddy or parabolic shear deformation and Levinson beam theories were employed based on the nonlocal differential constitutive relations of Eringen by Reddy (2007). Li *et al.* (2015) developed an analytic model of small-scaled functionally graded (FG) beams for the flexural wave propagation analysis based on the nonlocal strain gradient

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theory. The size-dependent wave propagation analysis of double-piezoelectric nano-beam-systems (DPNBSs) based on Euler–Bernoulli beam model was carried out by Ghorbanpour Arani *et al*. (2014). They concluded that the imposed external voltage is an effective controlling parameter for wave propagation of the coupled system. Ke et al. investigated the dispersion behavior of waves in magneto-electro-elastic (MEE) nano-beams based on Euler and Timoshenko nano-beam model and calculated the cut-off frequency that was function of various loads (Ma *et al.* 2017). In the other research, a sandwich beam with periodic multiple dissipative resonators in the sandwich core material was investigated for broadband wave mitigation and/or absorption by Chen *et al*. (2017). Joglekar and Mitra (2016) presented an analytical–numerical method, based on the use of wavelet spectral finite elements (WSFE), in order to study the nonlinear interaction of flexural waves with a breathing crack present in a slender beam. (Ding *et al.* 2016) studied the propagation and attenuation properties of waves in ordered and disordered periodic composite Timoshenko beams, which consider the effects of axial static load and structural damping They assumed that beam is resting on elastic foundations and subjected to moving loads of constant amplitude with a constant velocity.

Recently, the influences of the length scale parameters were considered by the researchers in nano and micro structures. In aforementioned studies in order to incorporate the small scales in equations of motions, various theories such as the strain gradient theory and Eringen's differential nonlocal model were used (Larbi *et al.* 2015, Lim *et al.* 2015, Zemri *et al.* 2015). Classical continuum models (Shakeri *et al.* 2006, Zhang and Paulino 2007), nonlocal continuum theory (Belkorissat *et al.* 2015; Besseghier *et al.* 2015, Bounouara *et al.* 2016, Ebrahimy and Hosseini 2016, Sari 2016, Shafiei *et al.* 2016, Waksmanski and Pan 2016, Bouafia *et al*. 2017, Ebrahimi and Barati 2017), strain gradient theory (Gholami *et al.* 2014, Rahmani and Jandaghian 2015, Ahouel *et al.* 2016), and modified couple stress models (Nateghi and Salamat-talab 2013, Ansari *et al.* 2014, Al-Basyouni *et al*. 2015) have been used by researchers for analysis of nano/micro systems. Generally, In contrast to the classical continuum theory, the stress of the nonlocal continuum theory at a reference point is assumed to depend not only on the strain at the reference point, but also on the strains at all other points in the body (Eringen 1983). In addition, regard to the strain gradient theory, the strain energy is a function of the strain and curvature components tensors (Yang *et al*. 2002). Lim *et al.* (2015) carried out an investigation to show that the nonlocal and strain gradient parameters basically described two different physical properties of the structures in nano and micro scales. They have presented a new approach and theory to relate the strain gradient and nonlocal theories named the nonlocal strain gradient theory (NSGT). This theory is a combination of the two aforementioned theories that incorporated both small scales parameters namely nonlocal and strain gradient parameters (Arefi *et al*. 2017). Based on the NSGT, Liew *et al.* (2008) analyzed the wave propagation in a SWCNT by molecular dynamics simulations.

The flexoelectricity is related to a particular electromechanical coupling phenomenon between polarization and strain gradients (Ebrahimi and Barati 2017). Zhang *et al.* (2014) *in*vestigated the flexoelectric effect on the electroelastic responses and the free vibration behaviors of a piezoelectric nano-plate (PNP). Also, a modified Kirchhoff plate model with the consideration of residual surface stress, surface elasticity, surface piezoelectricity, and flexoelectricity was developed to investigate the electroelastic responses and vibration behaviors of a piezoelectric nano-plate (PNP) by Zhang and Jiang (Zhang and Jiang 2014). Other researchers in several investigations studied effects of the flexoelectricity on various behaviors of the beam (Yan and Jiang 2013, Liang *et al.* 2014, Zhang *et al*. 2016). In other hand surface effects have significant influence in nano and micro structures (Nazemnezhad *et al.* 2012, Asgharifard Sharabiani and Haeri Yazdi 2013, Faraji Oskouie and Ansari 2017). Nonlinear free vibration of simply supported FG nano-scale beams with considering surface effects (surface elasticity, tension and density) and balance condition between the FG nano-beam bulk and its surfaces was investigated by Hosseini-Hashemi and Nazemnezhad (2013). Free vibration of current-carrying nano-scaled beams incorporating Rayleigh, Timoshenko, and higher-order beam models considering the surface energy were investigated by Kiani (2016). He studied effects of surface and shear deformation, electric current, magnetic field strength, and geometric parameters of the nano-beam on the first ten natural frequencies.

Regard to literature review mentioned above and author's knowledge, we can conclude that there is no published work about wave propagation analysis of the functionally graded (FG) nano-beams based on the nonlocal elasticity theory considering surface and flexoelectric effects. The nano-beam was rested on Winkler-Pasternak foundation and also the material properties of the nano-beam changes continuously along the thickness direction according to simple power-law form. The effects of surface elasticity, dielectricity and piezoelectricity as well as bulk flexoelectricity for considering coupling of strain gradients and electrical polarizations, first shear deformation beam theory (FSDBT), residual surface stresses and Hamilton's principle have been used to derive governing equations of motion. Then analytical method is used to calculate phase velocity of wave propagation in flexoelectric FG Timoshenko nano-beam as well as cut-off frequency. Furthermore, the comprehensive numerical results are presented to investigate the influence of important parameters such as flexoelectricity of the surface and bulk, residual surface stresses, Winkler and shear coefficients, power index, and geometric parameters on the wave propagation characteristics of FG nano-beam.

2. Material properties of FG nano-beams

In this section, the material properties of FG nano-beam are expressed in detail. The schematic of our problem is presented in Fig. 1.

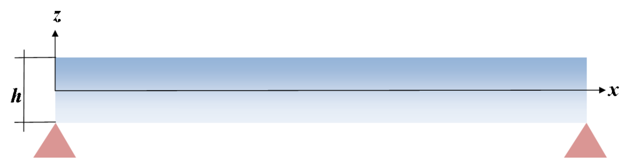


Fig. 1 FG nano-beam and attached coordinate system

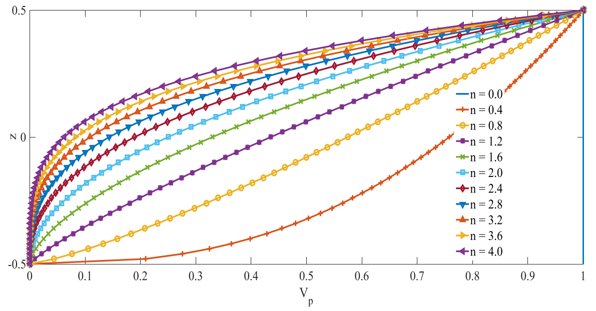


Fig. 2 Distribution of materials volume fraction along thickness coordinates for a FG nano- beam

According to this figure it is assumed that a simple power-law is used for distribution of the properties in FG nano-beam along the thickness direction. Based on the power-law distribution, the volume fraction of constituent PZH-5H V_p is written in the following form (Tornabene 2009, Tornabene and Viola 2009, Tornabene and Viola 2013, Tornabene *et al*. 2016, Fantuzzi and Tornabene 2017)

$$V_m = \left(\frac{2z+h}{2h}\right)^n, \quad -\frac{1}{2}h \le z \le -\frac{1}{2}h_H \tag{1}$$

In the Eq. (1), h is the thickness of FG nano-beam and z is the coordinate along thickness of it. Also, n is known as power gradient index for FG materials. Shown in Fig. 2 is the distribution of PZH-5H volume fraction V_p in thickness direction.

The non-homogeneous material properties of FG nano-beam are obtained using the mixture Voigt rule (Zhong and Yu 2007, Yao and Shi 2011, Arefi *et al*. 2016). Based on the mixture Voigt rule, a symbolic material property P(z) is assumed as

$$P(z) = P_b + P_{pb}\left(\frac{2z+h}{2h}\right)^n, \quad -\frac{1}{2}h \le z \le \frac{1}{2}h \tag{2}$$

In which, P_p and P_b are the PZH-5H and $BaTio_3$ properties, respectively, and also $P_{pb} = P_p - P_b$. In the present work, we assume that the elasticity modulus E and density ρ are described by Eq. (2), while Poisson's ratio υ is considered to be constant.

3. Formulation

If deformation of the structure made of a nano-dielectric material with the flexoelectricity consider infinitesimal the electric Gibbs free energy density function U_b can be written as (Zhang and Jiang 2014, Zhang *et al*. 2014)

$$U_b = -\frac{1}{2}a_{ij}E_iE_j + \frac{1}{2}C_{ijk}\varepsilon_{ij}\varepsilon_{kl} - e_{ijk}E_i\varepsilon_{kl} - f_{ijkl}E_i\eta_{ikl} \tag{3}$$

Where in above equation, f is the fourth-order flexoelectricity tensor, a is the second-order dielectric constant tensor, c is the fourth-order elastic constant tensor, and e is the classical piezoelectric tensor. It is worthy mentioned the fifth and sixth order terms are neglected(Zhang and Jiang 2014, Zhang *et al*. 2014). Indeed, representing the strain gradient and polarization coupling (direct flexoelectricity). With considering Eq. (3) the constitutive equations for the bulk can be derived as

$$\begin{aligned}
\sigma_{ij} &= \frac{\partial U_b}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{ij} - e_{ijk}E_i + f_{klij}\frac{\partial E_k}{\partial x_l} \\
\tau_{jkl} &= \frac{\partial U_b}{\partial \eta_{ikl}} = -f_{ijkl}E_i \\
D_i &= -\frac{\partial U_b}{\partial E_i} = a_{ij}E_j + e_{ijk}\varepsilon_{jk} + f_{ijkl}\eta_{jkl}
\end{aligned} \tag{4}$$

Where, σ_{ij} is the Cauchy stress tensor, D_i is the electric displacement vector, and τ_{jkl} is the higher order stress (moment stress), respectively. In other hand, considering the surface effects such as the surface elasticity, the residual surface stress, and the surface piezoelectricity, the surface internal energy density U_s can be specified as the following form (Liang *et al*. 2014, Ebrahimi and Barati 2017)

$$U_s = \Gamma_{\alpha\beta}\varepsilon^s_{\alpha\beta} - \frac{1}{2}\alpha^s_{\gamma\kappa}E^s_\gamma E^s_\kappa + \frac{1}{2}c^s_{\alpha\beta\gamma\kappa}\varepsilon^s_{\alpha\beta}\varepsilon^s_{\gamma\kappa} - e^s_{\kappa\alpha\beta}E^s_\kappa\varepsilon^s_{\alpha\beta} \tag{5}$$

In above equation, $\Gamma_{\alpha\beta}, \alpha^s_{\gamma\kappa}, c^s_{\alpha\beta\gamma\kappa}, e^s_{\kappa\alpha\beta}$ and E_κ are characterized the surface residual stress tensor, the surface permittivity, elastic constants, surface piezoelectric tensor and surface electric field, respectively. With considering Eq. (5) the constitutive equations for the surface of the beam can be derived as

$$\begin{aligned}
\sigma^s_{ij} &= \frac{\partial U_s}{\partial \varepsilon^s_{ij}} = \Gamma_{\alpha\beta} + C^s_{\alpha\beta\gamma\kappa}\varepsilon^s_{\gamma\kappa} - e^s_{\kappa\alpha\beta}E^s_\kappa \\
D^s_\gamma &= -\frac{\partial U_s}{\partial E^s_\gamma} = a^s_{\gamma\kappa}E^s_\kappa + e^s_{\gamma\alpha\beta}\varepsilon_{jk}
\end{aligned} \tag{6}$$

In which, $\sigma^s_{\alpha\beta}$ and D^s_γ are the surface Cauchy stress and surface electric displacement. The displacement field based on the first shear deformation beam theory (FSDBT) is expressed as

$$\begin{aligned}
u(x,z,t) &= u_0(x,t) - z\phi(x,t) \\
w(x,z,t) &= w_0(x,t)
\end{aligned} \tag{7}$$

Where, u and w are axial and transverse displacement components, u_0 and w_0 are axial and transverse displacement of mid-surface. In addition ϕ is the rotation angle of the beam cross section due to the pure bending that in Euler–Bernoulli beam is defined as $\phi = \frac{\partial w}{\partial x}$. The relations between displacement and strain as well as gradient strain are expressed as the following forms

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{j,i}+u_{i,j}\right),\quad \eta_{jkl}=\varepsilon_{kl,j}=\frac{1}{2}\left(u_{k,ji}+u_{l,jk}\right) \tag{8}$$

In which, u_i are the components of the mechanical displacement. In addition, for surface of the beam strain and displacement relation is defined as

$$\varepsilon^s_{\alpha\beta}=\frac{1}{2}\left(u^s_{\beta,\alpha}+u^s_{\alpha,\beta}\right) \tag{9}$$

Regarding to Eq. (8), nonzero strains and strain-gradients using Eq. (7) are calculated as

$$\begin{aligned}
&\varepsilon_{xx}=\frac{\partial u_0}{\partial x}-z\frac{\partial\phi}{\partial x},\\
&\varepsilon_{xz}=\frac{1}{2}\left(\frac{\partial w}{\partial x}-\phi\right),\\
&\eta_{111}=\frac{\partial\varepsilon_{xx}}{\partial x}=\frac{\partial^2 u_0}{\partial x^2}-z\frac{\partial^2\phi}{\partial x^2},\\
&\eta_{113}=\frac{\partial\varepsilon_{xx}}{\partial z}=-\frac{\partial\phi}{\partial x},\\
&\eta_{131}=\eta_{311}=\frac{\partial\varepsilon_{xz}}{\partial x}=\frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial\phi}{\partial x}\right),
\end{aligned} \tag{10}$$

According to nonlocal elasticity theory (Arani *et al.* 2014, Waksmanski and Pan 2016, Arefi and Zenkour 2017a), the constitutive relations presented in Eqs. (4) and (6) are rewritten as the following form

$$\begin{aligned}
&\left(1-(e_0a)^2\nabla^2\right)\sigma_{ij}=\frac{\partial U_b}{\partial\varepsilon_{ij}}=C_{ijkl}\varepsilon_{kl}-e_{kij}E_k+\frac{1}{2}f_{klij}\frac{\partial E_k}{\partial x_l}\\
&\left(1-(e_0a)^2\nabla^2\right)\tau_{jkl}=\frac{\partial U_b}{\partial\eta_{ikl}}=-\frac{1}{2}f_{ijkl}E_i\\
&\left(1-(e_0a)^2\nabla^2\right)D_i=-\frac{\partial U_b}{\partial E_i}=a_{ij}E_j+e_{ijk}\varepsilon_{jk}+\frac{1}{2}f_{ijkl}\eta_{jkl}\\
&\left(1-(e_0a)^2\nabla^2\right)\sigma^s_{ij}=\frac{\partial U_s}{\partial\varepsilon^s_{ij}}=\Gamma_{\alpha\beta}+C^s_{\alpha\beta\gamma\kappa}\varepsilon^s_{\gamma\kappa}-e^s_{\kappa\alpha\beta}E^s_\kappa\\
&\left(1-(e_0a)^2\nabla^2\right)D^s_\gamma=-\frac{\partial U_s}{\partial E^s_\gamma}=a^s_{\gamma\kappa}E^s_\kappa+e^s_{\gamma\alpha\beta}\varepsilon_{jk}
\end{aligned} \tag{11}$$

In Eq. (11), $\nabla^2=\partial^2/\partial x^2$ is the Laplacian operator and e_0a is the nonlocal parameter. Because of small value of thickness of FG nano-beam with respect to length, therefore it can be assumed that there are electric field only in thickness direction, which means that $E_1=E_2=0$ and also $E_3\neq 0$. It is worthy to mention that with the open circuit condition, the electric displacement on the surface is zero. Using Eqs. (10) and (11), the electric field can be obtained as

$$\begin{aligned}
E_3=&-\frac{e_{31}}{a_{33}}\frac{\partial u_0}{\partial x}+\left(z\frac{e_{31}}{a_{33}}+\frac{1}{2}\frac{f_{111}}{a_{33}}+\frac{1}{2}\frac{f_{14}}{a_{33}}\right)\frac{\partial\phi}{\partial x}-\\
&\frac{1}{2}\frac{f_{14}}{a_{33}}\frac{\partial^2 w}{\partial x^2}
\end{aligned} \tag{12}$$

And consequently the electric field gradient is defined as

$$\frac{\partial E_3}{\partial z}=\frac{e_{31}}{a_{33}}\frac{\partial\phi}{\partial x} \tag{13}$$

According to Eqs. (10), (11) and (13), the non-zero stress and the higher order stress (moment stresses) can be calculated as the following forms

$$\begin{aligned}
&\left(1-(e_0a)^2\nabla^2\right)\sigma_{11}=\left(C_{11}+\frac{e_{31}^2}{a_{33}}\right)\frac{\partial u}{\partial x}+\\
&\left(\frac{1}{2}f_{14}\frac{e_{31}}{a_{33}}-C_{11}z-e_{31}\left(z\frac{e_{31}}{a_{33}}+\frac{1}{2}\frac{f_{111}}{a_{33}}+\frac{1}{2}\frac{f_{14}}{a_{33}}\right)\right)\frac{\partial\phi}{\partial x}+\\
&\frac{1}{2}e_{31}\frac{f_{14}}{a_{33}}\frac{\partial^2 w}{\partial x^2}\left(1-(e_0a)^2\nabla^2\right)\sigma_{13}=kC_{44}\left(\frac{\partial w}{\partial x}-\phi\right)+\\
&\frac{1}{2}f_{111}\frac{e_{31}}{a_{33}}\frac{\partial\phi}{\partial x}\\
&\left(1-(e_0a)^2\nabla^2\right)\tau_{113}=\frac{1}{2}\frac{e_{31}f_{14}}{a_{33}}\frac{\partial u}{\partial x}-\\
&\frac{1}{2}\left(\frac{f_{111}f_{14}}{a_{33}}+\frac{f_{14}^2}{a_{33}}+\frac{e_{31}f_{14}}{a_{33}}\right)\frac{\partial\phi}{\partial x}+\frac{1}{2}\frac{f_{14}^2}{a_{33}}\frac{\partial^2 w}{\partial x^2}\\
&\left(1-(e_0a)^2\nabla^2\right)\tau_{131}=\left(1-(e_0a)^2\nabla^2\right)\tau_{311}=\frac{1}{2}\frac{e_{31}f_{111}}{a_{33}}\frac{\partial u}{\partial x}\\
&-\frac{1}{2}\left(\frac{f_{111}f_{14}}{a_{33}}+\frac{f_{111}^2}{a_{33}}+\frac{e_{31}f_{111}}{a_{33}}\right)\frac{\partial\phi}{\partial x}+\frac{1}{2}\frac{f_{111}f_{14}}{a_{33}}\frac{\partial^2 w}{\partial x^2}
\end{aligned} \tag{14}$$

Where, k is the shear correction factor, which in this investigation is selected $k=5/6$. The Hamilton's principle is used to drive governing equation of motion as (Komijani *et al.* 2014)

$$0=\int_0^T\left(\delta T-\delta U_s-\delta U_f+\delta W\right)d\bar{t} \tag{15}$$

Where δU_s, δU_f, δT and δW are the variations of strain energy, foundation reaction, kinetic energy and external works, respectively. Variation of strain energy δU_s is calculated as

$$\delta U_s=\int_0^L\int_A\left(\begin{aligned}&\sigma_{11}\delta\varepsilon_{11}+2\sigma_{13}\delta\varepsilon_{13}+\tau_{113}\delta\eta_{113}+\\&2\tau_{131}\delta\eta_{131}\end{aligned}\right)dAdx \tag{16}$$

Variation of kinetic energy is represented as

$$\delta T=\int_0^L\int_A\rho(z)\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial t}+\frac{\partial w}{\partial t}\frac{\partial w}{\partial t}\right)dAdx \tag{17}$$

Variations of work done by the external forces and Winkler- Pasternak foundation are written as (Arani *et al.* 2012, Kanani *et al.* 2014, Komijani *et al.* 2014)

$$\begin{aligned}
&\delta W=\int_0^L\left(F\delta u_0+Q\delta w_0+\bar{N}_0\frac{\partial w_0}{\partial\bar{x}}\frac{\partial\delta w_0}{\partial\bar{x}}\right)dx\\
&\delta U_f=\int_0^L\int_0^b\left(K_w w_0\delta w_0+K_s\frac{\partial w_0}{\partial x}\delta\left(\frac{\partial w_0}{\partial x}\right)\right)d\bar{y}d\bar{x}
\end{aligned} \tag{18}$$

Where F and Q are the axial and transverse forces per unit length respectively and $\bar{N}_0$ is the axial compressive or pretension force. Also, K_w and K_s are linear spring and shear coefficients of the foundation, respectively. Substituting Eqs. (10) and (14) into Eqs. (16)-(18) and consequently into Eq. (15), yields the governing equations of motions as

$$\begin{aligned}
&\delta u_0:\ \frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xx}^s}{\partial x}+F=I_0\frac{\partial^2 u_0}{\partial t^2}-I_1\frac{\partial^2\phi}{\partial t^2}\\
&\delta w_0:\ \frac{\partial N_{xz}}{\partial x}-\frac{\partial^2\bar{N}_{131}}{\partial x^2}-\bar{K}_w w_0+\bar{K}_s\frac{\partial^2 w_0}{\partial x^2}+\\
&\sigma_0^s\frac{\partial^2 w_0}{\partial x^2}+Q=I_0\frac{\partial^2 w_0}{\partial t^2}\\
&\delta\phi:\ -\frac{\partial M_{xx}^s}{\partial x}+N_{xz}-\frac{\partial M_{xx}}{\partial x}-\frac{\partial\bar{N}_{131}}{\partial x}-\\
&\frac{\partial\bar{N}_{113}}{\partial x}=I_1\frac{\partial^2 u_0}{\partial t^2}+I_2\frac{\partial^2\phi}{\partial t^2}
\end{aligned}\tag{19}$$

Where, $N_{\bar{x}}$, $Q_{\bar{x}\bar{z}}$, $M_{\bar{x}}$ and $M_{\bar{x}}^h$ are the resultants of forces and the moments and σ_0 is the surface residual stress (Ebrahimi and Barati 2017). The resultants of forces and the moments are expressed as

$$\begin{aligned}
&\left(1-(e_0a)^2\nabla^2\right)N_{xx}=\\
&A_{xx}\frac{\partial u}{\partial x}+\left(A_{x2}+B_{x1}\right)\frac{\partial\phi}{\partial x}+A_{x3}\frac{\partial^2 w}{\partial x^2},\\
&\left(1-(e_0a)^2\nabla^2\right)M_{xx}=\\
&B_{x2}\frac{\partial u}{\partial x}+\left(D_{x1}+B_{x3}\right)\frac{\partial\phi}{\partial x}+B_{x4}\frac{\partial^2 w}{\partial x^2},\\
&\left(1-(e_0a)^2\nabla^2\right)N_{xz}=A_{xz1}\left(\frac{\partial w}{\partial x}-\phi\right)+A_{xz2}\frac{\partial\phi}{\partial x},\\
&\left(1-(e_0a)^2\nabla^2\right)\bar{N}_{131}=\\
&A_{131}^1\frac{\partial u}{\partial x}-\left(A_{131}^2+B_{131}^1\right)\frac{\partial\phi}{\partial x}+A_{131}^3\frac{\partial^2 w}{\partial x^2},\\
&\left(1-(e_0a)^2\nabla^2\right)\bar{N}_{113}=\\
&A_{113}^1\frac{\partial u}{\partial x}-\left(A_{113}^2+B_{113}^1\right)\frac{\partial\phi}{\partial x}+A_{113}^3\frac{\partial^2 w}{\partial x^2},\\
&\left(1-(e_0a)^2\nabla^2\right)N_{xx}^s=\\
&A_{x1}^s\frac{\partial u}{\partial x}-\left(A_{x2}^s+B_{x1}^s\right)\frac{\partial\phi}{\partial x}+A_{x3}^s\frac{\partial^2 w}{\partial x^2},\\
&\left(1-(e_0a)^2\nabla^2\right)N_{xx}^s=\\
&A_{x1}^s\frac{\partial u}{\partial x}-\left(A_{x2}^s+B_{x1}^s\right)\frac{\partial\phi}{\partial x}+A_{x3}^s\frac{\partial^2 w}{\partial x^2},\\
&\left(1-(e_0a)^2\nabla^2\right)M_{xx}^s=\\
&B_{x2}^s\frac{\partial u}{\partial x}-\left(B_{x3}^s+D_{x1}^s\right)\frac{\partial\phi}{\partial x}+B_{x4}^s\frac{\partial^2 w}{\partial x^2},
\end{aligned}\tag{20}$$

It is noted that, A_x, B_x and D_x in Eq. (20) are the stretching stiffness, stretching-bending coupling stiffness and bending stiffness coefficients, respectively, which can be found in appendix A. The integration other constants presented in Eq. (20) can be calculated as

$$\begin{Bmatrix} I_0\\ I_1\\ I_2\end{Bmatrix}=\int_{-h/2}^{h/2}\begin{Bmatrix}1\\ z\\ z^2\end{Bmatrix}\rho(z)dz,\tag{21}$$

To obtain the equations of motion, Eq. (20) should be substituted into Eq. (19). Therefore, three coupled equations of motion are obtained

$$\begin{aligned}
&\delta u_0:\left(A_{x1}+A_{x1}^s\right)\frac{\partial^2 u_0}{\partial x^2}+\left(A_{x2}+A_{x2}^s\right)\frac{\partial^2\phi}{\partial x^2}+\left(B_{x1}+B_{x1}^s\right)\times\\
&\frac{\partial^2\phi}{\partial x^2}+\left(A_{x3}+A_{x3}^s\right)\frac{\partial^3 w}{\partial x^3}=\left(1-(e_0a)^2\nabla^2\right)I_0\frac{\partial^2 u_0}{\partial t^2}\\
&-\left(1-(e_0a)^2\nabla^2\right)I_1\frac{\partial^2\phi}{\partial t^2}\\
&\delta w:-A_{131}^1\frac{\partial^3 u_0}{\partial x^3}-A_{xz1}\frac{\partial\phi}{\partial x}+A_{xz2}\frac{\partial^2\phi}{\partial x^2}+\left(B_{131}^1+A_{131}^2\right)\frac{\partial^3\phi}{\partial x^3}+\\
&A_{xz1}\frac{\partial^2 w}{\partial x^2}-A_{131}^3\frac{\partial^4 w}{\partial x^4}=\left(1-(e_0a)^2\nabla^2\right)I_0\frac{\partial^2 w}{\partial t^2}\\
&\delta\phi:\left(-B_{x2}-B_{x2}^s-A_{131}^1-A_{113}^1\right)\frac{\partial^2 u_0}{\partial x^2}-A_{xz1}\phi+A_{xz2}\frac{\partial\phi}{\partial x}+\\
&\left(-B_{x3}-D_{x1}+D_{x1}^s+B_{x3}^s+B_{131}^1+A_{131}^1+B_{113}^1+A_{113}^1\right)\times\\
&\frac{\partial^2\phi}{\partial x^2}+A_{xz1}\frac{\partial w}{\partial x}-\left(B_{x4}+B_{x4}^s+A_{131}^3+A_{113}^3\right)\frac{\partial^3 w}{\partial x^3}\\
&=-\left(1-(e_0a)^2\nabla^2\right)I_0\frac{\partial^2 u_0}{\partial t^2}+\left(1-(e_0a)^2\nabla^2\right)I_2\frac{\partial^2\phi}{\partial t^2}
\end{aligned}\tag{22}$$

4. Wave propagation analysis

In order to Wave propagation analysis in sandwich nano-beam, the harmonic solution is considered for Eq. (22) as the following form (Arani *et al*. 2014)

$$d_j(x,t)=d_{0j}e^{i(\bar{k}x-\omega t)},\qquad d=u,w,\phi\tag{23}$$

In above equation, $\bar{k}$ and ω are wave number and frequency of the wave propagation, respectively (Arani *et al.* 2015, Arani e*t al.* 2016). Consequently, replacing Eq. (23) into Eq. (22) produces the following matrix equations for FG flexoelectric Timoshenko nano-beam

$$\begin{bmatrix} C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}\begin{bmatrix} u\\ w\\ \phi\end{bmatrix}=0\tag{24}$$

In which, $C_{ij}=(i,j=1,2,3)$ are calculated as following form

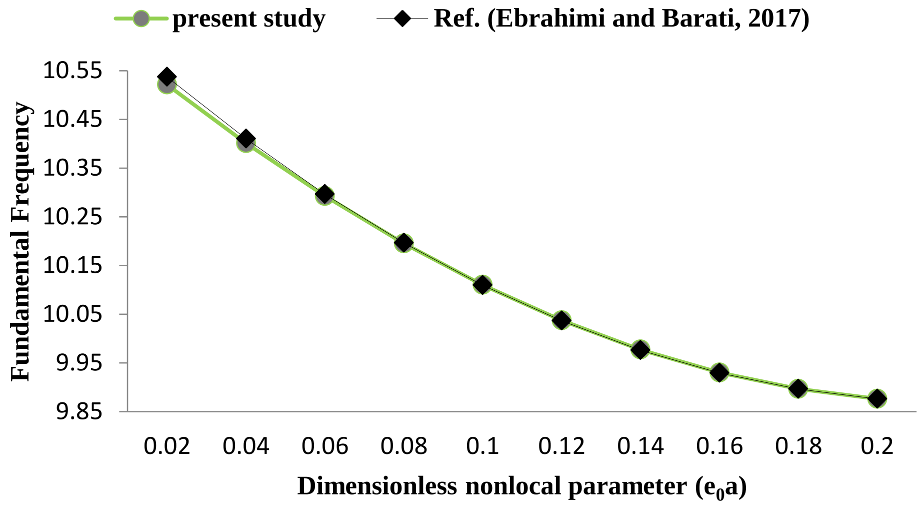


Fig. 3 Fundamental frequency versus Dimensionless nonlocal parameter (e_0a)

$$
\begin{aligned}
&C_{11} = -\bar{k}^2\left(A_{x1} + A_{x1}^s\right) + \left(1 + \bar{k}^2\left(ea\right)^2\right)I_0\omega^2 - \\
&\quad \left(1 + \bar{k}^2\left(ea\right)^2\right)I_1\omega^2, \\
&C_{12} = -i\bar{k}^3\left(A_{x3} + A_{x3}^s\right) \\
&C_{13} = -\bar{k}^2\left(A_{x2} + A_{x2}^s + B_{x1} + B_{x1}^s\right) \\
&C_{21} = A_{131}^1\bar{k}^3 i \\
&C_{22} = -\bar{k}^2\left(A_{xz1} + A_{131}^1\bar{k}^2\right)\left(1 + \bar{k}^2\left(ea\right)^2\right)I_0\omega^2 \\
&C_{23} = -\left(i\bar{k}A_{xz1} + \left(\bar{k}^2 A_{xz2} + i\bar{k}^3\left(B_{131}^1 + A_{131}^2\right)\right)\right) \\
&C_{31} = \bar{k}^2\left(B_{x2} + B_{x2}^s + A_{131}^1 + A_{113}^1\right) \\
&C_{32} = i\bar{k}A_{xz1} + i\bar{k}^3\left(B_{x4} + B_{x4}^s + A_{131}^3 + A_{113}^3\right) \\
&C_{33} = \left(-A_{xz1} + i\bar{k}A_{xz2} - \bar{k}^2\left(-B_{x3} - D_{x1} + D_{x1}^s\right.\right. \\
&\quad \left.\left. + B_{x3}^s + B_{131}^1 + A_{131}^1 + B_{113}^1 + A_{113}^1\right)\right) - \\
&\quad \left(1 + \bar{k}^2\left(ea\right)^2\right)I_0\omega^2 + \left(1 + \bar{k}^2\left(ea\right)^2\right)I_2\omega^2
\end{aligned}
\tag{25}
$$

In order to obtain a nontrivial solution, it is necessary to set the determinant of the coefficient matrix in Eq. (24) equal to zero (Arani *et al.* 2014, Arani *et al.* 2015, Arani *et al.* 2016). The cut-off frequency of the FG nano-beam can be calculated by setting $\bar{k} \to \infty$. In other words, at a certain frequency, the flexural wave number tends to infinite and the corresponding wave velocity tends to zero at that frequency, this frequency is called as cut-off frequency (Arani *et al.* 2015).

5. Numerical results and discussion

In this section, a parametric study is implemented to indicate nonlocal parameter, power gradient index, geometric dimensions of the beam, surface effects, flexoelectric property, foundation property and other important parameters in designing and controlling the phase velocity and cut-off frequency. The material properties and geometrical specifications of the FG nano-beam are presented in Table 1. It is found that the flexoelectric coefficient for BT rises to about 10 μC/m at room temperature, and reaches to 50 μC/m near tetragonal-cubic phase transition point.

5.1 Validation of results

To justify the accuracy and trueness of the governing equations extracted in this study, a comparison with existing reference using semi analytical solutions to calculate fundamental frequency of Euler-Bernoulli nano-beam is represented. Fig. 3 shows comparison between the obtained results by solving the governing equations extracted in this study and equation of motion obtained in Ref. (Ebrahimi and Barati 2017). It is worthy noted that the flexoelectricity and surface effects were considered in Ref. (Ebrahimi and Barati 2017). According to this comparison it is deduced that the present results are in a good agreement with the obtained results by Ref. (Ebrahimi and Barati 2017) (Fig. 3). Thus an exact comparison of the results obtained for wave propagation in this work with existing experimental results is impossible. However, the present work could be partially validated based on investigation presented by Li et al (Li *et al*. 2015, Li *et al.* 2016). As can be seen in Fig. 4, in both study (present investigation and works accomplished by Li *et al.*), nonlocal parameters have the same effects on the phase velocity behavior of wave propagation. This can be concluded that with increasing nonlocal parameter phase velocity of the wave propagation

Table 1 The material and geometrical properties of the constituent material of the FG nano-beam (Ke *et al.* 2010, Rafiee *et al.* 2013)

Material	C_{11}(Pa)	C_{44}(Pa)	Density[kg/m3]	e_{31} C/m2	Geometric dimensions(nm)		$a_{33}(\frac{C}{Vm})$
PZH-5Z	102E(9)	35.5 E (9)	7600	4.4	h	b	1.76E (-8)
BaTio$_3$	167.55E(9)	44.7 E (9)	6020	17.05	2	2h	0.79E (-8)

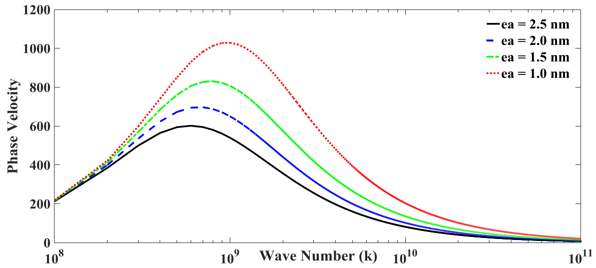


Fig. 4 Phase Velocity versus Wave Number for various nonlocal parameters

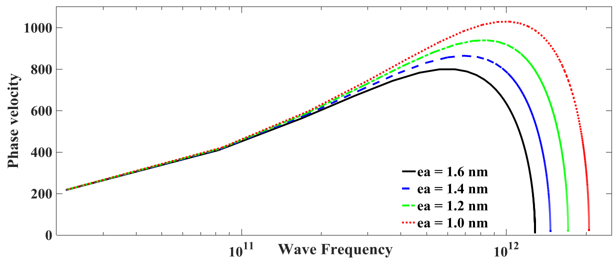


Fig. 5 Phase Velocity versus Wave Number for various nonlocal parameters

of FG nano-beam decreases for complete range of wave numbers.

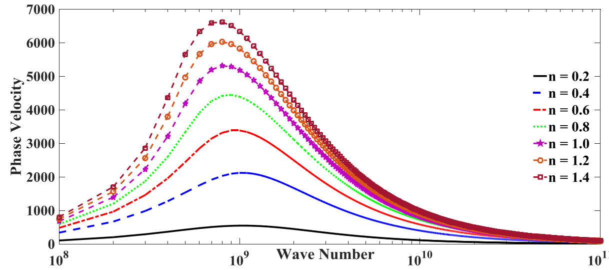
Fig. 5 illustrates corresponding wave frequency. According to results presented in aforementioned figure, increasing the wave frequency leads to raise the phase velocity until certain point and then started to decrease up to zero. The wave frequency where phase velocity is equal to zero named as the cut-off frequency. It can be also concluded that enhancing nonlocal parameter makes cut-off frequencies occurred in low wave frequency.

5.2 Materials property effects

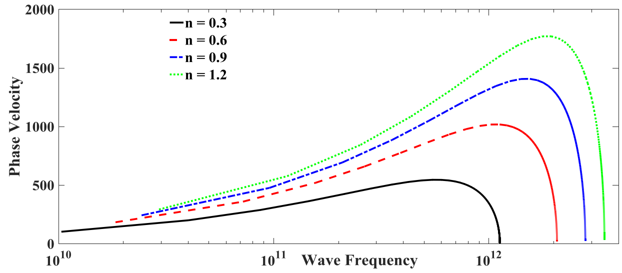
The influences of power gradient index for FG materials on phase velocity and cut-off frequency are investigated in Fig. 6. Regard to results represented in this figure can be concluded that power gradient index have significant effects on phase velocity and cut-off frequency.

5.3 Geometric effects

Fig. 7 shows the effects of thickness of the beam on the phase velocity and cut-off frequency. It is concluded that variation of the thickness has more effect on phase velocity in lower wave number in opposite variation of the thickness

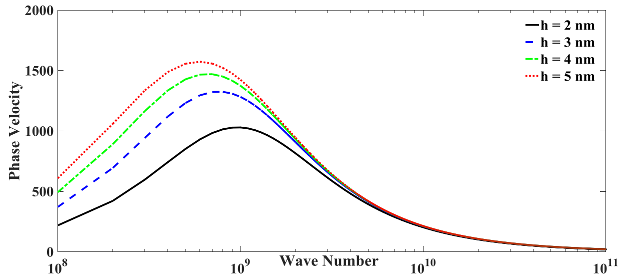


(a)

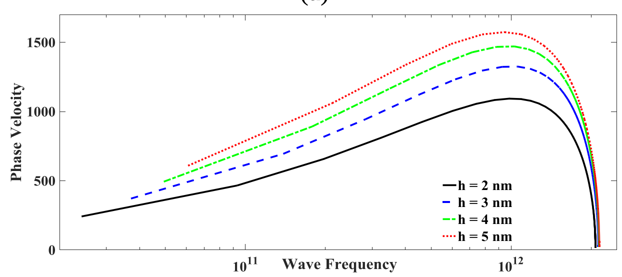


(b)

Fig. 6 (a) Phase Velocity versus Wave Number and (b) Phase Velocity versus Wave Frequency for various power gradient index

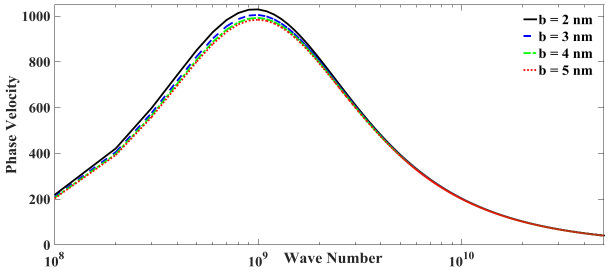


(a)

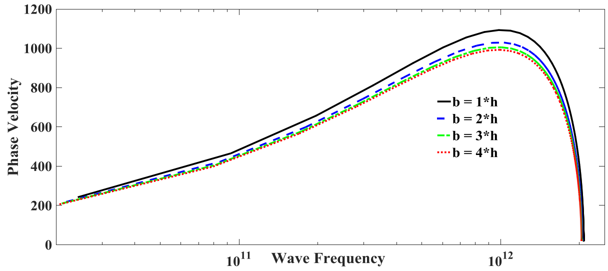


(b)

Fig. 7 (a) Phase Velocity versus Wave Number and (b) Phase Velocity versus Wave Frequency for various thickness of beam



(a)



(b)

Fig. 8 (a) Phase Velocity versus wave number and (b) Phase Velocity versus Wave Frequency for various width of the beam

has no effect in higher wave number range. Increasing thickness of the beam leads to enhance phase velocity but variation of thickness has no effects on cut-off frequency occurrence points.

Increasing the power gradient index causes the property of the beam leads to $BaTio_3$ property therefore it makes phase velocity increases. Also, increasing the power gradient index caused cut-off frequency occurs in higher wave frequency.

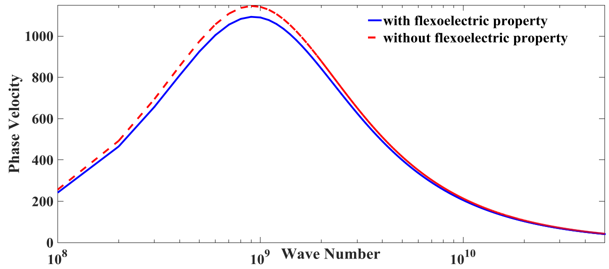
Fig. 8 illustrates the phase velocity respect to corresponding wave number and also versus wave frequency for various width of the FG nano-beam. It is concluded that increasing width of FG nano-beam leads to increase of the phase velocity in low wave number. Also it worthy noted that variation of width of the beam approximately has no effects on phase velocity in high wave number. It is also noted that cut-off frequency approximately occurs in a constant point for various width amounts.

5.4 Surface and flexoelectric effects

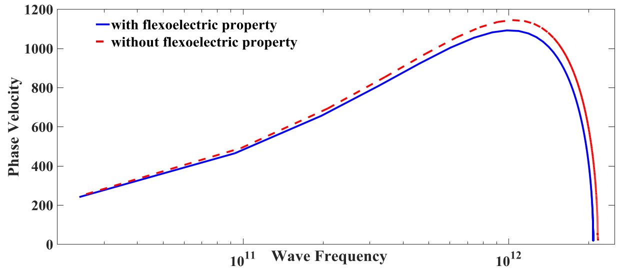
In order to investigate the effects of the flexoelectric property on wave propagation, phase velocity illustrated versus wave number and wave frequency with and without considering the flexoelectric property of the materials in Fig. 9. It is concluded that the flexoelectric property have significant effects on wave propagation as phase velocity decreases with considering aforementioned property in small wave number range. In addition, considering the flexoelectric property leads to small changes of cut-off occurrence point.

Fig. 10 shows variation of wave propagation of FG nano-beam in terms of variation of the residual surface stress. A more effects of the residual surface stress are in lower wave number range as increasing aforementioned parameter leads to enhance phase velocity. In addition it can be concluded that changing the residual surface stress has no effect on the cut-off occurrence point.

In order to investigate the surface effects on wave propagation, phase velocity illustrated versus wave number and wave frequency with and without considering surface effects in Fig. 11.

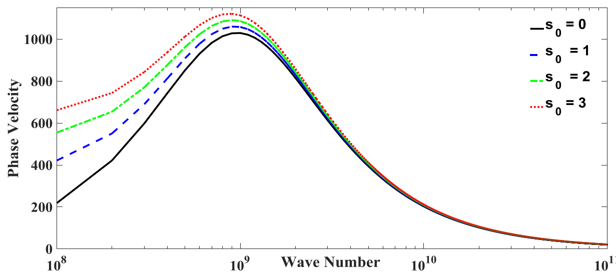


(a)

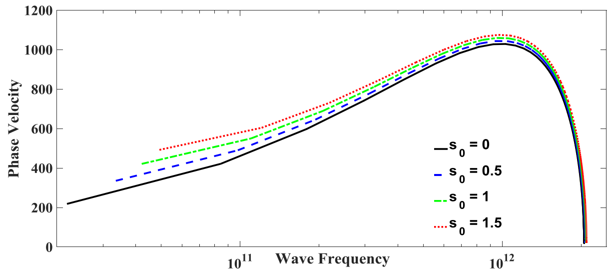


(b)

Fig. 9 (a) Phase Velocity versus wave number and (b) Phase Velocity versus Wave Frequency with and without considering flexoelectric property

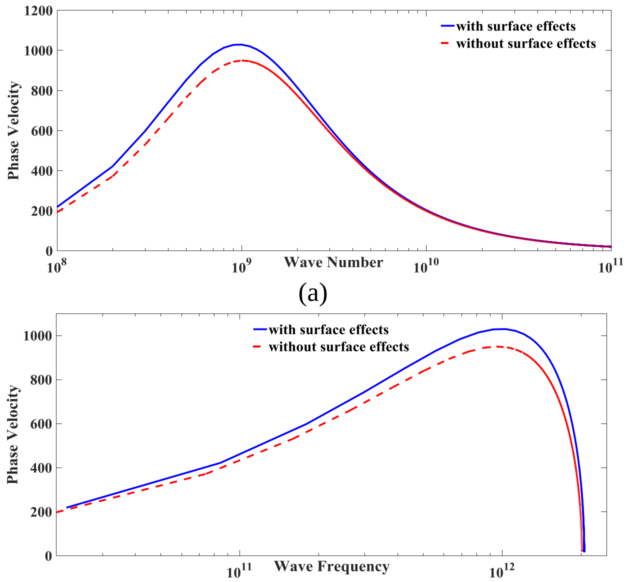


(a)



(b)

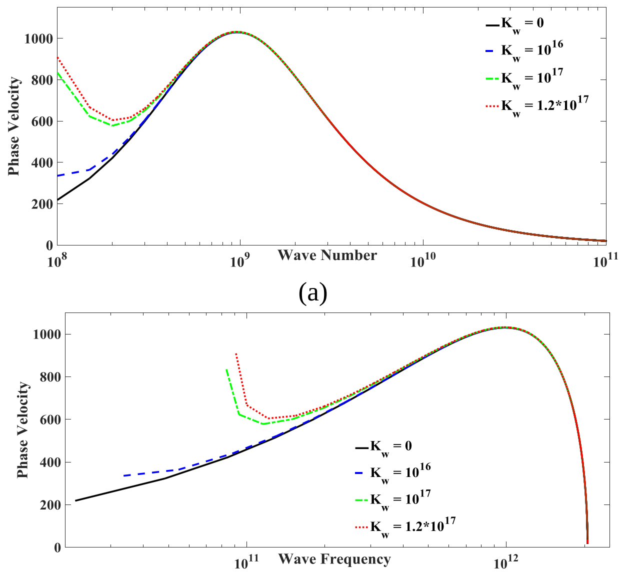
Fig. 10 (a) Phase Velocity versus wave number and (b) Phase Velocity versus Wave Frequency with and without considering flexoelectric property



(a)

(b)

Fig. 11 (a) Phase Velocity versus wave number and (b) Phase Velocity versus Wave Frequency with and without considering surface effects



(a)

(b)

Fig. 12 (a) Phase Velocity versus wave number and (b) Phase Velocity versus Wave Frequency for various Winkler coefficient

One can conclude that surface effects have important influences on wave propagation in which phase velocity increases with considering aforementioned property in small wave number range. In addition, considering the surface effects have no influence on cut-off occurrence point on the FG nano- beam.

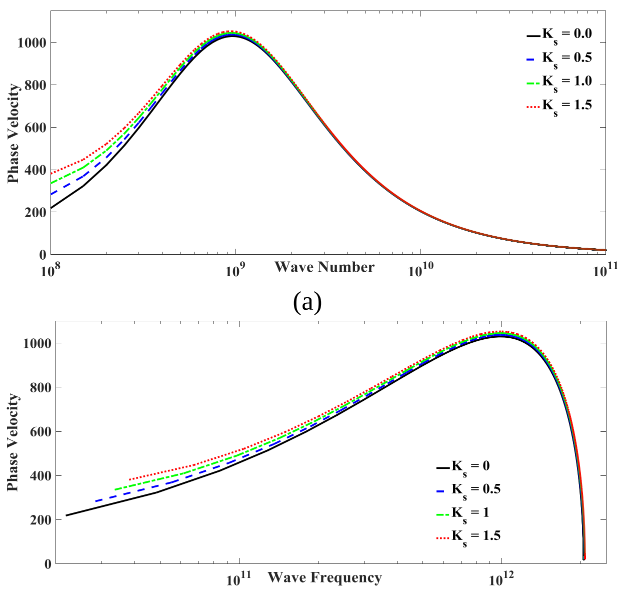
5.5 Foundation effects

Fig. 12 shows variation of phase velocity and cut-off frequency in terms of Winkler coefficient of the foundation. It is concluded that Winkler coefficient has more effect on phase velocity in lower wave number in opposite with variation of the thickness in which has no effect in higher wave number range. Increasing Winkler coefficient of the foundation leads to enhance phase velocity but variation of Winkler coefficient has no effects on cut-off frequency occurrence points.

The influences of Pasternak coefficient of the foundation on phase velocity and occurrence cut-off frequency point are investigated in Fig. 13. It is concluded that increasing Pasternak coefficient leads to enhance phase velocity in a low range of the wave number. Whilst variation Pasternak coefficient has no effect on occurrence cut-off frequency point.

6. Conclusions

Analysis of the wave propagation of the FG nano-beams was implemented in this investigation. The surface effects and flexoelectricity were considered in wave propagation analysis of the FG nano-beams resting on Winkler-Pasternak foundation. In addition the residual surface stress and the nonlocal elasticity theory were taken into account in derivtion of the governing equations of motion. The solution procedure for finding the phase velocity of wave propagation was perfomed based on the analytical. The effects of some main parameters such as nonlocal parameter, power gradient index, geometric parameter,



(a)

(b)

Fig. 13 (a) Phase Velocity versus wave number and (b) Phase Velocity versus Wave Frequency for various Pasternak coefficient surface effects, flexoelectric property and foundation characteristics and other important parameters in designing and controlling the phase velocity and cut-off frequency were studied in detail. The most important results of this study are presented as

1. The influence of small scale parameters of nonlocal strain gradient theory has been studied on the responses. One can conclude that increase of nonlocal parameter leads to decrease of phase velocity and cut-off frequency whilst increase of power index leads to significant inrease of aformentioned outputs.
2. Geometric dimensions of the FG nano-beam can strongly change the phase velocity in low wave number. The results indicate that phase velocity is increased with increasing the height of nano-beam but decreases by enhancing width. It must be noted the effects of aforementioned geometric dimensions on the occurrence cut-off frequency are very small and can be ignored.
3. The surface effects and flexoelectric property have significant and opposite influences on the phase velocity of the FG nano-beam. It is observed that considering surface effects and flexoelectric property caused phase velocity increases and decreases in low wave number range, respectively. The influences of aforementioned parameters on the occurrence cut-off frequency point are very small and can be disregarded.
4. The residual surface stress can strongly change the phase velocity in FG nano-beam in very low wave number and also it hasn't effects on the occurrence cut-off frequency.
5. Investigation on the effect of the Winkler-Pasternak foundation parameters on the wave propagation of the FG nano-beam leads to important conclusions. Increasing the Winkler (K_w) and shearing (K_s) coefficients caused that the phase velocity increases in lower wave number and also they have no effects in larger wave number range.

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Appendix A

$$A_{x1}=\int_h\left(C_{11}(z)+\frac{e_{31}(z)^2}{a_{33}(z)}\right)dzb,\qquad A_{x2}=\frac{1}{2}\int_h\left(-f_{111}(z)\frac{e_{31}(z)}{a_{33}(z)}\right)dzb,$$

$$B_{x1}=-\int_h\left(C_{11}(z)+\frac{e_{31}(z)^2}{a_{33}(z)}\right)zdzb,\qquad A_{x3}=\frac{1}{2}\int_h\left(-f_{14}(z)\frac{e_{31}(z)}{a_{33}(z)}\right)dzb,$$

$$B_{x2}=\int_h\left(C_{11}(z)+\frac{e_{31}(z)^2}{a_{33}(z)}\right)zdzb,\qquad B_{x2}=\frac{1}{2}\int_h\left(-f_{111}(z)\frac{e_{31}(z)}{a_{33}(z)}\right)zdzb,$$

$$B_{x3}=\frac{1}{2}\int_h\left(-f_{14}(z)\frac{e_{31}(z)}{a_{33}(z)}\right)zdzb,\qquad D_{x1}=-\int_h\left(C_{11}(z)+\frac{e_{31}(z)^2}{a_{33}(z)}\right)z^2dzb,$$

$$A^1_{131}=\frac{1}{2}\int_h\left(\frac{f_{111}(z)e_{31}(z)}{a_{33}(z)}\right)dzb,\; A^2_{131}=\frac{1}{2}\int_h\left(\frac{f_{111}(z)f_{14}(z)}{a_{33}(z)}+\frac{f_{111}(z)^2}{a_{33}(z)}\right)dzb,$$

$$A^3_{131}=\frac{1}{2}\int_h\left(\frac{f_{111}(z)f_{14}(z)}{a_{33}(z)}\right)dzb,\; B_{131}=\frac{1}{2}\int_h\left(\frac{f_{111}(z)e_{31}(z)}{a_{33}(z)}\right)zdzb,$$

$$A^1_{113}=\frac{1}{2}\int_h\left(\frac{f_{14}(z)e_{31}(z)}{a_{33}(z)}\right)dzb,\; A^2_{113}=\frac{1}{2}\int_h\left(\frac{f_{111}(z)f_{14}(z)}{a_{33}(z)}-\frac{f_{14}(z)^2}{a_{33}(z)}\right)dzb,$$

$$A^3_{113}=\frac{1}{2}\int_h\left(\frac{f_{14}(z)^2}{a_{33}(z)}\right)dzb,\qquad B_{113}=\frac{1}{2}\int_h\left(\frac{f_{14}(z)e_{31}(z)}{a_{33}(z)}\right)zdzb,$$

$$A^s{}_{x1}=\int_h\left(C^s{}_{11}(z)+\frac{e^s{}_{31}(z)e_{31}(z)}{a_{33}(z)}\right)dz+$$
$$\left[\left(C^s{}_{11}(z)+\frac{e^s{}_{31}(z)e_{31}(z)}{a_{33}(z)}\right)\Bigg|_{z=\frac{h}{2}}-\left(C^s{}_{11}(z)+\frac{e^s{}_{31}(z)e_{31}(z)}{a_{33}(z)}\right)\Bigg|_{z=-\frac{h}{2}}\right]\frac{h}{2},\qquad (1)$$

$$A^s{}_{x2}=\frac{1}{2}\int_h\left(\frac{f_{111}(z)}{a_{33}(z)}+\frac{f_{14}(z)}{a_{33}(z)}\right)e^s{}_{31}(z)dz\times$$
$$\left[\left(\frac{f_{111}(z)}{a_{33}(z)}+\frac{f_{14}(z)}{a_{33}(z)}\right)e^s{}_{31}(z)\Bigg|_{z=\frac{h}{2}}-\left(\frac{f_{111}(z)}{a_{33}(z)}+\frac{f_{14}(z)}{a_{33}(z)}\right)e^s{}_{31}(z)\Bigg|_{z=-\frac{h}{2}}\right]\frac{h}{2},$$

$$A^s{}_{x3}=\frac{1}{2}\int_h\left(f_{14}(z)\frac{e^s{}_{31}(z)}{a_{33}(z)}\right)dz+$$
$$\left[\left(\frac{e^s{}_{31}(z)f_{14}(z)}{a_{33}(z)}\right)\Bigg|_{z=\frac{h}{2}}-\left(\frac{e^s{}_{31}(z)f_{14}(z)}{a_{33}(z)}\right)\Bigg|_{z=-\frac{h}{2}}\right]\frac{h}{2},$$

$$B^s{}_{x1}=\int_h\left(C^s{}_{11}(z)-\frac{e^s{}_{31}(z)e_{31}(z)}{a_{33}(z)}\right)zdz+$$
$$\left[\left(C^s{}_{11}(z)-\frac{e^s{}_{31}(z)e_{31}(z)}{a_{33}(z)}\right)\Bigg|_{z=\frac{h}{2}}-\left(C^s{}_{11}(z)-\frac{e^s{}_{31}(z)e_{31}(z)}{a_{33}(z)}\right)\Bigg|_{z=-\frac{h}{2}}\right]\frac{bh^2}{2},$$

$$B^s{}_{x2}=\int_h\left(C^s{}_{11}(z)+\frac{e^s{}_{31}(z)e_{31}(z)}{a_{33}(z)}\right)zdz+$$
$$\left[\left(C^s{}_{11}(z)+\frac{e^s{}_{31}(z)e_{31}(z)}{a_{33}(z)}\right)\Bigg|_{z=\frac{h}{2}}-\left(C^s{}_{11}(z)+\frac{e^s{}_{31}(z)e_{31}(z)}{a_{33}(z)}\right)\Bigg|_{z=-\frac{h}{2}}\right]\frac{bh^2}{2},$$

$$D^s{}_{x1}=\int_h\left(C^s{}_{11}(z)-\frac{e^s{}_{31}(z)e_{31}(z)}{a_{33}(z)}\right)z^2dz+$$
$$\left[\left(C^s{}_{11}(z)-\frac{e^s{}_{31}(z)e_{31}(z)}{a_{33}(z)}\right)\Bigg|_{z=\frac{h}{2}}-\left(C^s{}_{11}(z)-\frac{e^s{}_{31}(z)e_{31}(z)}{a_{33}(z)}\right)\Bigg|_{z=-\frac{h}{2}}\right]\frac{bh^3}{8},$$

$$B^s{}_{x3}=\frac{1}{2}\int_h\left(\frac{f_{111}(z)}{a_{33}(z)}+\frac{f_{14}(z)}{a_{33}(z)}\right)ze^s{}_{31}(z)dz$$
$$\left[\left(\frac{f_{111}(z)}{a_{33}(z)}+\frac{f_{14}(z)}{a_{33}(z)}\right)e^s{}_{31}(z)\Bigg|_{z=\frac{h}{2}}-\left(\frac{f_{111}(z)}{a_{33}(z)}+\frac{f_{14}(z)}{a_{33}(z)}\right)e^s{}_{31}(z)\Bigg|_{z=-\frac{h}{2}}\right]\frac{bh^2}{2},$$

$$B^s{}_{x4}=\frac{1}{2}\int_h\left(f_{14}(z)\frac{e^s{}_{31}(z)}{a_{33}(z)}\right)zdz+$$
$$\left[\left(\frac{e^s{}_{31}(z)f_{14}(z)}{a_{33}(z)}\right)\Bigg|_{z=\frac{h}{2}}-\left(\frac{e^s{}_{31}(z)f_{14}(z)}{a_{33}(z)}\right)\Bigg|_{z=-\frac{h}{2}}\right]\frac{bh^2}{2},$$