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An iterative method for damage identification of skeletal structures utilizing biconjugate gradient method and reduction of search space

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Abstract. This paper is devoted to proposing a new approach for damage detection of structures. In this technique, the biconjugate gradient method (BCG) is employed. To remedy the noise effects, a new preconditioning algorithm is applied. The proposed preconditioner matrix significantly reduces the condition number of the system. Moreover, based on the characteristics of the damage vector, a new direct search algorithm is employed to increase the efficiency of the suggested damage detection scheme by reducing the number of unknowns. To corroborate the high efficiency and capability of the presented strategy, it is applied for estimating the severity and location of damage in the well-known 31-member and 52-member trusses. For damage detection of these trusses, the time history responses are measured by a limited number of sensors. The results of numerical examples reveal high accuracy and robustness of the proposed method.

Keywords: damage detection; model updating; biconjugate gradient; preconditioning; approximated pseudo inverse; condition number

1. Introduction

Recently, the damage detection of structures has considerably attracted the attention of researchers. Note that; estimating the severity and location of damage provides the possibility of avoiding the damage propagation in structures and repairing at the right time. By this way, the occurrence of damage is prevented during the remaining useful life of structures. Consequently, the continuous and periodic inspection of important structures is essential to identify their damages after the extreme events. Accordingly, the damage detection of structures is necessary for health monitoring and maintaining the structures.

Damage detection is a process in which the location and severity of damage are estimated. In general, damage detection techniques are divided into groups, namely local and global tactics (Doebling *et al*. 1998). Visual inspection is one of the simplest local approaches. To remove the limitations of local schemes, the global ones were developed. In these strategies, the location and severity of damage are estimated by using the structural responses. Due to the fact that various structural responses are available, different global methods have been presented up to now.

In practice, different devices are applied for measuring the structural responses (Wu and Casciati 2014). Usually, these responses are noisy. As a result, it is not possible to reach the actual responses. To defeat this difficulty, various approaches have been suggested for providing the stability of the damage detection process with noisy measurements (Simon *et al.* 2015). It is worthwhile to mention that the model updating and sensitivity analysis techniques are the most efficient tactics utilized for damage detection of structures (Simon *et al.* 2015). In these schemes, the finite element model of the healthy structure is sequentially updated until its responses match with those of the actual damaged structure.

Messina *et al.* (1998) took advantage of a method based on the sensitivity analysis for damage detection of structures. Moreover, Nasser-Alavi *et al*. (2011) suggested a new sensitivity analysis approach for estimating the location and severity of damage in structures. For this purpose, they utilized an algorithm previously applied to solve subset problem. Also, Esfandiari (2017) and Esfandiari *et al.* (2017) presented different sensitivity analysis approaches for damage identification. In these tactics, some of the structural frequencies were used. In addition, Cheng *et al.* (2009) proposed a method in order to determine optimal sensor locations and damaged areas utilizing sensitivity analysis and model updating. It is worthwhile to remark that the least square method has been extensively utilized for solving the system of equations of the damage detection problem by some researchers. Additionally, Entezami and Shariatmadar (2014) determined the location of damage and its severity by using the techniques based on combining the model updating strategies and regularization tactics.

Recently, various researchers employed the dynamic responses of the structures, including the acceleration, for damage detection of structures. In this regard, Yu and Chung (2012) take advantage of measured acceleration

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responses and FE model updating and least square method for damage identification of reinforced concrete structures. In this process, a nonlinear over-determinate system of equations was required to be solved. Note that; this system of equations is established by matching the responses of the analytical model with those of the damaged structure. Up to now, various algorithms have been proposed to solve this system of equations. Li and Law (2010) mixed the Tikhonov regularization method with the least square scheme for finding the solution of the aforesaid system of equations. This technique was stable against the noisy measurements. In addition, by using a penalty function and stabilization methods, Lu and Law (2007) proposed a sensitivity analysis strategy in which noisy acceleration responses were applied for damage detection of structures. These researchers remarked that they utilized the accelerations due to their easy measurements. In another study, Wang and Yang (2012) suggested a modified Tikhonov regularization (MTR) method by imposing the constraints on the physical parameters. Besides, Sarvi *et al*. (2014) proposed a new damage identification strategy. To achieve this goal, they employed Levenberg-Marquardt algorithm and transformed the nonlinear system of equations into the linear one. Furthermore, Nasser-Alavi *et al.* (2016) presented a tactic in which the gradient and Tikhonov regularization methods were utilized.

As previously mentioned, a nonlinear system of equations is required to be solved in the damage detection methods based on the sensitivity analysis and model updating. Various methods have been suggested for solving nonlinear system of equations. They could be categorized into direct and iterative methods. The least square with regularization technique belongs to the former one (Esfandiari *et al*. 2017, Li and Law 2010, Wang and Yang 2012, Entezami and Shariatmadar 2014, 2015, Lei 2013), and conjugate gradient (CG) and biconjugate gradient (BCG) approaches are the iterative ones whose rate of convergence is high (Saad 2003, Yang 2009). Safari *et al.* (2012, 2014) applied various iterative algorithms for nonlinear analysis of structures. According to their results, BCG method is a proper technique for finding the solution of the system of equations whose coefficient matrix is asymmetric. It is worth emphasizing that the iterative methods may not converge when they are applied for solving the linear system of equations due to ill-posedness of the set of questions. To remedy this difficulty, preconditioning approaches are usually utilized. Also, these techniques can provide the stability and improve the efficiency of numerical methods (Saad 2003). For estimating the inverse of a matrix as a preconditioner matrix, various formulas have been proposed (Toutounian and Soleymani 2013, Li and Li 2010, Li *et al*. 2011). Additionally, Safari *et al.* (2014) employ the preconditioning technique for nonlinear geometric analysis of space frames. It is worthwhile to highlight that the current paper employs a new strategy to provide the stability conditions and estimating the inverse of sensitivity matrix.

Based on the model updating and sensitivity analysis, this paper presents a new technique for damage detection of structures. For this purpose, an iterative method is utilized for finding the solution of the system of equations. By applying the iterative algorithms for calculating the matrix inverse and finding the eigenvalues, the computational costs are considerably reduced. It should be mentioned that the structural accelerations are utilized as the outputs of the damaged structure. In this work, the structural damage is modeled as a reduction in elasticity modulus of the elements.

The remaining text is organized as follows. In section 2, the formulation of the damage detection problem is briefly presented. Section 3 deals with introducing the BCG approach, preconditioning technique, the procedure used for reducing the size of the search space and calculating the pseudo-inverse of the sensitivity matrix. Afterwards, the authors' new algorithm is presented in section 4. Then, numerical examples corroborate the high accuracy and efficiency of the suggested approach in section 5. Finally, the conclusions are summarized in section 6.

2. Basic formulation of damage detection

In this section, the general form of the damage detection problem based on the model updating and sensitivity analysis is presented. In matrix notation, the differential equation governing the dynamic behavior of a multi-degree of freedom system is presented as follows (Li and Law 2010)

$$\mathbf{M\ddot{u}} + \mathbf{C\dot{u}} + \mathbf{Ku} = \mathbf{f(t)} \tag{1}$$

Where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are $n \times n$ matrices which denote mass, damping and stiffness matrices of the structure, respectively. Moreover, $\mathbf{\ddot{u}}$, $\mathbf{\dot{u}}$ and $\mathbf{u}$ are acceleration, velocity and displacement vectors, correspondingly. These vectors have n entries, in which n denotes the number of degrees of freedom. Also, the external force vector is shown by $\mathbf{f(t)}$. By solving Eq. (1), the structural responses can be obtained at each time step which can be employed in the damage detection process.

On the contrary to Eq. (1), damage detection problem includes a nonlinear system of equation. In solution procedure of this problem, the suggested structural model is updated for matching its responses with those of the measured ones (by sensors). This process is expressed mathematically as below

$$\mathbf{R_d} = \mathbf{R(X)} \tag{2}$$

In this equation, $\mathbf{X} = \{x_1, x_2, ..., x_i, ..., x_{ne}\}$ is the vector of unknowns. It should be added that ne denotes the numbers of structural members. Furthermore, $\mathbf{R_d}$ and $\mathbf{R(X)}$ are acceleration time history of the actual damaged structure and its model, respectively.

It is obvious that Eq. (2) is nonlinear (Li and Law 2010, Naseralavi *et al.* 2016). By omitting its higher order sentences, it can be changed into the coming linear system of equations

$$\mathbf{R_d} - \mathbf{R_h} = \frac{\partial \mathbf{R}}{\partial \mathbf{X}}(\mathbf{X}_d - \mathbf{X}_h) \quad (3)$$

$$\Delta\mathbf{R} = \mathbf{S}.\Delta\mathbf{X} \quad (4)$$

Note that; the damage localization and severity estimation can be conducted by solving this system of equation whose unknown parameters are related to the magnitude of damage in each member. In Eq. (3), $\mathbf{R_h}$ is the structural response of the healthy structure. Moreover, $\mathbf{X}_h$ and $\mathbf{X}_d$ denote the damage vectors of the healthy and damaged structure, respectively. It should be reminded that the entries of $\mathbf{X}_h$ are equal to zero. $\mathbf{S} = {\partial \mathbf{R}}/{\partial \mathbf{X}}$ is named the sensitivity matrix, and its entries are the known coefficients of the equations. To calculate the entries of this matrix, the finite difference approach can be applied. To compute the i-th column of the aforesaid matrix, a structural member is artificially damaged with the severity of Se. This parameter is assumed to be 0.001 in this paper. Accordingly, the sensitivity matrix can be calculated as below (Naseralavi *et al.* 2016)

$$\mathbf{S} = [\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_i, ..., \mathbf{s}_{ne}] \ , \quad \mathbf{s}_i = \frac{\mathbf{R}_{di} - \mathbf{R}_h}{Se} \quad (5)$$

In this equation, the i-th column of the sensitivity matrix and the response of the i-th structural element due to artificial damage are denoted by $\mathbf{s}_i$ and $\mathbf{R}_{di}$, respectively. If the "nsr" sensors installed on the structure measure the accelerations at nt iterations, the number of entries of $\Delta\mathbf{R}$ is $(nt \times nsr) \times 1$, and the number of columns and rows of the sensitivity matrix are equal to ne and $(nt \times nsr)$, respectively. In general, the system of equations of the damage detection procedure is over-determinate. In other words, number of equations is greater than the number of unknowns.

As previously mentioned, Eq. (4) can be obtained by linearizing Eq. (2). This process causes errors in the solution procedure. To minimize the error due to linearization, model updating methods are usually employed (Li and Law 2010, Lu and Law 2007). At the first stage, the damage detection system of equations is established and solved. Then, by using the unknown vector achieved from the first iteration, the model is updated and a new system of equation is set up and solved. This procedure is iteratively repeated until reaching the proper accuracy. This procedure can be mathematically expressed as follows

$$\begin{cases} \Delta\mathbf{R}_0 = \mathbf{S}_0.\Delta\mathbf{X}_1 \\ \Delta\mathbf{R}_1 = \mathbf{S}_1.\Delta\mathbf{X}_2 \\ \vdots \\ \Delta\mathbf{R_{i-1}} = \mathbf{S_{i-1}}.\Delta\mathbf{X}_i \end{cases} \longrightarrow \mathbf{X} = \Delta\mathbf{X}_1 + \Delta\mathbf{X}_2 + ... + \Delta\mathbf{X_i} \quad (6)$$

in which $\Delta\mathbf{X}_i$, $\mathbf{X}$ and $\mathbf{S_{i-1}}$ are the damage vector in i-th iteration, the ultimate damage vector and sensitivity matrix obtained from the damage vector in the i-1 iteration, respectively. After introducing the system of equation of the damage detection problem, it is required to select an appropriate solution routine. In general, the least square scheme with the regularization methods have been widely utilized for finding the solution of the over-determinate system of equations in which it was required to invert the coefficient matrix directly and calculate the eigenvalues (Esfandiari 2017, Li and Law 2010, Wang and Yang 2012, Naseralavi *et al*. 2016). Hence, increasing the size of the system of equations leads to considerable increment in computational costs.

In the next section, a new iterative method is presented. In this approach, it is not required to compute the inverse and the eigenvalues of the coefficient matrix. As a result, the authors' technique is very efficient for structures with large number of members. This approach is able to solve the system of equations with rectangular coefficient matrices. Also, it should be stable against the noisy measurements.

3. The proposed method

In this section, the suggested strategy is mathematically expressed. First, CG and BCG approaches are briefly reviewed. Afterwards, the preconditioning tactic and the approximate method for calculating pseudo-inversion of matrices are presented.

3.1 Conjugate gradient method

As mentioned before, various methods have been proposed for solving the linear and nonlinear system of equations by other researchers. They are categorized into direct and iterative approaches (Saad 2003). The latter ones utilize an initial guess to produce a sequence of improving approximate solutions for a class of problems. An iterative procedure is convergent when the corresponding sequence converges, i.e., the convergence criterion is satisfied. In contrast, the direct tactics solves the problem by a finite sequence of operations. It should be reminded that usage of direct methods for large system of equations is expensive from mathematical point of view. Additionally, they sometimes are not able to find the solution (Saad 2003). Due to the fact that the iterative approaches are simpler and faster than the direct ones, they are generally employed in practice. CG and BCG iterative tactics are more popular and efficient than other iterative strategies for solving the large system of equations (Saad 2003, Yang 2009, Saffari *et al*. 2012). Usually, the conjugate gradient technique is applied for finding the solution of the linear system of equations whose coefficient matrix is real and symmetric. In contrast, this matrix is not required to be real, positive-definite and symmetric in the BCG approaches. Both of these tactics find the solution of $\mathbf{AX} = \mathbf{b}$ by minimizing the following second-order equation (Lu and Chen 2017)

$$f(\mathbf{X}) = \frac{1}{2}\mathbf{X}^T\mathbf{AX} - \mathbf{X}^T\mathbf{b} \quad (7)$$

3.2 Biconjugate gradient method

The BCG approach has been suggested for solving the system of equations whose coefficient matrices are un-symmetric. This tactic employs a dual scheme for solving the system of equations in which both $\mathbf{AX}=\mathbf{b}$ and $\mathbf{A}^*\mathbf{X}^*=\mathbf{b}^*$ systems of equations are established and solved. It should be added that $\mathbf{A}^*$ is the conjugate transpose of matrix $\mathbf{A}$. In other words, this well-known approach is developed based on projecting $\mathbf{AX}=\mathbf{b}$ into the Krylov subspace (Saffari *et al.* 2012). The BCG algorithm solves the system of Eq. (4) as follows (Saad 2003):

1. Guess the initial $\Delta\mathbf{X}_i^0$.
2. Compute $\mathbf{r}_0=\Delta\mathbf{R}-\mathbf{S}.\Delta\mathbf{X}_i^0$ and choose $\mathbf{r}_0^*$ with condition inner product $(\mathbf{r}_0,\mathbf{r}_0^*)\neq 0$.
3. Set $\mathbf{P}_0=\mathbf{r}_0$ and $\mathbf{P}_0^*=\mathbf{r}_0^*$.
4. Until reaching to convergence criteria and for $k=1,2,\ldots,$ repeat following steps,

4.1. $\alpha_k=\dfrac{(\mathbf{r}_k,\mathbf{r}_k^*)}{(\mathbf{SP}_k,\mathbf{P}_k^*)}$

4.2. $\Delta\mathbf{X}_i^{k+1}=\Delta\mathbf{X}_i^k+\alpha_k.\mathbf{P}_k$

4.3. $\mathbf{r}_{k+1}=\mathbf{r}_k-\alpha_k\mathbf{SP}_k$

4.4. $\mathbf{r}^*{}_{k+1}=\mathbf{r}_k^*-\alpha_k\mathbf{SP}_k^*$

4.5. $\beta_k=\dfrac{(\mathbf{r}_{k+1},\mathbf{r}_{k+1}^*)}{(\mathbf{r}_k,\mathbf{r}_k^*)}$

4.6. $\mathbf{P}_{k+1}=\mathbf{r}_{k+1}+\beta_k\mathbf{P}_k$

4.7. $\mathbf{P}_{k+1}^*=\mathbf{r}_{k+1}^*+\beta_k\mathbf{P}_k^*$

Due to the fact that a large system of equations governs the damage detection problem of structures, it is suggested to utilize the iterative methods for finding its solution. It should be reminded that this system of equation is rectangular if the structural accelerations are employed in damage detection process. Prior to using the iterative BCG approaches for solving this system of equations, it is essential to reshape its coefficient matrix into a square one (Saad 2003). For this purpose, a preconditioner matrix is required. In addition, this matrix should be able to establish the well-posedness of the new square coefficient matrix. In other words, the condition number of the square matrix ought to be as small as possible. In what follows, the selection process of a suitable preconditioner matrix is discussed in depth.

It is worthwhile to highlight that using the preconditioner matrix reshape the coefficient matrix into an asymmetric square one. For this reason, it is necessary to take advantage of the BCG method in damage detection of structures. Recall that; this tactic can be utilized for problems whose coefficient matrices are asymmetric.

3.3 Preconditioning

Preconditioning is typically related to transforming the system of equations into a simpler form suitable for numerical solving strategies (Saad 2003). This can be achieved with the help of the preconditioner matrix ($\hat{\mathbf{M}}$). In general, the iterative methods applied for solving the linear system of equations are prone to divergence. Consequently, they are usually mixed with the preconditioning approaches to solve **AX**=**b**. Pre-multiplying the system of equations by $\hat{\mathbf{M}}$ leads to $\hat{\mathbf{M}}.\mathbf{AX}=\hat{\mathbf{M}}.\mathbf{b}$. The solution of this new form is similar to that of the original one. Note that; the solution procedure of the new system of equation is more stable in comparison to that of the initial one, and its rate of convergence is higher. It is worth remarking that the selected preconditioner matrix should be calculated and established easily at each iteration. Moreover, an appropriate preconditioner matrix is able to increase the rate of convergence.

Ill-conditioning and ill-posedness are two main difficulties which exist in solving the system of Eq. (4). It should be reminded that coefficient matrix is ill-conditioned if its condition number is very large. When the solution of a system of equation is highly sensitive to the small changes in its right side, it is named ill-posed. It should be mentioned that the condition number can be used for measuring the stability and convergence rate of an ill-posed system of equations. Note that; a system of equations whose condition number is very large is highly sensitive to the small changes in its right side. In addition, numerical instability and divergence may occur in its solution procedure.

To avoid instability, regularization tactics are usually used (Hanke and Groetsch 2017). Recently, Tikhonov regularization method has been widely applied in damage detection problems (Li and Law 2010, Lu and Law 2007). In this paper, a preconditioner matrix is presented for guaranteeing the stability of the numerical solution procedure as substitute for stabilizing approaches. In other words, by employing the authors' preconditioner matrix, the condition number of the system of equations of the damage detection problem is considerably reduced as comprehensively shown in section 5. Additionally, this matrix can be easily set up.

The suggested preconditioner is the approximation of the pseudo inverse of the sensitivity matrix ($\mathbf{S}^{\otimes}$) whose size is $(nt\times nsr)\times ne$. Hence, its pseudo-inverse is a $ne\times(nt\times nsr)$ matrix. At this stage, pre-multiplying the system of equations of the damage detection problem by $\mathbf{S}^{\otimes}$ results in a square system of equations. Furthermore, numerical investigations prove that the condition number of the new system of equations is significantly reduced. This new system of equations has the subsequent appearance:

$$\mathbf{S}_{i-1}^{\otimes}\ \mathbf{S}_{i-1}\ \Delta\mathbf{X}_i=\mathbf{S}_{i-1}^{\otimes}\ \Delta\mathbf{R}_{i-1} \tag{8}$$

Where $\Delta\mathbf{R}_{i-1}$, $\mathbf{S}_{i-1}$ and $\mathbf{S}_{i-1}^{\otimes}$ denote the difference between the responses of the damaged and undamaged structures at the (i-1)-th iteration, the sensitivity matrix obtained from the damage vector of the (i-1)-th iteration and the approximate pseudo-inverse matrix.

3.4 The approximate of pseudo-inverse

Inversion of a large matrix is expensive from mathematical point of view. As a result, applying the iterative methods for finding the approximate inversion of matrices, which can be used as a preconditioner, has attracted the attention of researchers (Toutounian and Soleymani 2013, Li and Li 2010, Li *et al.* 2011). To approximately invert a matrix, $\hat{\mathbf{M}}.\mathbf{A}$ should be equal to an identity matrix. One of the most stable methods for calculating the approximate inverse of a matrix has been proposed by Schulz (Toutounian and Soleymani 2013). The formulation of this approach has the succeeding shape

$$\mathbf{N}_{m+1} = \mathbf{N}_m(2\mathbf{I} - \mathbf{A}\mathbf{N}_m) \qquad m = 0,1,2,... \tag{9}$$

In the Eq. (9), $\mathbf{N}_m$ is an approximate inverse of a matrix at the m-th iteration and $\mathbf{I}$ denotes the identity matrix whose size is similar to that of $\mathbf{N}_m$. It is noticeable that the convergence rate of Eq. (9) is quadratic. Other researchers suggested methods with higher order convergence rates. In 2013, Toutounian presented a tactic with the convergence of the higher order. Moreover, the high efficiency and convergence rate of this strategy in computing the approximate inverse of a rectangular matrix was proved (Toutounian and Soleymani 2013).

$$\mathbf{N}_{m+1} = \frac{1}{2}\times\mathbf{N}_m\left[9\mathbf{I} - \mathbf{A}\mathbf{N}_m(16\mathbf{I} - \mathbf{A}\mathbf{N}_m(14\mathbf{I} - \mathbf{A}\mathbf{N}_m(6\mathbf{I} - \mathbf{A}\mathbf{N}_m)))\right], m = 0,1,2,... \tag{10}$$

The size of the system of equations corresponding to a damage detection problem is dependent on the measured responses, the structure's topology and model. For structures made of large number of members, this system of equation is large. Therefore, it is strongly suggested to use iterative methods for finding the approximate inversion of its coefficient matrix.

To approximately calculate the pseudo-inverse of the sensitivity matrix, the coming equation can be utilized

$$\mathbf{S}_{m+1}^{\otimes} = \frac{1}{2}\times\mathbf{S}_m^{\otimes}\left[9\mathbf{I} - \mathbf{S}\mathbf{S}_m^{\otimes}(16\mathbf{I} - \mathbf{S}\mathbf{S}_m^{\otimes}(14\mathbf{I} - \mathbf{S}\mathbf{S}_m^{\otimes}(6\mathbf{I} - \mathbf{S}\mathbf{S}_m^{\otimes})))\right], m = 0,1,2,... \tag{11}$$

In this equation, $\mathbf{S}_m^{\otimes}$ and $\mathbf{I} \in R^{(nt\times nsr)\times(nt\times nsr)}$ are the approximate pseudo-inverse of the sensitivity matrix at the m-th iteration and the corresponding identity matrix, respectively. It is obvious that increasing the number of iterations leads to more accurate results, and selection of a proper $\mathbf{S}_0^{\otimes}$ improves the convergence rate. On the other hand, choosing an inappropriate $\mathbf{S}_0^{\otimes}$ results in divergence. In this paper, the coming formula is applied for finding the suitable $\mathbf{S}_0^{\otimes}$ which guarantee the convergence (Toutounian and Soleymani 2013)

$$\mathbf{S}_0^{\otimes} = \frac{\mathbf{S}^T}{\sigma_1^2} \tag{12}$$

In which the transpose of matrix $\mathbf{S}$ and its maximum eigenvalues are denoted by $\mathbf{S}^T$ and σ_1, respectively. Various methods, such as power technique, are available for computing σ_1. Due to the fact that the accurate value of σ_1 is not essential for Eq. (12), it is calculated with the help of approximate equation presented in the next section.

3.5 Gershgorin circle theorem

Based on this theorem, the spectrum of a square matrix can be bounded (Golub and Van 2012). For a square matrix $\mathbf{A} \in R^{n\times n}$, R_i can be computed for each of its rows. Then, by computing R_i for a square matrix $\mathbf{A} \in R^{n\times n}$ and depicting circles with center of a_{ii} and radius of R_i, it can be shown that each eigenvalues of this matrix is placed into at least one of these circles (Golub and Van 2012). In other words, the maximum values of $a_{ii} + R_i$ is an approximation of the maximum eigenvalue as follows

$$R_i = \sum_{j=1,\ j\neq i}^{n} \left|a_{ij}\right| \tag{13}$$

$$\sigma_1 = Max\left\{a_{ii} + R_i\right\} \quad , \quad i = 1,2,...,n \tag{14}$$

In each iteration, the sensitivity matrix ($\mathbf{S}$) is rectangular. For this reason, $\mathbf{S}^T\mathbf{S}$ should be applied prior to usage of Gershgorin circle theorem. Finally, the approximation value of the eigenvalue of the sensitivity matrix can be achieved by utilizing Eq. (14). Then, it can be applied in Eq. (11). Note that; the convergence of Eq. (11) is guaranteed when the eigenvalue is obtained by using Eq. (14).

3.6 Reduction of search space

The entries of the damage vector (the unknowns vector) corresponding to the damage detection problem are ranged from zero to one. Moreover, most of its entries are zero (Li and Law 2010). The latter characteristic of this vector has been used in two-step approaches. At first, these tactics identified the probable damage members. Afterwards, the damage detection procedure is conducted on these members. In practice, most of the structure's members are undamaged and few number of them are damaged. Hence, the search space can be reduced.

This paper takes advantage of the aforementioned characteristics of the damage vector. In this way, the computational efforts are considerably reduced, and the accuracy of the suggested damage detection algorithm is increased when the measurements are noisy. In the authors' damage detection algorithm, the entries of the damage vector which are less than a specific value (such as 0.01) are set to zero. Actually, in this way, most of the entries of the damage vector are zero. At this stage, the structure is re-updated based on the modified damage vectors, and the new system of equations is obtained. Afterwards, the members corresponding to the zero entries of the damage vector are removed from the system of equations. As a result, the number of unknowns is significantly reduced. This process iteratively repeated until the end of the solution procedure. The steps of this method are summarized as below:

1- The calculated damage vector divided into two sections: the probable damaged members (more than 0.01) and the members with negligible damage (less than 0.01).

$$\mathbf{X}=\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{ne} \end{Bmatrix} \rightarrow \mathbf{X}=\begin{Bmatrix} \mathbf{X}_{\mathbf{damagedpart}} \\ \mathbf{X}_{\mathbf{healty\ part}} \end{Bmatrix} \tag{15}$$

These sub-vectors are denoted by $\mathbf{X}_{\mathbf{damagedpart}}$ and $\mathbf{X}_{\mathbf{healty\ part}}$, respectively.

2- The entries of $\mathbf{X}_{\mathbf{healty\ part}}$ are set to zero.

$$\mathbf{X}_{\mathbf{healty\ part}}=0 \tag{16}$$

3- The elasticity modulus of members belongs to $\mathbf{X}_{\mathbf{healty\ part}}$ are modified.

$$E_i=\begin{cases} 0 & if \quad x_i \in \mathbf{X}_{\mathbf{healty\ part}} \\ E(1-x_i) & if \quad x_i \in \mathbf{X}_{\mathbf{damagedpart}} \end{cases} \tag{17}$$

The elasticity modulus of the i-th element and its damage severity are shown by E and x_i, correspondingly.

4- The structure is updated based on the elasticity modulus calculated in the previous iteration, and a new system of equation is established by using Eq. (8).

5- The unknowns corresponding to the entries of $\mathbf{X}_{\mathbf{healty\ part}}$ are set to zero and removed from the new system of equations. Then, the remaining unknowns are obtained by using the BCG algorithm, and the solution procedure return to its first step.

$$\mathbf{X}=\begin{Bmatrix} \mathbf{X}_{\mathbf{damagedpart}} \\ \mathbf{X}_{\mathbf{healty\ part}} \end{Bmatrix} \xrightarrow{\textbf{Decrease of search space}} \mathbf{X}=\{\mathbf{X}_{\mathbf{damagedpart}}\} \tag{18}$$

$$\mathbf{S}^{\otimes}\mathbf{S}=[\mathbf{s}^{\otimes}\mathbf{s}_{\mathbf{d}} \quad \mathbf{s}^{\otimes}\mathbf{s}_{\mathbf{h}}] \xrightarrow{\textbf{Construction of new S}} \mathbf{S}^{\otimes}\mathbf{S}=[\mathbf{s}^{\otimes}\mathbf{s}_{\mathbf{d}}] \tag{19}$$

where $\mathbf{s}^{\otimes}\mathbf{s}_{\mathbf{d}}$ and $\mathbf{s}^{\otimes}\mathbf{s}_{\mathbf{h}}$ are sub-matrices of the sensitivity matrix corresponding to $\mathbf{X}_{\mathbf{damagedpart}}$ and $\mathbf{X}_{\mathbf{healty\ part}}$, respectively.

It is worthwhile to mention that when the iteration number reaches to 20% of the structure's elements, the process of search space reduction begins. In other words, after a relative convergence, the convergence rate is become more rapid by reducing the search space and modifying the elasticity modulus of the elements which are vividly healthy.

$$T \cong 0.2 \times ne \tag{20}$$

in which T is the minimum number of iterations required for starting reduction search space procedure. Also, ne denotes the number of elements.

4. The proposed algorithm

In this section, the issues discussed in the previous sections are summarized in a flowchart. This flowchart is illustrated in Fig. 1. At each iteration, the system of equations is established based on the responses of the structural model and the damaged structure. With the help of Eq. (11), the preconditioner matrix can be obtained by computing the pseudo-inverse of the sensitivity matrix. By pre-multiplying this matrix by the system of equations, the new system of equations is obtained (Eq. (8)). Due to the asymmetry of the coefficient matrix of this new system of equation, the BCG approach is employed for finding its solution. At this stage, the structural model is updated based on the damage vector (vector includes the damage severity of the members) achieved from the previous iteration. Then, this procedure is repeated. In this process, the reduction of the search space begins when the iteration number reaches to the values achieved from Eq. (20). Note that; this procedure is conducted by utilizing Eqs. (15)-(19).

It is worthwhile to highlight that investigating the convergence and accuracy of the suggested algorithm in damage detection problem are important issues. For this purpose, the following convergence and accuracy criteria are used (Lu and Law 2007, Naseralavi *et al.* 2016)

$$\mathbf{C}=\left\|\mathbf{X_i}-\mathbf{X_{i-1}}\right\|_2 \tag{21}$$

$$\textbf{error } 1=\left\|\mathbf{R_d}-\mathbf{R_{model}}\right\|_2 \tag{22}$$

$$\textbf{error } 2=\sum_{\mathbf{i=1}}^{ne}\left|\mathbf{x_i^{calculated}}-\mathbf{x_i^{exact}}\right| \times 100 \tag{23}$$

where $\mathbf{X_{i-1}}$ and $\mathbf{X_i}$ are the damage vector of iteration i and i-1, respectively. Eq. (21) is utilized for assessing the convergence of the proposed method. It is the second norm of the vector equal to the difference between unknown vectors of two sequential iterations. Eqs. (22) and (23) are applied for evaluating the error of the proposed damage detection technique. It should be added that $\mathbf{R_{model}}$ contains the responses of the updated structural model in each iteration. Moreover, $\mathbf{x_i^{exact}}$ and $\mathbf{x_i^{calculated}}$ are the actual and calculated damage vector of the i-th element.

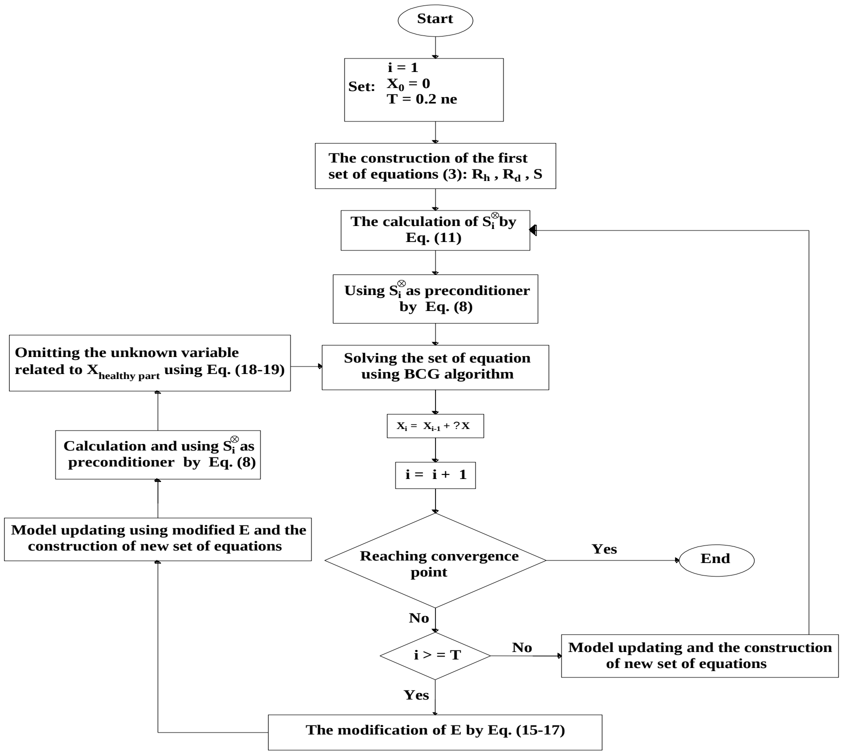


Fig. 1 The flowchart of the proposed method

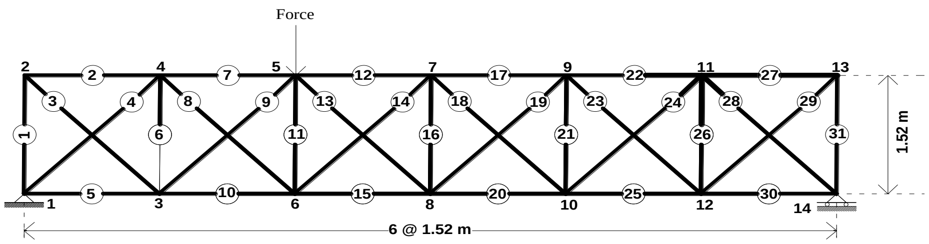


Fig. 2 31-member plane structure (Li and Law 2010)

5. Numerical examples

To assess the capability and efficiency of the suggested method, it is applied for damage detection of two trusses. Note that; the noisy measurements are employed. Due to the fact that experimental data are not available, the aforesaid errors should be induced to the responses of the damaged structure's model. To achieve this goal, the coming equation is employed (Li and Law 2010).

$$\mathbf{R}_{\mathbf{measured}} = \mathbf{R}_{\mathbf{calculated}} + E_{\mathbf{p}}.\mathbf{N}.\mathbf{R}_{\mathbf{calculated}} \tag{24}$$

In this equation, the responses with and without noise are denoted by $\mathbf{R}_{\mathbf{measured}}$ and $\mathbf{R}_{\mathbf{calculated}}$, respectively. Also, the induced error level and the error vector with standard normal distribution are shown by $E_{\mathbf{p}}$ and $\mathbf{N}$, correspondingly. The mean and standard deviation are assumed to be equal to 0 and 1, respectively.

5.1 Plane truss

Herein, a 31-member truss is assessed (Li and Law 2010). As shown in Fig. 2. It has 14 nodes and 25 degrees of freedom. This truss is made of a material with the density of 2770 kg/m^3 and elasticity modulus of 70 Gpa. Moreover, the cross-sectional area of its members is 0.0025 m^2. The parameters required for calculating the accelerations and load patterns are assumed to be equal to those of Li and Law (2010), for comparison purposes. Members 18, 19, 20 and 22 are damaged. The elasticity modulus of member 20 is reduced by 15%, and the elasticity modulus of the other damaged members is decreased by 10%. To perform the damage detection process, an impulsive load is applied to node 5. The impulsive load duration is 0.005 sec, and its maximum value is equal to 320.4 kN. The vertical accelerations of nodes 2, 4, 6 and 7 are recorded. In other words, only four sensors are used for measuring the vertical accelerations.

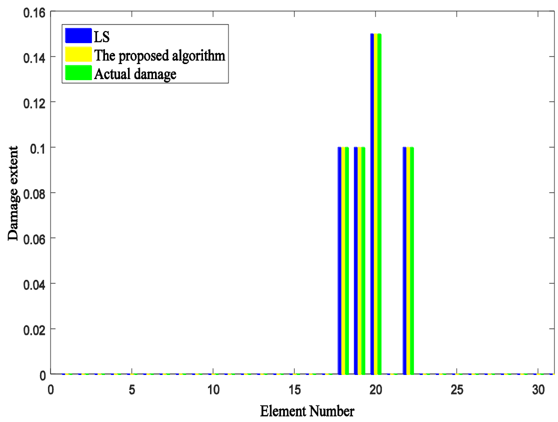


Fig. 3 The results of damage detection for noise 0%

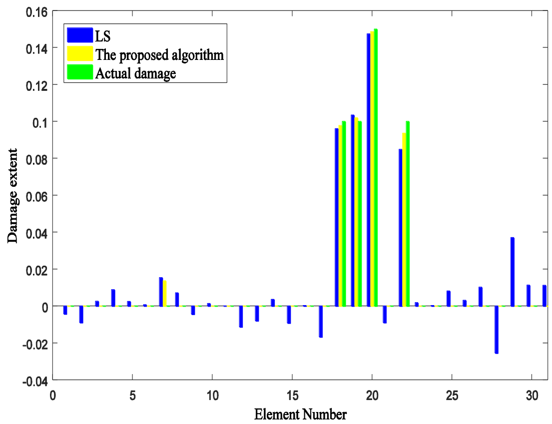


Fig. 4 The results of damage detection for noise 5%

5.1.1 Detection of damaged elements

To prove the stability of the suggested approach against the noisy measurement, responses with 5% and 10% noise levels are applied. The least square technique has been widely employed in researches dealing with solving the system of equations corresponding to the damage detection problem. In this paper, to assess the efficiency (reducing the condition number of the coefficient matrix) of the suggested technique, the obtained results are compared with those of the least square method. In Fig. 3, the damage detection chart of the 31-member truss with noise-free measurements is presented. Due to the fact that the measurements are noise-free and the sensitivity matrix is not large, the location and severity of damage can be obtained accurately with the help of both approaches. Now, the measurements with 5% noise level are used. Based on Fig. 4, it is obvious that the suggested approach can estimate the location and severity of damage with high accuracy by utilizing these measurements. With the help of Eq. (23), the error existing in the damage vector is computed, and it is equal to 2.58%. In a similar manner, the damage detection process is conducted by employing measurements with 10% noise level as depicted in Fig. 5. In this case, the error index is equal to 8.2%, and the authors' tactic performs well. In other words, fortunately, the suggested method and the tactic used for stabilizing the damage detection process against noisy measurements are successful.

At this stage, the obtained results are compared with those of Li and Law (2010). In this way, the comparison between the suggested stabilizing approach and other techniques, such as Tikhonov regularization is conducted. It should be added that this reference proposed an adaptive Tikhonov regularization algorithm for solving the system of equations corresponding to the damage detection problem. In Fig. 6, the results achieved by locating and estimating severity of damage in 31-member truss are presented for measurements with noise 5% and noise 10%. Obviously, the authors' approach performs more accurately in determination of damage location than the other one. Similarly, the proposed algorithm is more successful in estimating the damage severity in comparison to the other researchers' scheme. On the other hand, the damage magnitude in members wrongly identified as the damaged members by the suggested tactic is less than those calculated by other method.

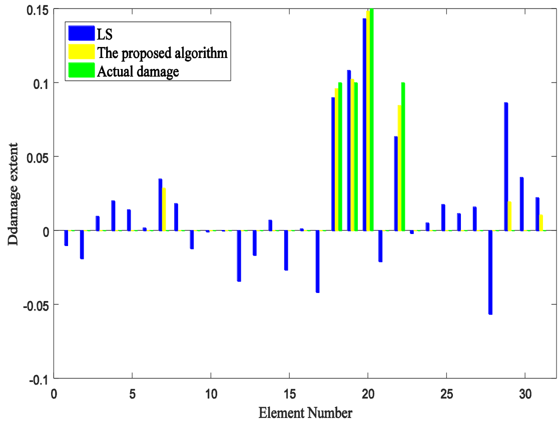


Fig. 5 The results of damage detection for noise 10%

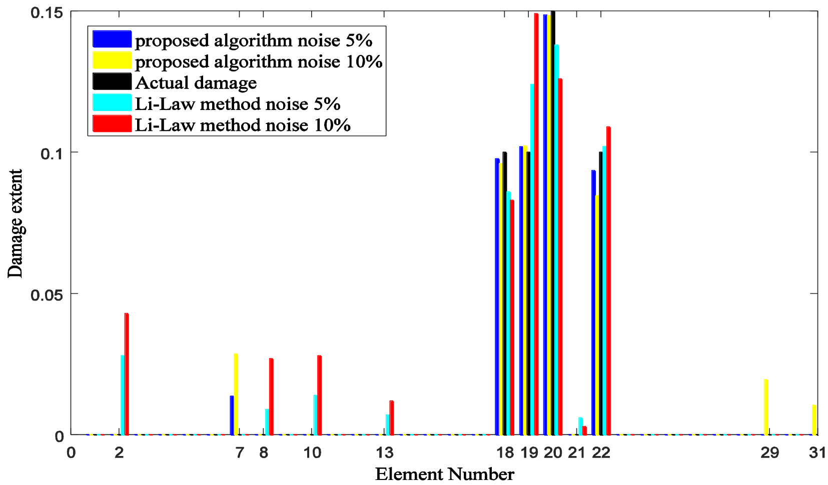


Fig. 6 The comparison between the results of proposed method and Li-Law method

Table 1 Condition number of coefficient matrix

	0% noise		5% noise		10% noise	
Iteration	$\mathbf{S}^T.\mathbf{S}$	$\mathbf{S}^{\otimes}.\mathbf{S}$	$\mathbf{S}^T.\mathbf{S}$	$\mathbf{S}^{\otimes}.\mathbf{S}$	$\mathbf{S}^T.\mathbf{S}$	$\mathbf{S}^{\otimes}.\mathbf{S}$
1	33819.887	25.271	33819.887	25.271	33819.887	25.271
2	54701.198	41.182	49455.806	37.248	46945.412	35.466
3	32221.821	24.112	34012.908	25.552	37310.915	28.192
4	34887.025	26.512	37392.322	28.525	40783.546	30.577
5	34844.773	26.486	37756.628	28.298	42067.584	31.126
6	34692.069	26.432	37901.030	28.410	41539.402	30.308
7	-	-	38318.494	6.071	40457.868	10.836
8	-	-	37770.040	3.088	41460.728	7.057
9	-	-	37967.261	3.070	40983.143	6.074
10	-	-	37932.250	-	41462.395	6.034
11	-	-	37832.587	-	40617.493	6.074

5.1.2 Evaluation of the condition number of matrix

Now, the capability of the approach proposed for reducing the condition number of the system of equations and its effect on the authors' stabilizing process is investigated. After preconditioning procedure, the coefficient matrix of the system of equations for presented

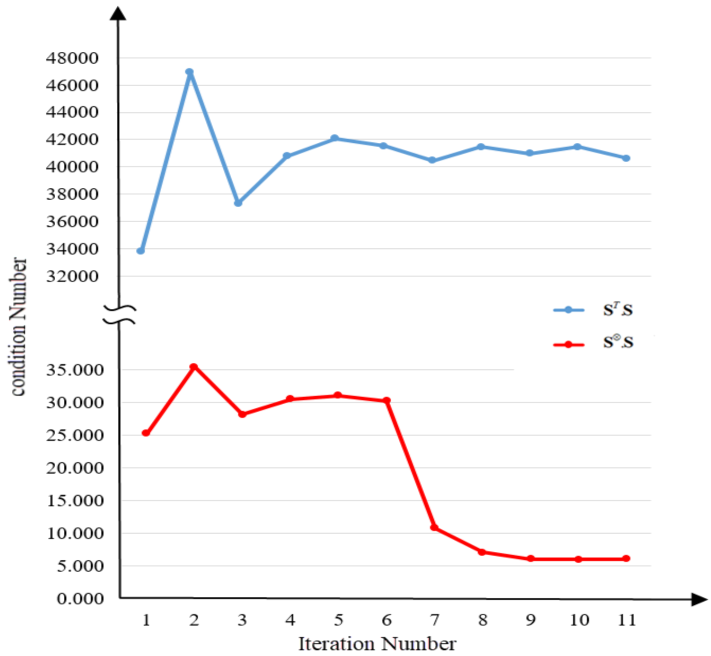


Fig. 7 The comparison of condition number for noise 10%

method changes into $\mathbf{S}^{\otimes}.\mathbf{S}$. In contrast, for least square method, the coefficient matrix is in the form of $\mathbf{S}^{T}.\mathbf{S}$. For both techniques, the condition numbers corresponding to various steps of the model updating process with different noise levels are listed in Table 1. It is clear that the condition number of the suggested algorithm is continuously decreased.

As a consequence, the stability of the coefficient matrix is guaranteed when the noisy measurements are applied. In the first iteration of the damage detection process of the 31-member truss, the sensitivity matrices of various noise levels, which are calculated based on the responses of the undamaged structure, are the same. It is worthwhile to mention that the condition numbers of matrices $\mathbf{S}^{\otimes}.\mathbf{S}_d$ and $\mathbf{S}^{T}.\mathbf{S}$ are 25.271 and 33819.88, respectively. In fact, due to the usage of stabilizing method, a considerable reduction can be observed in condition number. For noise 5%, the condition numbers are shown in Fig. 7. By reducing the search space, the condition numbers are decreased more considerably. As a result, the solution approach becomes more stable against noisy input data.

5.1.3 Convergence test

This section deals with evaluating the accuracy and convergence rate of the suggested tactic with the help of Eq. (22). Moreover, the effect of the process, which is introduced for reducing the search space, on the convergence rate is investigated. Based on Fig. 8(a), the damage detection of the 31-member truss is finished prior to the beginning of the process of reducing search space when the measurements are noise-free. For this case, the number of iterations of the model updating and establishing the new system of equations procedures is equal to 6, and the second norm of the vector equal to the difference between the damaged and modeled structural responses is $error1 = 7.8e-5$. For noises 5% and 10%, the reduction process of search space begins. According to Eq. (20), 6 iterations are required for the beginning of the aforesaid procedure. In Figs. 8(b) and 8(c), the computational operations of Eq. (22) are presented for both noise levels. For noise 5%, the search space is reduced in two stages, and 26 members are omitted. Furthermore, for noise 10%, the reduction of search space is conducted in 4 stages, and 7 members are finally removed. Based on Figs. 8(b) and 8(c), it is obvious that the accuracy of the solution method is decreased after the beginning of the first stage of the search space reduction process. But in the end, the convergence rate is enhanced.

In Fig. 9, the calculated damage magnitudes are presented for noise 10%. For instance, in the first stage of damage detection method, the magnitude of damages corresponding to members 30 and 31 is equal to 0.042 and 0.207, respectively. In the coming iterations, the damage magnitude is 0.019 and 0.01 for member 29 and 31, correspondingly.

In Fig. 10, the solution space is illustrated, and it is divided into two search subspace and removed one. Note that; for noise-free measurements, the damage detection process is completed prior to reducing the search space. For the case with noise 5%, 86% reduction can be observed in the search space. In the case with noise 10%, 82% of the initial space is omitted. It is worthwhile to highlight that reducing the search space decreases the required computational efforts.

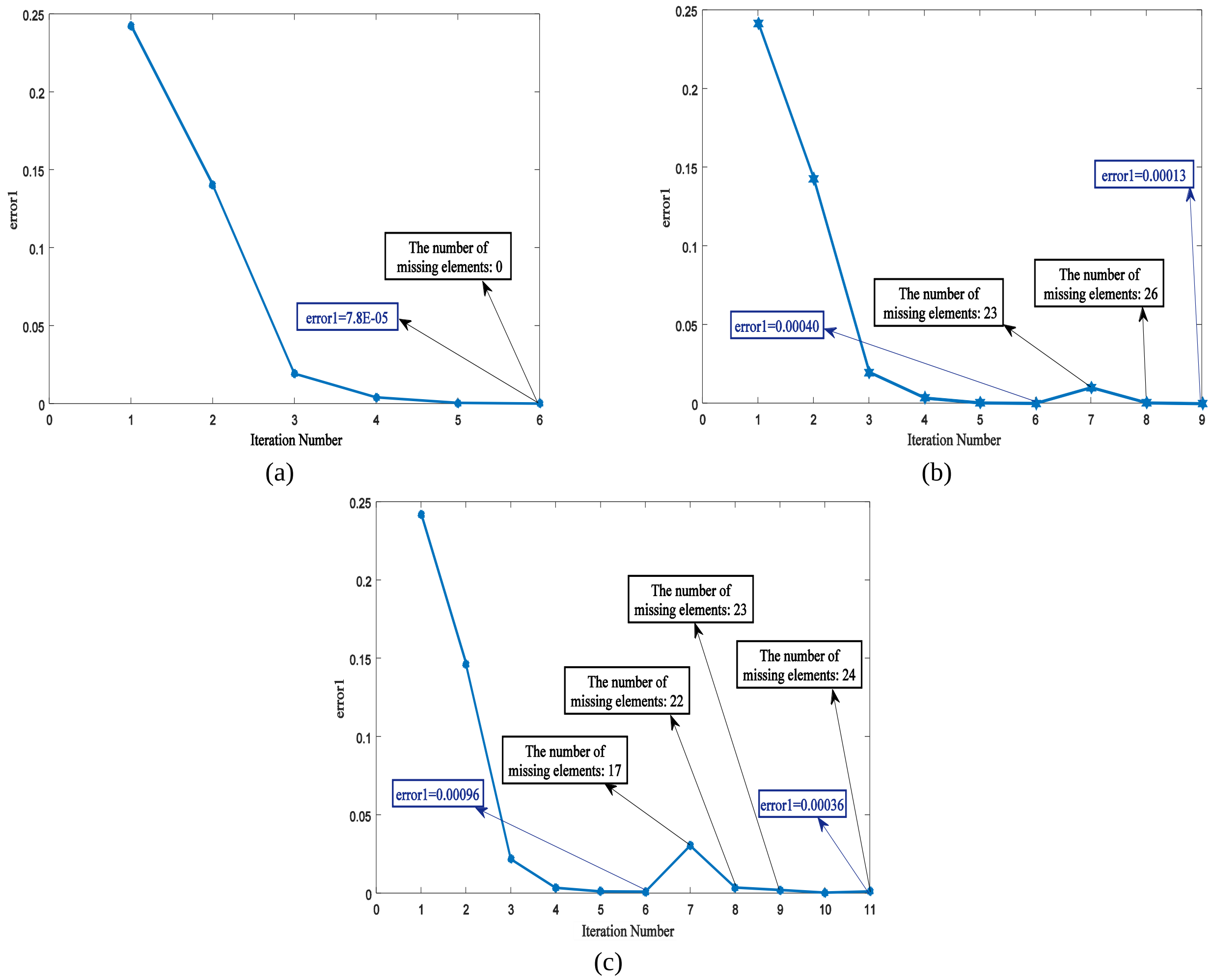


Fig. 8 The amount of norm of residual: (a) for noise 0%, (b) for noise 5% and (c) for noise 10%

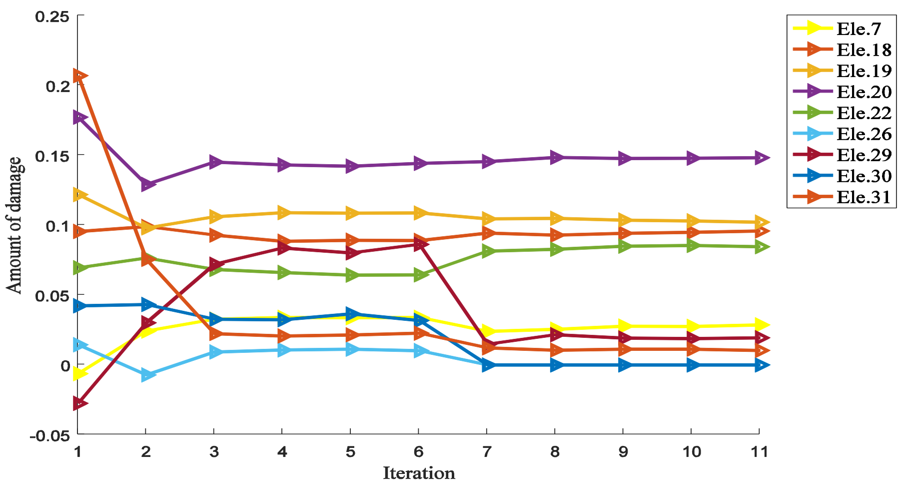


Fig. 9 Iteration results of damage detection for noise 10%

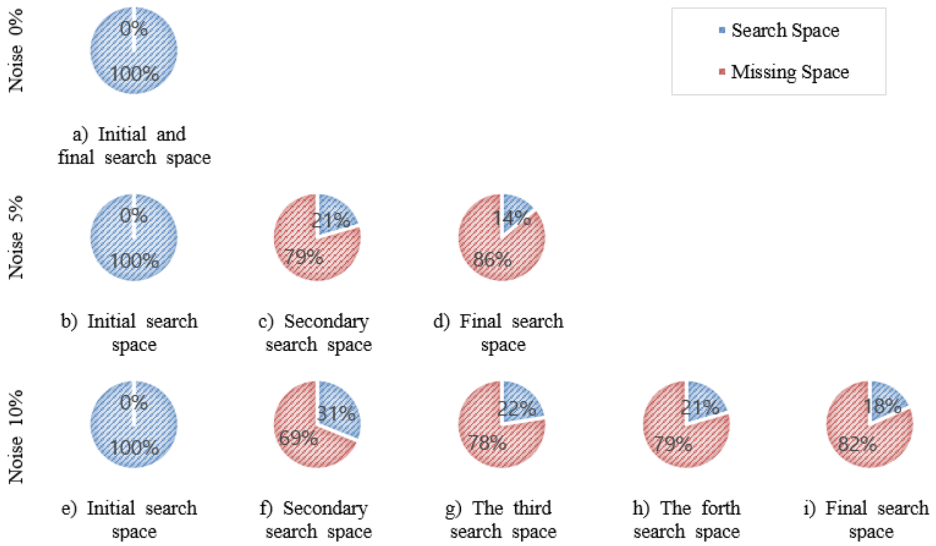
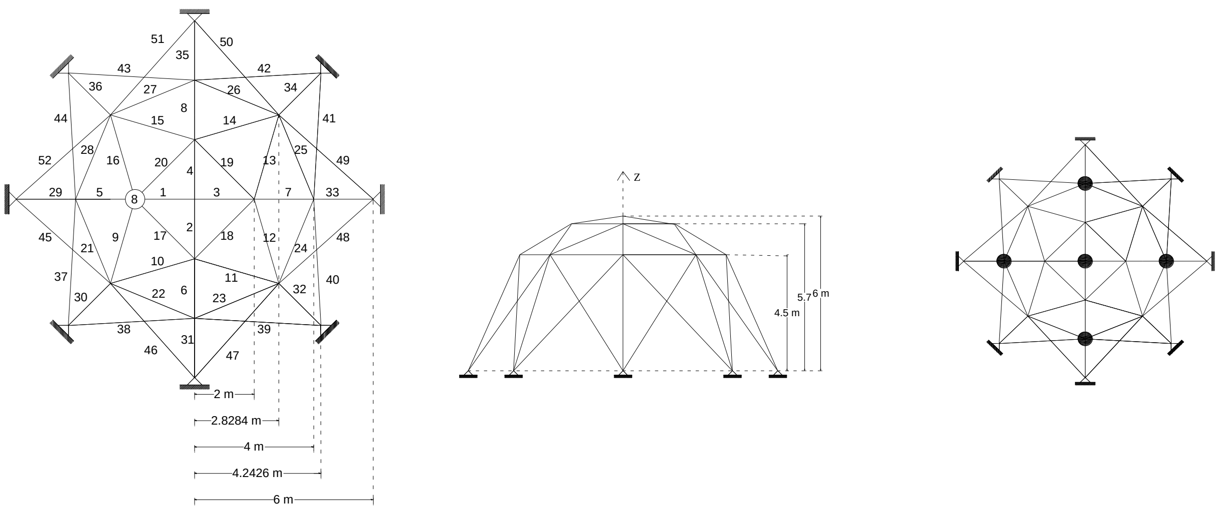


Fig. 10 The amount of search and missing space (%)



(a) Element numbers (b) Front view (c) Pattern of sensors

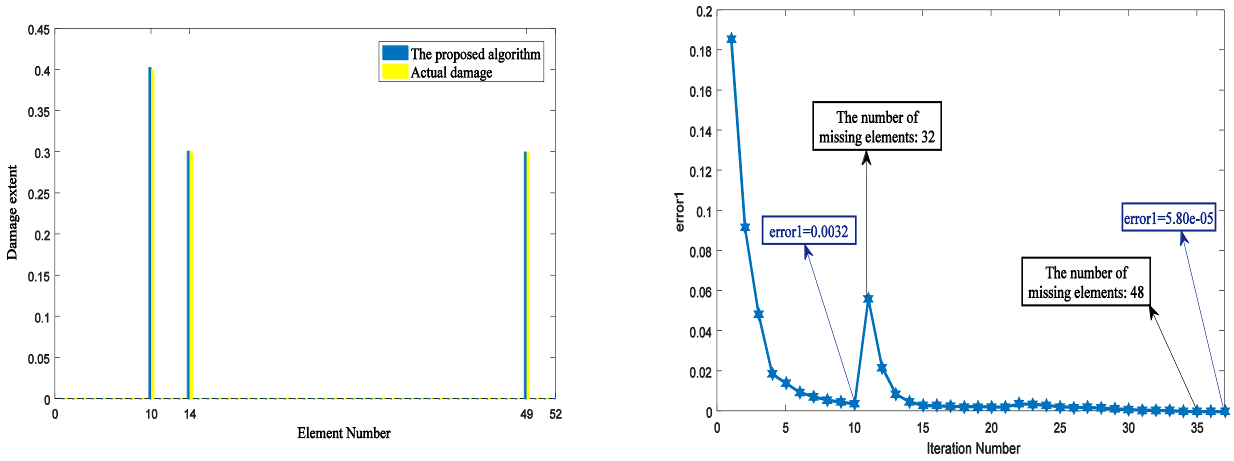
Fig. 11 52-bar dome-like space truss (Kaveh and Zolghadr 2011)

5.2 Space truss

In this section, a 52-space truss with different damage scenarios is assessed. For this purpose, a limited number of sensors and noisy measurements are applied. In Figs. 11(a) and 11(b), the finite element model of this structure is depicted. It includes 21 nodes and 39 degrees of freedom. Besides, the sensors installed on this truss are shown in Fig. 11(c). In specified places, the acceleration is measured in x, y and z directions. The cross-sectional area, elasticity modulus and material density of the structural elements are 0.0002 m^2, 210 Gpa and 7800 kg/m^3, respectively (Kaveh and Zolghadr 2011). Note that, only the sensors are arranged based on Naseralavi 2016. Two different damage scenarios and two noise levels are investigated. They are presented in Table 2. In the first damage scenario, the elements are considerably damaged, and in the second scenario, damage severity is restricted to 20%. Additionally, an impulsive load with the maximum value of 320.4 kN and duration of 0.005 sec is used. It is applied to node 8 in z-direction.

Table 2 Damage scenarios

Scenario	Element no.	Damage extent (%)	Noise (%)
	10	40	
1	14	30	5 %
	49	30	
	4	15	
2	9	20	6 %
	23	20	
	50	10	



(a) (b)

Fig. 12 The results for scenario 1: (a) damage detection results and (b) the amount of norm of residual

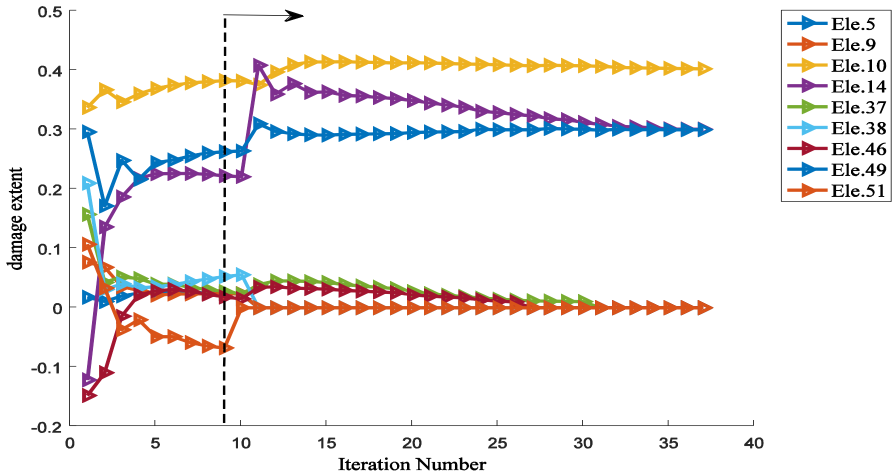


Fig. 13 iteration results of damage detection for scenario 1

In Fig. 12(a), the results of damage detection of the space truss for scenario 1 is presented. Findings prove the efficiency and stability against noisy measurement**s** of the authors' tactic. In both damage scenarios, the suggested scheme successfully locates and estimates the severity of damage.

Based on Eq. (22), the convergence curve is obtained and presented in Fig. 12(b). After iteration 10, the reduction process of the search space begins. It can be seen from Fig. 12(b) that at the end of damage detection process 48 members are omitted from the search space. In iteration 11, after reducing the search space, the convergence cannot be achieved. To solve this problem, the solution procedure moves to a point whose search space includes 3 members. In this way, the convergence can be gained.

For several members, the process of finding the damage severity is presented in Fig. 13. It is clear that by executing the procedure of reducing the search space, the estimated severities of some elements become more accurate. For instance, the severity of damage of member 14 is firstly 22%. nevertheless, after implementing the search space reduction, this value is changed into 30% which is the actual magnitude of damage. For the first damage scenario of the truss space, the search space division into search subspaces and omitted ones are conducted, based on Fig. 14. Note that; 10 iterations are required for reducing the search space. After iteration 10, the search space is significantly decreased. Consequently, the accuracy is increased, and the required time for solving the system of equations is reduced. Finally, the missing subspace includes 48 members. Accordingly, the damage detection method correctly identified the location of the damaged members.

For the second damage scenario of the space truss, the corresponding results of damage detection are presented in Figs. 15(a) and 15(b) and 16. Based on Fig. 15(a), it is obvious that the proposed approach is successful in locating and estimating the severity of damage. As can be seen from Fig. 15(a), After reducing the search space, the convergence path of the numerical method tends zero, and the final search space includes 4 members.

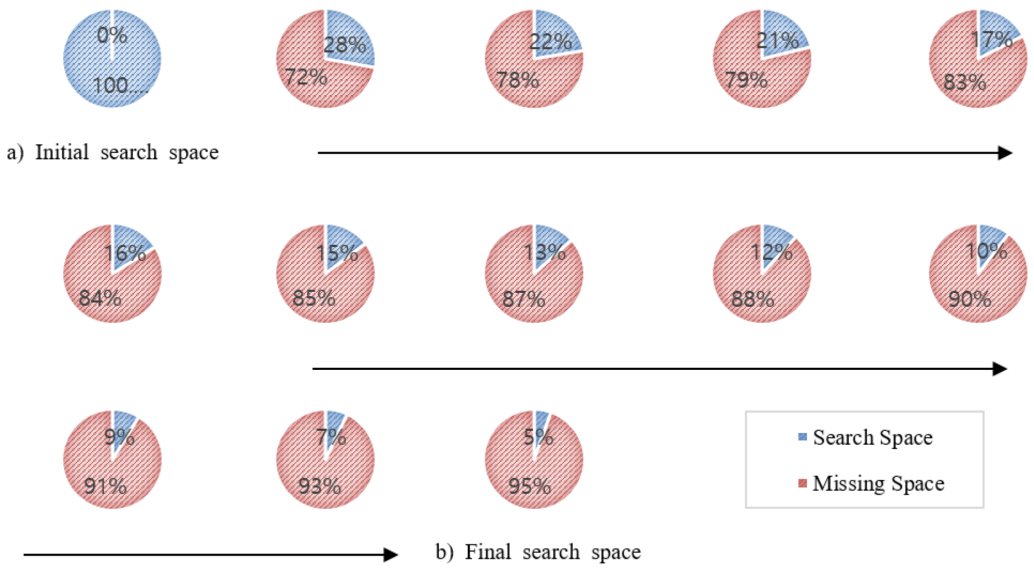
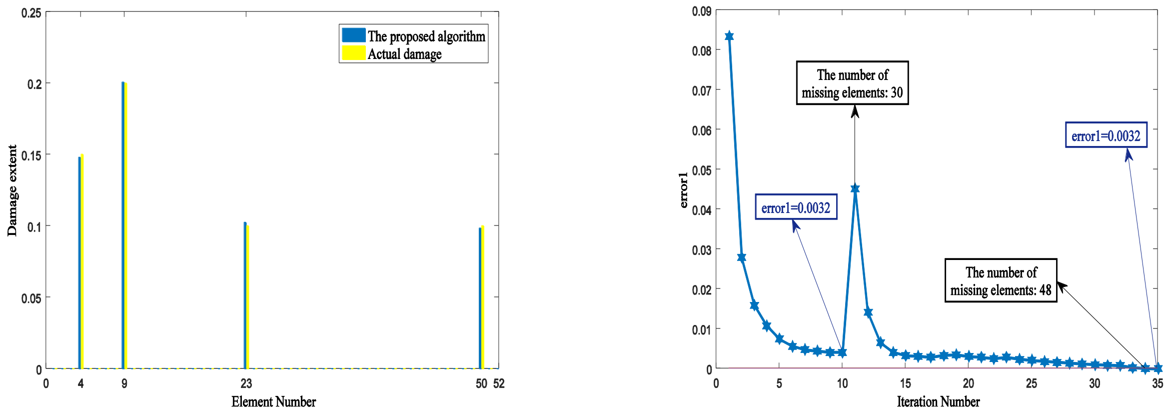


Fig. 14 The amount of search and missing space (%) for scenario 1



(a) (b)

Fig. 15 The results for scenario 2: (a) damage detection results and (b) the amount of norm of residual

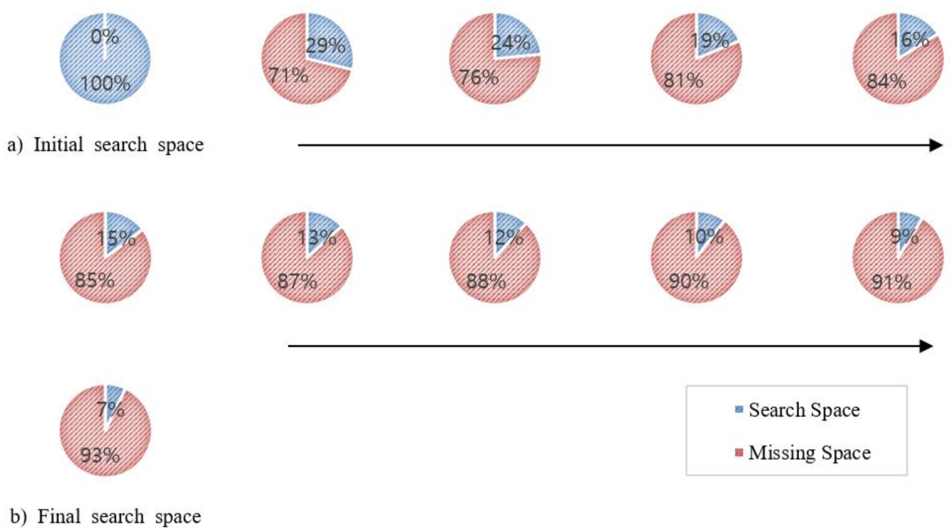


Fig. 16 The amount of search and missing space (%) for scenario 2

As Fig. 16 reveals, the reduction of search space occurs in 11 steps, and the last missing space consists of 48 members. Based on the findings of these two numerical examples, it is observed that the capability of the damage detection process using time history responses, such as nodal accelerations, is dependent on the location and magnitude of the applied load. Also, the number of sensors and their arrangement are important factors. Due to the fact that the number of sensors utilized in the second example is more than that of the first one, the damage detection method leads to more accurate results. It is worthwhile to highlight that the suggested scheme is suitable for solving the nonlinear large system of equations. The preconditioning method and the procedure of reducing the search space stabilize the authors' damage detection algorithm against noisy measurements.

6. Conclusions

This paper deals with proposing a new method for the damage detection problems. This technique is developed by integration of recent methods designed to improve accuracy and save computational costs in solving equations. The method takes advantage of the iterative BCG method, a proper preconditioner matrix obtained by approximately calculating the pseudo-inverse of the sensitivity matrix and a procedure of reducing the search space. The BCG technique was applied due to the asymmetry of the sensitivity matrix of the system of damage detection equations. Moreover, for providing the numerical stability against the input errors due to noisy measurements, the preconditioner was utilized. In addition, a new process for reducing the search space was presented based on the characteristics of the damage vector. The authors' method employs iterative strategies instead of direct ones. In other words, the direct inversion of matrix and calculation of the eigenvalues are not required. As a result, this approach is applicable to the structures including a large number of elements.

To demonstrate the performance and accuracy of the presented method, two numerical examples were deployed. To measure the nodal accelerations, a limited number of sensors was applied. Findings proved the high accuracy and stability of the presented scheme for damage identification of the numerical examples in the presence of noisy data. Furthermore, the proposed preconditioner matrix decreased the condition number of the coefficient matrix of system. As a result, the ill-conditioned problem was upgraded to the well-conditioned problem. The convergence diagrams revealed that executing the search space reduction led to a higher convergence rate. Besides, the estimated magnitudes of the damaged elements became more accurate and the vast majority of elements, which were wrongly recognized as damaged elements, were eliminated from the search space.

Finally, based on the discussed results, it can be concluded: (1) the method is capable of estimating the sites and severities of damages with high precision, (2) the limited number of sensors does not have a considerable influence on the damage detection process using proposed method, (3) using an appropriate preconditioner matrix along with the reduction of search space stabilizes the damage detection process against the noisy data.

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