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Variable properties thermopiezoelectric problem under fractional thermoelasticity

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Abstract. The dynamic response of a finite length thermo-piezoelectric rod with variable material properties is investigated in the context of the fractional order theory of thermoelasticity. The rod is subjected to a moving heat source and fixed at both ends. The governing equations are formulated and then solved by means of Laplace transform together with its numerical inversion. The results of the non-dimensional temperature, displacement and stress in the rod are obtained and illustrated graphically. Meanwhile, the effects of the fractional order parameter, the velocity of heat source and the variable material properties on the variations of the considered variables are presented, and the results show that they significantly influence the variations of the considered variables.

Keywords: fractional order theory of thermoelasticity; thermo-piezoelectric coupling; Laplace transform; moving heat source; variable properties

1. Introduction

The classical coupled thermoelasticity theory predicts an infinite speed for heat propagating in elastic medium, which is not consistent with physical observations. To overcome such shortcoming, Lord and Shulman (L-S) (1967) and Green and Lindsay (G-L) (1972) introduced the generalized thermoelastic theories respectively. In L-S theory, a new wave-type heat conduction law was postulated to replace the classical Fourier's law by introducing the heat flux rate and one thermal relaxation time. In G-L theory, both the energy equation and the Duhamel-Neumann relation were modified by introducing two relaxation times, and the heat conduction equation was also modified by introducing the temperature-rate term. In both theories, the governing equations are of hyperbolic type, which can describe the so-called second sound effect, i.e., heat propagates in medium with a finite speed.

Nowadays, many attentions have been devoted to studying the applications of piezoelectric materials. The counterparts of our problem have been considered in the context of the generalized thermoelastic theories by semi-analytical or numerical methods (He *et al*. 2002, Karamany and Ezzat 2005, He *et al*. 2007). Farzad and Mohsen (2017) studied the nonlocal thermo-electro-mechanical vibration problem of smart curved FG piezoelectric Timoshenko nanobeam. Xiong and Tian (2017) researched the transient thermo-piezo-elastic responses of a functionally graded piezoelectric plate under thermal shock. Generally speaking, material properties, such as the modulus of elasticity, the coefficient of thermal expansion and the thermal conductivity etc., would vary with temperature, which in turn influence the thermoelastic coupling behaviors. To explore the effect of temperature-dependent properties on thermoelastic behaviors and extend the applicability of the solutions to certain range of temperature, many contributions have done by (Othman and Song 2008, Othman and Kumar 2009, Othman and Lotfy 2009, Xiong and Tian 2011) based on the generalized thermoelasticity respectively.

Ever since the first application of fractional calculus to solving an integral equation by Abel, fractional calculus has been successfully used to modify many existing models in various fields, especially in the field of heat conduction, diffusion, viscoelasticity, mechanics of solids. In the application of fractional calculus, Povstenko (2005, 2009, 2011) made a review of thermo-elasticity that uses fractional heat conduction equation and investigated new models by employing fractional derivative. Recently, Youssef (2010) and Youssef and Lehaibi (2010a) formulated the theory of fractional order generalized thermoelasticity by introducing the Riemann-Liouville fractional integral operator into the generalized heat conduction equation. Subsequently, this theory was applied to investigating a two-dimensional thermal shock problem by Youssef (2012) as well as a half-space problem by Youssef and Lehaibi (2010b). Very recently, a completely new model on fractional order generalized thermoelasticity was introduced by Sherief *et al.* (2010). By employing this theory, Shweta and Santwana (2011) solved an elastic half space problem with Laplace transform and state-space method.

So far, there are few works involving the dynamic response for thermo-piezoelectric problems with variable

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material properties in the context of fractional order theory of thermoelasticity. In present work, the dynamic response of a thermo-piezoelectric problem with variable properties and subjected to a moving heat source is investigated in the context of the fractional order theory of thermoelasticity.

2. Basic equations

In the absence of body force and free charge, the piezoelectric- thermoelastic governing equations for linear thermo-piezoelectric media are as follows

Motion equation

$$\sigma_{ij,j} = \rho \ddot{u}_i \tag{1}$$

Energy equation

$$\rho\left(\dot{S}T_0 - Q\right) + q_{i,i} = 0 \tag{2}$$

Gauss equation and electric field relation

$$D_{i,i} = 0, \quad E_i = -\phi_{,i} \tag{3}$$

Strain-displacement relations

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{4}$$

Constitutive equations

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - h_{ijk}E_k - \lambda_{ij}\theta \tag{5a}$$

$$D_i = h_{ijk}\varepsilon_{jk} + \tau_{ij}E_j + p_i\theta \tag{5b}$$

$$\rho S = \frac{\rho C_E}{T_0}\theta + \lambda_{ij}\varepsilon_{ij} + p_i E_i \tag{5c}$$

The fractional order heat conduction equation advocated by Sherief *et al.* (2010)

$$q_i + \tau\frac{\partial^\alpha}{\partial t^\alpha}q_i = -\kappa_{ij}\theta_{,j} \tag{6}$$

In the above equations, a comma followed by a suffix denotes material derivative and a superposed dot denotes the derivative with respect to time. u_i are the components of displacement vector, ε_{ij} the components of strain tensor, D_i the components of electric displacement, E_i the components of electric field vector, ρ mass density, S entropy, q_i the components of heat flux vector, Q strength of the applied heat source per unit mass, c_{ijkl} elastic constants, h_{ijk} piezoelectric constants, λ_{ij} thermal modulus, τ_{ij} dielectric constants, p_i pyroelectric constant, C_E specific heat at constant deformation, ϕ electric potential function, κ_{ij} the coefficients of thermal conductivity, $\theta\left(\theta = T - T_0\right)$ temperature increment, T_0 initial reference temperature, T absolute temperature, t time, τ thermal relaxation time, α fractional order parameter such that $0 < \alpha \le 1$.

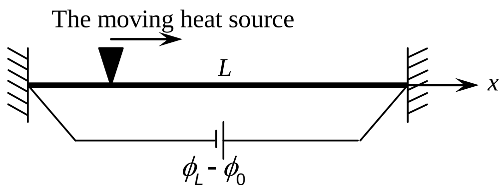


Fig.1 The schematic of the thermo-piezoelectric problem

We consider the dynamic response of a thermo-piezoelectric rod with variable properties and finite length L. The rod is fixed at both ends and subjected to a moving heat source (Fig. 1). The problem can be treated as one-dimensional and the one-dimensional coordinate system is assumed to be aligned along the *x*-axis. All considered variables are thus only functions of *x* and *t*, and the only remaining displacement component is $u_x = u\left(x,t\right)$.

Moreover, the only non-vanishing components of heat flux, stress and electric displacement are also in *x*-direction, and all the material derivatives are zero except that with respect to *x*. Thus, the above equations are reduced to

$$\sigma_{xx,x} = \rho\ddot{u} \tag{7}$$

$$\rho\left(\dot{S}T_0 - Q\right) + q_{x,x} = 0 \tag{8}$$

$$D_{x,x} = 0, \quad E_x = -\phi_{,x} \tag{9}$$

$$\varepsilon_{xx} = u_{,x} \tag{10}$$

$$\sigma_{xx} = c_{11}\varepsilon_{xx} - h_{11}E_x - \lambda_{11}\theta \tag{11a}$$

$$D_x = h_{11}\varepsilon_{xx} + \tau_{11}E_x + p_1\theta \tag{11b}$$

$$\rho S = \rho C_E\theta/T_0 + \lambda_{11}\varepsilon_{xx} + p_1E_x \tag{11c}$$

$$q_x + \tau\frac{\partial^\alpha}{\partial t^\alpha}q_x = -\kappa_{11}\theta_{,x} \tag{12}$$

Substituting Eqs. (11(a)) into (7), (11(b)) into (9), (11(c)) and (12) into (8) respectively, we get

$$c_{11}\frac{\partial^2 u}{\partial x^2} + h_{11}\frac{\partial^2\phi}{\partial x^2} - \lambda_{11}\frac{\partial\theta}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \tag{13}$$

$$h_{11}\frac{\partial^2 u}{\partial x^2} - \tau_{11}\frac{\partial^2\phi}{\partial x^2} + p_1\frac{\partial\theta}{\partial x} = 0 \tag{14}$$

$$\kappa_{11}\frac{\partial^2\theta}{\partial x^2} + \left(1 + \tau\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\rho Q - \rho C_E\frac{\partial\theta}{\partial t} - \lambda_{11}T_0\frac{\partial^2 u}{\partial x\partial t} + p_1T_0\frac{\partial^2\phi}{\partial x\partial t}\right) = 0 \tag{15}$$

where $c_{11}=(\lambda+2\mu),\lambda_{11}=(3\lambda+2\mu)\alpha_t$, α_t is the coefficient of the linear thermal expansion, λ and μ are the Lame's constants.

To consider the temperature-dependence of material properties, we assume that

$$\lambda=\lambda_0 f(\theta),\ \mu=\mu_0 f(\theta),\ \kappa_{11}=\kappa_0 f(\theta) \tag{16}$$

where λ_0 and μ_0 are constants, $f(\theta)$ is a given non-dimensional function of temperature. In case of temperature-independent properties, $f(\theta)\equiv 1$.

Rishin, Lyashenko, Akinin and Nadezhdin (1973) investigated the relationship between modulus of elasticity of several sprayed coatings and temperature, and they reported the modulus of elasticity decreases monotonically with the increasing of temperature. To linearize the governing partial differential equations of the problem, taking into account the condition $|\theta|/T_0 << 1$, we assume that $f(\theta)$ takes the following form without loss of generality

$$f(\theta)\approx 1-\chi T_0 \tag{17}$$

where χ is an empirical material constant.

For convenience, the following non-dimensional quantities are introduced

$$\begin{aligned}&x^*=c_0\eta_0 x,\ u^*=c_0\eta_0 u,\ L^*=Lc_0\eta_0,\ t^*=c_0^2\eta_0 t,\\&\tau^*=c_0^2\eta_0\tau,\ \theta^*=\frac{\theta}{T_0},\ \sigma_{xx}^*=\frac{\sigma_{xx}}{c_{11}},\ D_x^*=\frac{D_x}{h_{11}},\\&Q^*=\frac{Q}{\kappa_{11}T_0c_0^2\eta_0^2},\ c_0^2=\frac{c_{11}}{\rho},\ \eta_0=\frac{\rho C_E}{\kappa_{11}},\ \phi^*=\frac{\tau_{11}}{h_{11}L}\phi\end{aligned} \tag{18}$$

In terms of these non-dimensional quantities, Eqs. (13)-(15) take the forms respectively (dropping the asterisks for convenience)

$$\frac{\partial^2 u}{\partial x^2}+\beta A_1\frac{\partial^2\phi}{\partial x^2}-A_2\frac{\partial\theta}{\partial x}=\beta\frac{\partial^2 u}{\partial t^2} \tag{19}$$

$$\frac{\partial^2 u}{\partial x^2}-A_3\frac{\partial^2\phi}{\partial x^2}+B_1\frac{\partial\theta}{\partial x}=0 \tag{20}$$

$$\frac{\partial^2\theta}{\partial x^2}+\left(1+\tau\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\rho Q-\frac{\partial\theta}{\partial t}-B_2\frac{\partial^2 u}{\partial x\partial t}+\beta B_3\frac{\partial^2\phi}{\partial x\partial t}\right)=0 \tag{21}$$

where

$$\begin{aligned}&A_1=\frac{c_0\eta_0 Lh_{11}^2}{\tau_{11}c_{11}},\ A_2=\frac{\lambda_{11}T_0}{c_{11}},\ A_3=c_0\eta_0 L,\\&B_1=\frac{p_1T_0}{h_{11}},\ B_2=\frac{\lambda_{11}}{\kappa_0\eta_0},\ B_3=\frac{p_1c_0Lh_{11}}{\kappa_0\tau_{11}},\ \beta=\frac{1}{1-\chi T_0}\end{aligned} \tag{22}$$

The initial conditions are assumed as

$$\begin{aligned}&u(x,0)=\dot u(x,0)=0,\\&\theta(x,0)=\dot\theta(x,0)=0,\\&\phi(x,0)=\dot\phi(x,0)=0.\end{aligned} \tag{23}$$

The boundary conditions are given as

$$\begin{aligned}&u(0,t)=u(L,t)=0,\\&\phi(0,t)=\phi(L,t)=0,\\&\frac{\partial\theta(0,t)}{\partial x}=\frac{\partial\theta(L,t)}{\partial x}=0\end{aligned} \tag{24}$$

The piezoelectric rod is subjected to a moving heat source along the positive direction of *x*-axis with a constant velocity υ. The non-dimensional heat source has the form

$$Q=Q_0\delta(x-\upsilon t) \tag{25}$$

where Q_0 is constant and δ is the delta function. υ is the non-dimensional velocity of the heat source.

3. Solutions in Laplace domain

By applying the Laplace transform

$$L[f(t)]=\bar f(s)=\int_0^\infty e^{-st}f(t)\mathrm{d}t,\quad \mathrm{Re}(s)>0 \tag{26}$$

to Eqs. (19)-(21) together with Eq. (23), we can obtain

$$\mathscr{D}^2\bar u-\beta s^2\bar u+\beta A_1\mathscr{D}^2\bar\phi-A_2\mathscr{D}\bar\theta=0 \tag{27}$$

$$\mathscr{D}^2\bar u-A_3\mathscr{D}^2\bar\phi+B_1\mathscr{D}\bar\theta=0 \tag{28}$$

$$\begin{aligned}&\left[\mathscr{D}^2-s(1+\tau s^\alpha)\right]\bar\theta-B_2\ s(1+\tau s^\alpha)\mathscr{D}\bar u\\&+\beta B_3\ s(1+\tau s^\alpha)\mathscr{D}\bar\phi=-(1+\tau s^\alpha)\vartheta e^{-\frac{s}{\upsilon}x}\end{aligned} \tag{29}$$

where $\mathscr{D}=\mathrm{d}/\mathrm{dx},\vartheta=\rho Q_0/\upsilon$.

Eliminating $\bar\theta$ and $\bar\phi$ from Eqs. (27)-(29), we obtain the equation satisfied by $\bar u$

$$\mathscr{D}^4\bar u+\frac{m_2}{m_1}\mathscr{D}^2\bar u+\frac{m_3}{m_1}\bar u=\frac{m}{m_1}e^{-\frac{s}{\upsilon}x} \tag{30}$$

where

$$\begin{aligned}&m_1=\frac{-B_1(\beta A_1+A_3)}{(\beta A_1B_1-A_2A_3)s(1+\tau s^\alpha)},m_3=\frac{(B_1\beta B_3-A_3)\beta B_1s^2}{(\beta A_1B_1-A_2A_3)},\\&m_2=1-\frac{B_1^2\beta(A_1B_2-B_3)+A_3B_1(1-A_2B_2)+A_2(A_3-B_1\beta B_3)}{(\beta A_1B_1-A_2A_3)}\\&\quad-\frac{\beta B_1s(2\beta A_1B_1-A_2A_3)}{A_2(\beta A_1B_1-A_2A_3)(1+\tau s^\alpha)},\ m=\frac{\vartheta B_1}{\upsilon}.\end{aligned} \tag{31}$$

The general solution including particular solution for $\bar u$ can be expressed as

$$\bar u=C_1e^{\lambda_1x}+C_2e^{\lambda_2x}+C_3e^{\lambda_3x}+C_4e^{\lambda_4x}+Ke^{-\frac{s}{\upsilon}x} \tag{32}$$

where C_1 , C_2 , C_3 and C_4 are parameters to be determined from the boundary conditions and K is

$$K = m\Big/\left[m_1 \left(s/\upsilon \right)^4 + m_2 \left(s/\upsilon \right)^2 + m_3 \right] \tag{33}$$

$\lambda_1, \lambda_2, \lambda_3$ and λ_4 are the roots of the characteristic equation

$$\lambda^4 + \frac{m_2}{m_1}\lambda^2 + \frac{m_3}{m_1} = 0 \tag{34}$$

which are given by

$$\lambda_1 = -\lambda_2 = \sqrt{\frac{-m_2 + \sqrt{m_2^2 - 4m_1m_3}}{2m_1}},$$
$$\lambda_3 = -\lambda_4 = \sqrt{\frac{-m_2 - \sqrt{m_2^2 - 4m_1m_3}}{2m_1}}$$

Eliminating $\bar{\theta}$ from Eqs. (27) and (28), we get

$$\left(\vartheta + A_2 \right)\frac{\partial^2 \bar{u}}{\partial x^2} - \beta B_1 s^2 \bar{u} + \left(\beta A_1 B_1 - A_2 A_3 \right)\frac{\partial^2 \bar{\phi}}{\partial x^2} = 0 \tag{35}$$

Substituting Eq. (32) into Eq. (35), we can obtain the solution for $\bar{\phi}$

$$\begin{aligned} \bar{\phi} &= a_1C_1\mathrm{e}^{\lambda_1 x} + a_2C_2\mathrm{e}^{\lambda_2 x} + a_3C_3\mathrm{e}^{\lambda_3 x} + a_4C_4\mathrm{e}^{\lambda_4 x} + aK\mathrm{e}^{-\frac{s}{\upsilon}x} \\ &+ D_1x + D_2 \end{aligned} \tag{36}$$

Substituting Eqs. (32) and (36) into Eq. (28), we obtain the solution for $\bar{\theta}$

$$\begin{aligned} \bar{\theta} &= b_1C_1\mathrm{e}^{\lambda_1 x} + b_2C_2\mathrm{e}^{\lambda_2 x} + b_3C_3\mathrm{e}^{\lambda_3 x} + b_4C_4\mathrm{e}^{\lambda_4 x} + bK\mathrm{e}^{-\frac{s}{\upsilon}x} \\ &+ \frac{A_3}{B_1}D_1x + D_3 \end{aligned} \tag{37}$$

where D_1, D_2 and D_3 are undetermined parameters.

Substituting Eqs. (32), (36) and (37) into Eq. (11(a)), we obtain

$$\begin{aligned} \bar{\sigma}_{xx} &= C_1\left(\lambda_1\left(1 + a_1\frac{A_1}{c_0\eta_0} \right) - A_2b_1 \right)\mathrm{e}^{\lambda_1 x} + C_2\left(\lambda_2\left(1 + a_2\frac{A_1}{c_0\eta_0} \right) - A_2b_2 \right)\mathrm{e}^{\lambda_2 x} \\ &+ C_3\left(\lambda_3\left(1 + a_3\frac{A_1}{c_0\eta_0} \right) - A_2b_3 \right)\mathrm{e}^{\lambda_3 x} + C_4\left(\lambda_4\left(1 + a_4\frac{A_1}{c_0\eta_0} \right) - A_2b_4 \right)\mathrm{e}^{\lambda_4 x} \\ &+ K\left(-\frac{s}{\upsilon}\left(1 + a\frac{A_1}{c_0\eta_0} \right) - A_2b \right)\mathrm{e}^{-\frac{s}{\upsilon}x} + D_1\left(\frac{A_1}{c_0\eta_0} - \frac{g_3}{f_1}A_2 \right) - A_2D_3 \end{aligned} \tag{38}$$

where

$$\begin{aligned} a &= \frac{B_1\beta\upsilon^2 - \left(B_1 + A_2 \right)}{B_1\beta A_1 - A_2A_3},\ a_1 = a_2 = \frac{\left[B_1\beta s^2/\lambda_1^2 - \left(B_1 + A_2 \right) \right]}{B_1\beta A_1 - A_2A_3}, \\ a_3 &= a_4 = \frac{\left[B_1\beta s^2/\lambda_3^2 - \left(B_1 + A_2 \right) \right]}{B_1\beta A_1 - A_2A_3},\ \ b = \frac{s\left(1 - aA_3 \right)}{\upsilon B_1}, \\ b_1 &= -b_2 = \frac{\lambda_1\left(A_3a_1 - 1 \right)}{B_1},\ b_3 = -b_4 = \frac{\lambda_3\left(A_3a_2 - 1 \right)}{B_1} \end{aligned} \tag{39}$$

Substituting Eqs. (32), (36) and (37) into Eq. (29), we obtain

$$\frac{\beta B_1B_3 - A_3}{B_1}D_1 - D_3 = 0 \tag{40}$$

In order to determine the parameters $C_i\left(i = 1,2,3,4 \right)$ and $D_j\left(j = 1,2 \right)$, we consider the boundary conditions in Eq. (24) and get

$$C_1 + C_2 + C_3 + C_4 + K = 0 \tag{41a}$$

$$\begin{aligned} &C_1\mathrm{e}^{\lambda_1 Lc_0\eta_0} + C_2\mathrm{e}^{\lambda_2 Lc_0\eta_0} + C_3\mathrm{e}^{\lambda_3 Lc_0\eta_0} + C_4\mathrm{e}^{\lambda_4 Lc_0\eta_0} \\ &+ K\mathrm{e}^{-(s/\upsilon)Lc_0\eta_0} = 0 \end{aligned} \tag{41b}$$

$$a_1C_1 + a_2C_2 + a_3C_3 + a_4C_4 + aK + D_2 = 0 \tag{41c}$$

$$\begin{aligned} &a_1C_1\mathrm{e}^{\lambda_1 Lc_0\eta_0} + a_2C_2\mathrm{e}^{\lambda_2 Lc_0\eta_0} + a_3C_3\mathrm{e}^{\lambda_3 Lc_0\eta_0} + a_4C_4\mathrm{e}^{\lambda_4 Lc_0\eta_0} \\ &+ aK\mathrm{e}^{-(s/\upsilon)Lc_0\eta_0} + D_1Lc_0\eta_0 + D_2 = 0 \end{aligned} \tag{41d}$$

$$\begin{aligned} &b_1C_1\lambda_1 + b_2C_2\lambda_2 + b_3C_3\lambda_3 + b_4C_4\lambda_4 - bK\left(s/\upsilon \right) \\ &+ A_3D_1/B_1 = 0 \end{aligned} \tag{41e}$$

$$\begin{aligned} &b_1C_1\lambda_1\mathrm{e}^{\lambda_1 Lc_0\eta_0} + b_2C_2\lambda_2\mathrm{e}^{\lambda_2 Lc_0\eta_0} + b_3C_3\lambda_3\mathrm{e}^{\lambda_3 Lc_0\eta_0} + b_4C_4\lambda_4\mathrm{e}^{\lambda_4 Lc_0\eta_0} \\ &- bK\left(s/\upsilon \right)\mathrm{e}^{-(s/\upsilon)Lc_0\eta_0} + A_3D_1/B_1 = 0 \end{aligned} \tag{41f}$$

The undetermined parameters C_i and D_j are obtained by solving the system of algebraic Eqs. from (41(a)) to (41(f)) in Matlab. After getting D_1, D_3 can be obtained from Eq. (40). Due to the complexity of the expressions of the obtained parameters, they are not presented here.

4. Numerical inversion of the Laplace transform

In order to determine the distributions of the non-dimensional temperature, displacement, stress and electric potential in the thermo-piezoelectric rod, $\bar{\theta}$, $\bar{u}$, $\bar{\sigma}_{xx}$ and $\bar{\phi}$ need to be inverted from Laplace domain into time domain. Due to the complexity of the solutions obtained in Laplace domain, it is not feasible to invert them analytically. Alternatively, a numerical technique that is the Riemann-sum approximation method could be employed to obtain numerical results. In this method, any function $\tilde{f}\left(x,s \right)$ in Laplace domain can be inverted into the time domain Durbin (1973) by the following formula

$$f\left(x,t \right) = \frac{e^{\beta_0 t}}{t}\left[\frac{1}{2}\tilde{f}\left(x,\beta_0 \right) + \mathrm{Re}\sum_{n=1}^{N}\tilde{f}\left(x,\beta_0 + \frac{in\pi}{t} \right)\left(-1 \right)^n \right] \tag{42}$$

where Re is the real part and i is the imaginary number unit. For faster convergence, numerous numerical experiments have shown that the value of β_0 satisfies the relation $\beta_0 t \approx 4.7$ (Honig and Hirdes 1984).

5. Numerical results and discussions

In view of the Riemann-sum approximation given in Eq. (42), numerical Laplace inversion is implemented to obtain the variations of the considered variables. In the calculation, the material properties of the thermo-piezoelectric rod are given as (He,Tian and Shen 2002)

$$\lambda_0 = 7.76\times10^{10}\,\mathrm{N\cdot m^{-2}},\quad \mu_0 = 3.86\times10^{10}\,\mathrm{N\cdot m^{-2}},\quad \rho = 7600\,\mathrm{Kg\cdot m^{-3}},$$
$$C_E = 420\,\mathrm{J\cdot Kg^{-1}\cdot K^{-1}},\quad h_{11} = 0.2\,\mathrm{C\cdot m^{-2}},\quad \tau_{11} = 0.392\times10^{-10}\,\mathrm{F\cdot m^{-1}},$$
$$p_1 = 4\times10^{-4}\,\mathrm{C\cdot K^{-1}\cdot m^{-2}},\ \kappa_0 = 1.4\,\mathrm{W\cdot K^{-1}\cdot m^{-1}},\ \alpha_t = 1.78\times10^{-5}\,\mathrm{K^{-1}}.$$

The other constants are taken as

$$Q_0 = 10,\quad T_0 = 293,\quad \tau = 0.05,\quad L = 10$$

In calculation, three cases are considered at time $t = 0.1$. The first case is to investigate how the non-dimensional temperature, displacement, stress and electric potential vary with different fractional order parameter when the velocity of heat source and the material properties remain constant.

The second case is to investigate how the considered variables vary with the material properties when the fractional order parameter and the velocity of heat source remain constant. The last case is to investigate how the considered variables vary with the velocity of heat source when the fractional order parameter and the material properties remain constant.

In the first case, the values of fractional order parameter are set as $\alpha = 0.25$, 0.5, 0.75 and 1.0 respectively while $\upsilon = 1, \beta = 1.2$. In the second case, the material properties are set as $\beta = 0.5$, 1.0, and 1.5 respectively while the constants $\upsilon = 1, \alpha = 0.5$. In the third case, the velocities of heat source are set as $\upsilon = 1$, 2 and 3 respectively while $\beta = 1.2, \alpha = 0.5$. The obtained results are illustrated in Figs. 2-13.

Figs. 2-4 show the distributions of the non-dimensional temperature. As shown in Fig. 2, the peak value of temperature increases as the fractional order parameter α increases under the same β and υ. As observed from Fig. 3, the peak value of the temperature increases with the increase of β under the same α and υ. From the definition of β in Eqs. (22) and (17), it can be deduced that the material properties decreases as β increases, which means that the peak value of the temperature decreases with the increase of the material properties. As seen from Fig. 4, the peak value of temperature decreases as the moving heat source velocity υ increases under the same α and β. The effect of fractional order parameter, temperature-dependent material properties and heat source velocity on temperature is significant. For each curve of the non-dimensional temperature, the value of temperature varies steadily at the first stage, then increases and reaches the peak, after that, decreases.

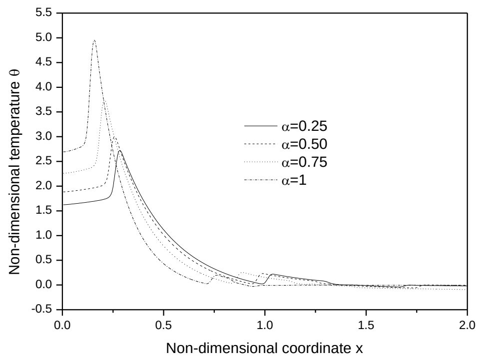


Fig. 2 Distributions of temperature with different α when $\upsilon = 1$ and $\beta = 1.2$

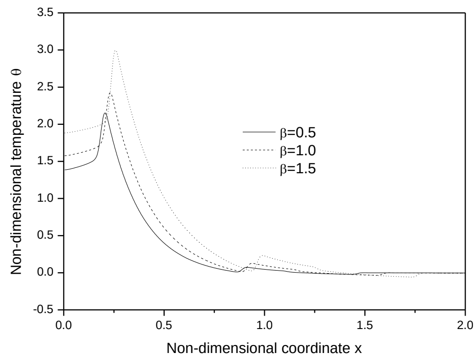


Fig. 3 Distributions of temperature with different β when $\upsilon = 1$ and $\alpha = 0.5$

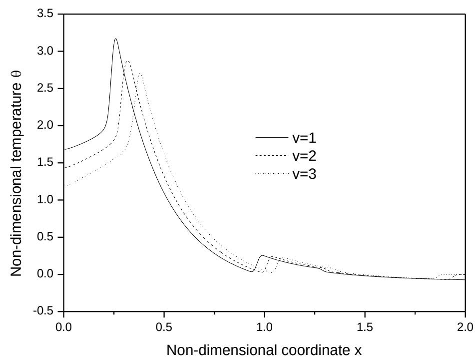


Fig. 4 Distributions of temperature with different υ when $\beta = 1.2$ and $\alpha = 0.5$

Figs. 5-7 show the distributions of the non-dimensional stress. As seen from Fig. 5, the peak value of stress increases with the increase of α under the same β and υ. As shown in Fig.6, the peak value of stress increases with the increase of β under the same α and υ. As observed from Fig.7, the peak value of stress decreases with the increase of the moving heat source velocity υ under the same α and β.

For each curve of the non-dimensional stress, the value of stress first varies steadily and reaches the peak and then goes down.

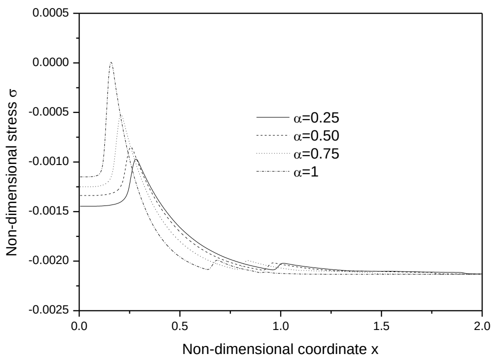


Fig. 5 Distributions of stress with different α when $\upsilon=1$ and $\beta=1.2$

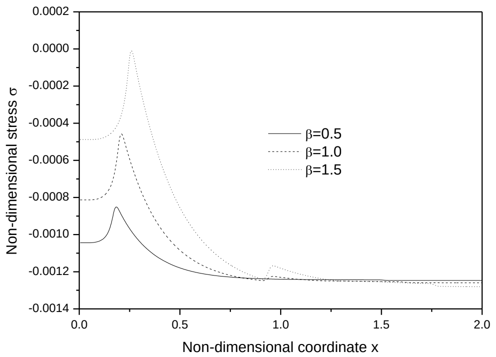


Fig. 6 Distributions of stress with different β when $\upsilon=1$ and $\alpha=0.5$

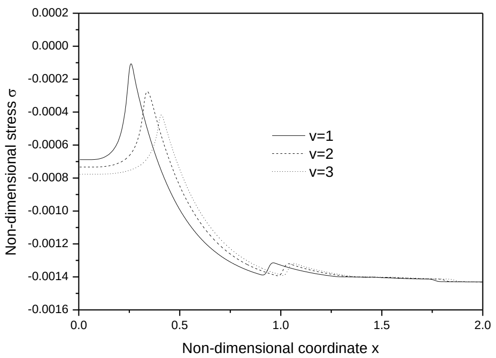
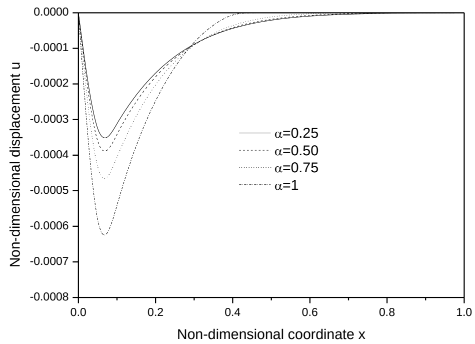


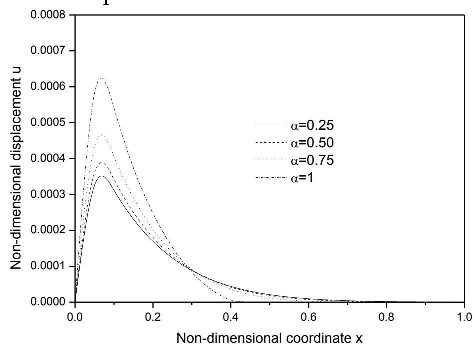
Fig. 7 Distributions of stress with different υ when $\beta=1.2$ and $\alpha=0.5$

Figs. 8-10 show the distributions of the non-dimensional displacement. As observed from Fig. 8, the absolute peak value of displacement increases as the fractional order parameter α increases under the same β and υ. As seen from Fig.9, the absolute peak value of displacement increases with the increase of β under the same α and υ.

As shown in Fig. 10, the absolute peak value of displacement decreases with the increase of the moving heat source velocity υ under the same α and β. For each curve of the non-dimensional displacement, the absolute value of displacement firstly goes up and then goes down.

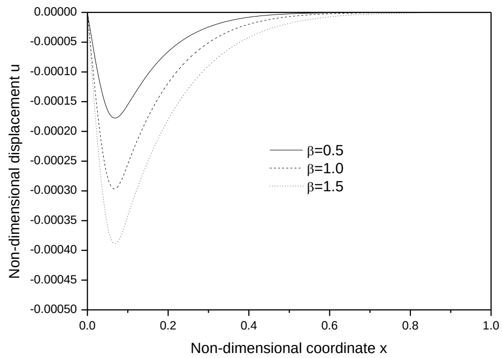


(a) The direction of applied electric field is opposite to the direction of polarization

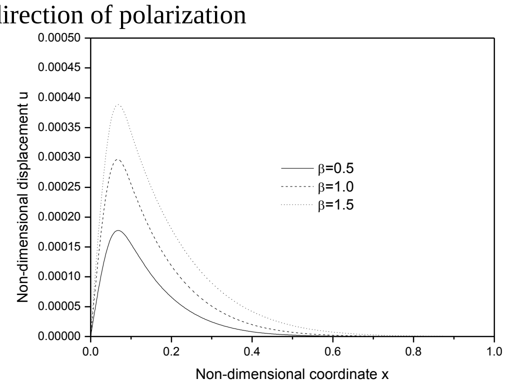


(b) The direction of the applied electric field is the same as the direction of polarization

Fig. 8 Distributions of displacement with different α at $\upsilon=1$ and $\beta=1.2$

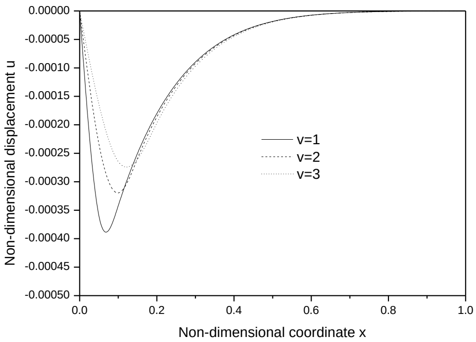


(a) The direction of applied electric field is opposite to the direction of polarization

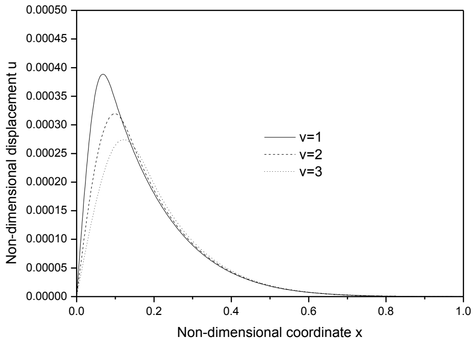


(b) The direction of the applied electric field is the same as the direction of polarization

Fig. 9 Distributions of displacement with different β when $\upsilon=1$ and $\alpha=0.5$

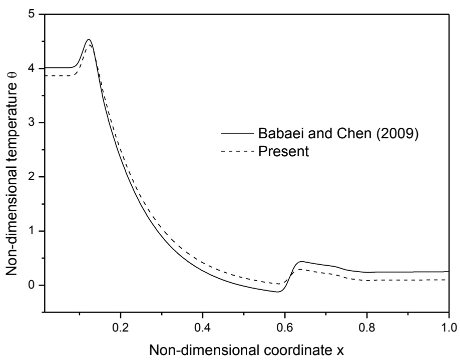


(a) The direction of applied electric field is opposite to the direction of polarization

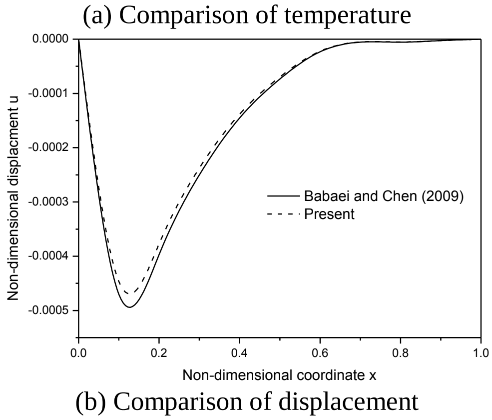


(b) The direction of the applied electric field is the same as the direction of polarization

Fig. 10 Distributions of displacement with different υ when $\beta = 1.2$ and $\alpha = 0.5$



(a) Comparison of temperature



(b) Comparison of displacement

Fig. 11 Comparisons made between present work and the work in reference (Babaei and Chen 2009)

Because the rod is subjected to the thermopiezoelectric effect, when the direction of applied electric field is opposite to the direction of polarization, the displacement in the rod is negative. On the contrary, the displacement in the rod is positive.

To compare the results in present work with others, three representative variables, i.e., the non-dimensional temperature and displacement, are chosen to be compared with those given in reference (Babaei and Chen 2009) in case and . The comparisons are presented in Figs. 11(a) and 11(b).

The comparisons show the results for the chosen variables in both works agree well with each other except some slight deviations maybe caused by the different methods used in both works respectively.

6. Conclusions

The dynamic response of a thermo-piezoelectric rod with variable properties and subjected to a moving heat source is investigated in the context of the fractional order theory of thermoelasticity. This work may be helpful for better understanding of the interactions among temperature field, mechanical field and electric field in smart structures made of piezoelectric ceramics and provide some guidelines in the optimal design of actuators or sensors made of piezoelectric ceramics serving in a thermoelastic environment. From the obtained results, the following conclusions can be arrived at

- The effects of the fractional order parameter, the temperature-dependent material properties and the moving heat source velocity on the considered variables are very significant.
- The peak values of the non-dimensional temperature, displacement, stress and increase with the increase of the fractional order parameter α under the same β and υ .
- The peak values of the non-dimensional temperature, displacement, stress and increase with the increase of β under the same α and υ , which in turn means that the peak values of the considered variables decrease with the increase the temperature-dependent properties.
- The peak values of the non-dimensional temperature, displacement, stress and decrease with the increase of the moving heat source velocity υ under the same α and β .

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