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Optimal placement and tuning of multiple tuned mass dampers for suppressing multi-mode structural response

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Abstract. The optimal design of multiple tuned mass dampers (multiple TMD's) to suppress multi-mode structural response of beams and floor structures was investigated. A new method using a numerical optimizer, which can effectively handle a large number of design variables, was employed to search for both optimal placement and tuning of TMD's for these structures under wide-band loading. The first design problem considered was vibration control of a simple beam using 10 TMD's. The results confirmed that for structures with widely-spaced natural frequencies, multiple TMD's can be adequately designed by treating each structural vibration mode as an equivalent SDOF system. Next, the control of a beam structure with two closely-spaced natural frequencies was investigated. The results showed that the most effective multiple TMD's have their natural frequencies distributed over a range covering the two controlled structural frequencies and have low damping ratios. Moreover, a single TMD can also be made effective in controlling two modes with closely spaced frequencies by a newly identified control mechanism, but the effectiveness can be greatly impaired when the loading position changes. Finally, a realistic problem of a large floor structure with 5 closely spaced frequencies was presented. The acceleration responses at 5 positions on the floor excited by 3 wide-band forces were simultaneously suppressed using 10 TMD's. The obtained multiple TMD's were shown to be very effective and robust.

Keywords: multiple tuned mass dampers; optimal design; robustness design; closely-spaced natural frequencies structures; nonlinear programming.

1. Introduction

Tuned mass dampers (TMD's) have been widely used for suppressing excessive structural vibration. In applications to continuous structures, a single TMD is traditionally designed by approximating the main structure as an equivalent single-degree-of-freedom (SDOF) system. The optimal parameters of the TMD can be found using an optimal design strategy for SDOF structures which has been well established for various types of loading and response of interest (Den Hartog 1956, Warburton 1982, Fujino and Abe 1993). Warburton and Ayrodine in 1980 demonstrated that this approach works

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successfully for structures having widely-spaced natural frequencies and the optimal location of the TMD is at an anti-node of the controlled mode. More recently, Abe and Igusa in 1995 have extended this approach to the design of multiple TMD's with distributed frequencies. For structures with closely-spaced natural frequencies, however, this traditional approach is no longer adequate. Ayrodine and Warburton (1980) showed that as the frequency ratio of adjacent modes of the main structure approaches unity, optimal parameters of TMD's deviate increasingly from those for SDOF structural models. That is, multi-mode structural response should be considered in such a case.

Design of multiple TMD's for structures with closely-spaced natural frequencies has been the subject of several researches. One of the earlier works is of Setareh and Hanson (1992) on the tuning of two TMD's for reducing vibration of a beam having two natural frequencies close to each other. In their study, the two TMD's were designed by assuming the locations at two fixed points on the beam and conducting numerical optimization for the stiffnesses and damping coefficients. Although considering only four design variables, their optimization procedure was not clearly described. Thus, even the obtained TMD's seem to be effective, the procedure may not be readily applicable to general complex structures. Subsequently, Abe and Igusa (1995) attempted to derive analytical expressions for vibration control of structures with m closely spaced frequencies using multiple TMD's. Based on certain simplifying assumptions, a perturbation technique was utilized in their study to get an approximate relation between the structural response and parameters of TMD's. Though closed-form solutions for the response could not be derived for $m > 2$, they were able to prove that at least m TMD's placing at different locations are required to achieve an effective control of the response.

The review so far reveals that a more powerful method for optimal design of multiple TMD's for general structures with many closely spaced frequencies has yet to be developed. Furthermore, it is noted that dynamic properties of a structure with closely spaced frequencies, especially the mode shapes, are quite sensitive to small changes in the structure. While optimal TMD's may be placed at the anti-node(s) of the controlled modes for structures with widely spaced frequencies, the adding of TMD's to a structure with closely spaced frequencies will modify the mode shapes, and it is very hard then to predict where the effective locations of TMD's are. It is therefore crucial to include locations of TMD's as design variables in the optimal design of TMD's for structures with closely spaced frequencies.

When both location and mechanical parameters of multiple TMD's are taken into account, the number of design variables becomes considerably large. To confront such a comprehensive design problem for structures with multiple closely spaced frequencies, numerical optimization is the only choice. A new method has been developed for this purpose by the authors (Hoang and Warnitchai 2005). It uses a numerical optimizer that employs a nonlinear programming technique to search for optimal parameters of TMD's. In the method, the target response to be controlled is formulated as a performance index which can be efficiently evaluated by a well-known algorithm. By that formulation, the gradient of the performance, which provides information about the best searching direction in the variable space, is explicitly computed. This helps to avoid numerical errors and speed up the convergence. The authors demonstrated, through various design problems of TMD's for SDOF structures, that this is a very powerful method by which a large number of design variables can be effectively handled without imposing any restriction before the analysis. Moreover, its framework is highly flexible and can be easily extended to general structures with different combinations of loading conditions and target response quantities.

In the present paper, the optimal design of multiple TMD's to suppress multi-mode response of continuous structures under wide-band excitation is investigated. The developed numerical optimization method is here employed to search for both optimal placement and tuning of TMD's. To validate the method, an optimal design of multiple TMD's for a simply supported beam having widely-spaced natural frequencies is first performed. The method is then used to determine TMD's which minimize total vibration energy of a spring-supported cantilever beam with two closely spaced frequencies. New insight into the control of structures with closely spaced frequencies using TMD's can be gained from this basic design problem. The remainder of the paper examines a badminton floor with multiple closely spaced frequencies. This is a realistic two-dimension structure with frequency ratios of several adjacent modes close to unity. The target is to minimize acceleration response of the floor in order to satisfy human comfort requirements. From the results, it is shown that multiple TMD's can effectively suppress multi-mode response in a robust manner.

2. Formulation of optimal design problem

The theoretical formulation for optimal design of multiple TMD's by using a numerical optimizer was described in detail in Hoang and Warnitchai (2005). The structural response to be suppressed can be displacement, velocity or acceleration at any point on the main structure. Moreover, the theoretical framework is so flexible that it can be easily modified to handle other types of structural response, such as stress or total vibration energy, or even to take the motions of TMD's into account. In this section, the formulation concerning the total vibration energy of continuous structures will be briefly stated.

2.1. Equations of motion

Consider a combined system of a main linear structure and multiple TMD's. The main structure is described by m vibration modes with shape function $\psi_i(x)$, modal mass m_{si}, modal frequency ω_{si} and modal damping ratio ξ_{si}, where $i=1,\cdots, m$. Attached to the structure is a set of n TMD's acting along the structural response direction. The j-th TMD ($j=1, \cdots, n$) has mass m_j, natural frequency ω_j, damping ratio ξ_j and is located at x_j. The equation of motion of the combined system subjected to a random excitation can be derived as

$$\mathbf{M}\ddot{\mathbf{z}}(t) + \mathbf{C}\dot{\mathbf{z}}(t) + \mathbf{K}\mathbf{z}(t) = \mathbf{f}u(t) \tag{1}$$

Here, $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ are mass, stiffness and damping matrices, respectively, with details given in Hoang and Warnitchai (2005); $\mathbf{z}(t)=\{z_{s1}\ z_{s2} \cdots z_{sm}\ z_1\ z_2 \cdots z_n\}^T$ in which the first m components correspond to modal coordinates of the main structure and the remaining are total displacements of n TMD's; $u(t)$ is a stationary zero-mean random force applied to the main structure and $\mathbf{f}$ is an influence vector where its components depend on the loading position x_u.

To facilitate further derivations, Eq. (1) needs to be converted into the state-space form. If Φ denotes a matrix of orthonormal eigenvectors obtained from an eigen-analysis of the undamped form of Eq. (1), that is

$$\Phi^T\mathbf{M}\Phi=\mathbf{I} \quad \text{and} \quad \Phi^T\mathbf{M}\Phi=\Lambda \tag{2}$$

where Λ is a diagonal matrix of eigenvalues, then by applying transformation $\mathbf{z}(t)=\Phi\mathbf{q}(t)$, Eq. (1) can be rewritten in terms of a new set of generalized coordinates $\mathbf{q}(t)$ as

$$\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \Lambda\mathbf{q}(t) \;=\; \Phi^T\mathbf{f}u(t) \tag{3}$$

Note that the damping matrix $\mathbf{D}=\Phi^T\mathbf{C}\Phi$ is usually not diagonal. The first-order state equation can be expressed by

$$\dot{\mathbf{x}}(t) \;=\; \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \tag{4}$$

where $$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix},\;\; \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\Lambda & -\mathbf{D} \end{bmatrix} \;\text{and}\;\; \mathbf{b} = \begin{Bmatrix} \mathbf{0} \\ \Phi^T\mathbf{f} \end{Bmatrix} \tag{5}$$

Considering Eq. (4), with a random excitation input $u(t)$, the random response in terms of the state vector can be described by a covariance matrix $\mathbf{X}(t)=E[\mathbf{x}(t)\mathbf{x}^T(t)]$, where the operator E denotes the expected value. In this case since $u(t)$ is a stationary process, the state covariance matrix is time-invariant and can be obtained by solving a Lyapunov equation. If $u(t)$ is a white-noise input with spectral intensity S_0, $\mathbf{X}$ is the solution of the following Lyapunov equation (Bryson and Ho 1975):

$$\mathbf{AX} + \mathbf{XA}^T + 2\pi S_0\,\mathbf{b}\,\mathbf{b}^T = \mathbf{0} \tag{6}$$

The Lyapunov equation can be efficiently solved by the algorithm of Bartels and Stewart (1972) using Schur matrix decomposition.

2.2. Optimization problem

Given the structure and the excitation, it is required to search for a set of location and mechanical parameters of TMD's that minimizes the structural response. This design problem can be solved by posing an optimization problem in which the target response quantity is indicated by a performance index J and the parameters of TMD's to be optimized are viewed as design variables. If the design variables are collectively represented by a design vector $\mathbf{p}$, then $J=J(\mathbf{p})$. In case of continuous structures, the total vibration energy is a logical performance index to be selected because it represents the entire structural response, not response at a single point. For a stationary random excitation in this study, J is chosen as

$$J \;=\; E[\mathrm{W}(t)] \tag{7}$$

where $\mathrm{W}(t)$ is the total vibration energy of the main structure at an instant time t, given by

$$\mathrm{W}(t) \;=\; \frac{1}{2}\sum_{i=1}^{m}(m_{si}\dot{z}_{si}^2 + k_{si}z_{si}^2) \tag{8}$$

which can be readily converted into the form of state vector $\mathbf{x}(t)$ as

$$\mathrm{W}(t) = \frac{1}{2}\mathbf{x}^T(t)\mathbf{R}\mathbf{x}(t) \tag{9}$$

where $$\mathbf{R} = \begin{bmatrix} \mathbf{S\Phi} & \mathbf{0} \\ \mathbf{0} & \mathbf{S\Phi} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_s \end{bmatrix} \begin{bmatrix} \mathbf{S\Phi} & \mathbf{0} \\ \mathbf{0} & \mathbf{S\Phi} \end{bmatrix}, \mathbf{S} = [\mathbf{I}_{m \times n} \quad \mathbf{0}_{m \times n}],$$

$$\mathbf{K}_s = diag\{k_{s1}, k_{s2}, \ldots, k_{sm}\}, \mathbf{C}_s = diag\{c_{s1}, c_{s2}, \ldots, c_{sm}\},$$

$$k_{si} = m_{si}\omega_{si}^2 \text{ and } c_{si} = 2m_{si}\omega_{si}\xi_{si} \tag{10}$$

Introducing Eq. (9) into Eq. (7) and making use of the trace operation lead to

$$J = \frac{1}{2}tr[\mathbf{RX}] \tag{11}$$

The major advantage of Eq. (11) is that, for any given vector **p**, J is straightforwardly evaluated in an efficient manner because **X**(t) is the solution of the Lyapunov equation (6). Furthermore, its gradient $\nabla J \equiv \partial J / \partial \mathbf{p}$ is also explicitly evaluated in a similar manner where only one additional Lyapunov equation has to be solved and the remaining calculations are merely matrix manipulations. Details of the derivation of the gradient ∇J are given in Hoang and Warnitchai (2005). Accordingly, a computer program to compute J and ∇J can be built and the optimal design problem can be solved by standard nonlinear programming techniques. In this problem, it is noted that there exist some constraints on design variables such as positive definite values for mechanical properties of TMD's and limited range for their locations. Nevertheless, all these constrained design variables can be converted to unconstrained design variables by applying typical transformations with square and sine/cosine functions. By this way, one of the most robust gradient-based unconstrained optimization techniques is employed herein. It is the technique following Davidon-Fletcher-Powell algorithm (Rao 1996).

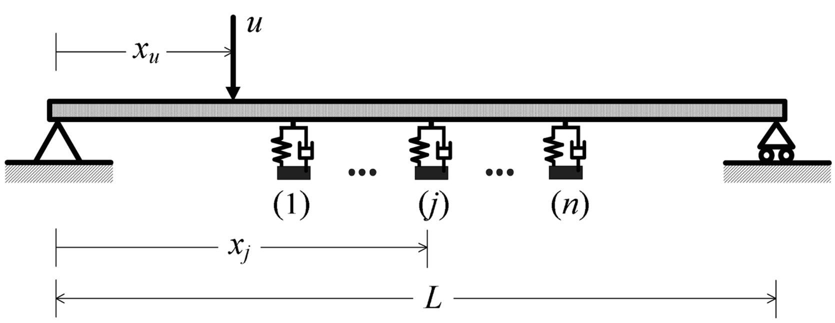


Fig. 1 A simply supported beam with attached multiple TMD's

3. Structures with widely spaced frequencies

This section deals with a simply supported beam which possesses widely-spaced natural frequencies. For such a structure, it is generally believed that TMD's can be adequately designed by treating each vibration mode of the main structure as an equivalent SDOF system (Warburton and Ayrodine 1980, Abe and Igusa 1995). To examine this point, the developed numerical optimization method is here employed to search for TMD's which minimize multi-mode response of the beam. A comparison between the obtained results and those based on the SDOF analogy then can be made.

The simple beam is given in Fig. 1 with length L, mass density ρ, cross-sectional area A and flexural rigidity EI. The natural frequencies of the beam are $\omega_{si} = (i\pi)^2\sqrt{EI/\rho AL^4}$ ($i=1, \ldots, m$). Its modal damping ratios ξ_{si} are all assumed to be 0.5% for simplicity. The beam is subjected to a point wide-band force modeled by white noise at quarter-span ($x_u=L/4$). The target is to control the total vibration energy of the beam, as expressed by Eq. (11). In this case, it can be proved that the first three modes are significant and sufficient to accurately describe the target energy index, and thus m is set to 3. A set of 10 TMD's is to be used where all TMD's are assigned to have equal masses. The total mass of TMD's is chosen to be 1% of the beam's mass. The design variables are then natural frequencies ω_j, damping ratios ξ_j and locations x_j of TMD's, where $j=1, \ldots, 10$.

The numerical optimization begins by setting the design variables to their initial values. These values are computer-generated random numbers with uniform distribution over specific ranges, e.g., $0.8\omega_{s1} < \omega_j < 1.2\omega_{s3}$, $1\% < \xi_j < 4\%$, and $0 < x_j < L$. With many different sets of initial values, the optimization process converges to many local minima. Excluding repeated local minima, distinct local minima are obtained. By this way, to identify the global minimum (the solution with lowest performance value), the numerical optimization needs to be executed many times, e.g., about 40 times (each with a different set of initial values). For each execution, the optimization process typically takes 10 minutes of running time on a Pentium 4 personal computer.

It was found that all these different solutions (global and local minima) do have some common features. Firstly, the frequencies of TMD's are in clusters around the three natural frequencies of the beam. An example of frequency clustering is: 5 TMD's are tuned to the first mode, 4 TMD's are for the second and 1 TMD is for the third. Secondly, the TMD's are located at the anti-nodes of the beam's mode shapes. These common features suggest that the concept of SDOF analogy can be used. In this simple beam case, while the first mode has only one anti-node at mid-span, the other two modes have several anti-nodes, which lead to many possible effective placements of TMD's, and consequently many local minima.

To interpret the obtained results, the local minimum solutions are sorted into groups according to the frequency clustering pattern of TMD's. Solutions in the same group may have different placements of

Table 1 Three groups of the most effective configurations of 10 TMD's in suppressing the total vibration energy of a simple beam

Group	Number of TMD's tuned to mode			$\min(J/J_0)$
	1st	2nd	3rd	
A	5	4	1	0.1733
B	6	3	1	0.1746
C	5	3	2	0.1747

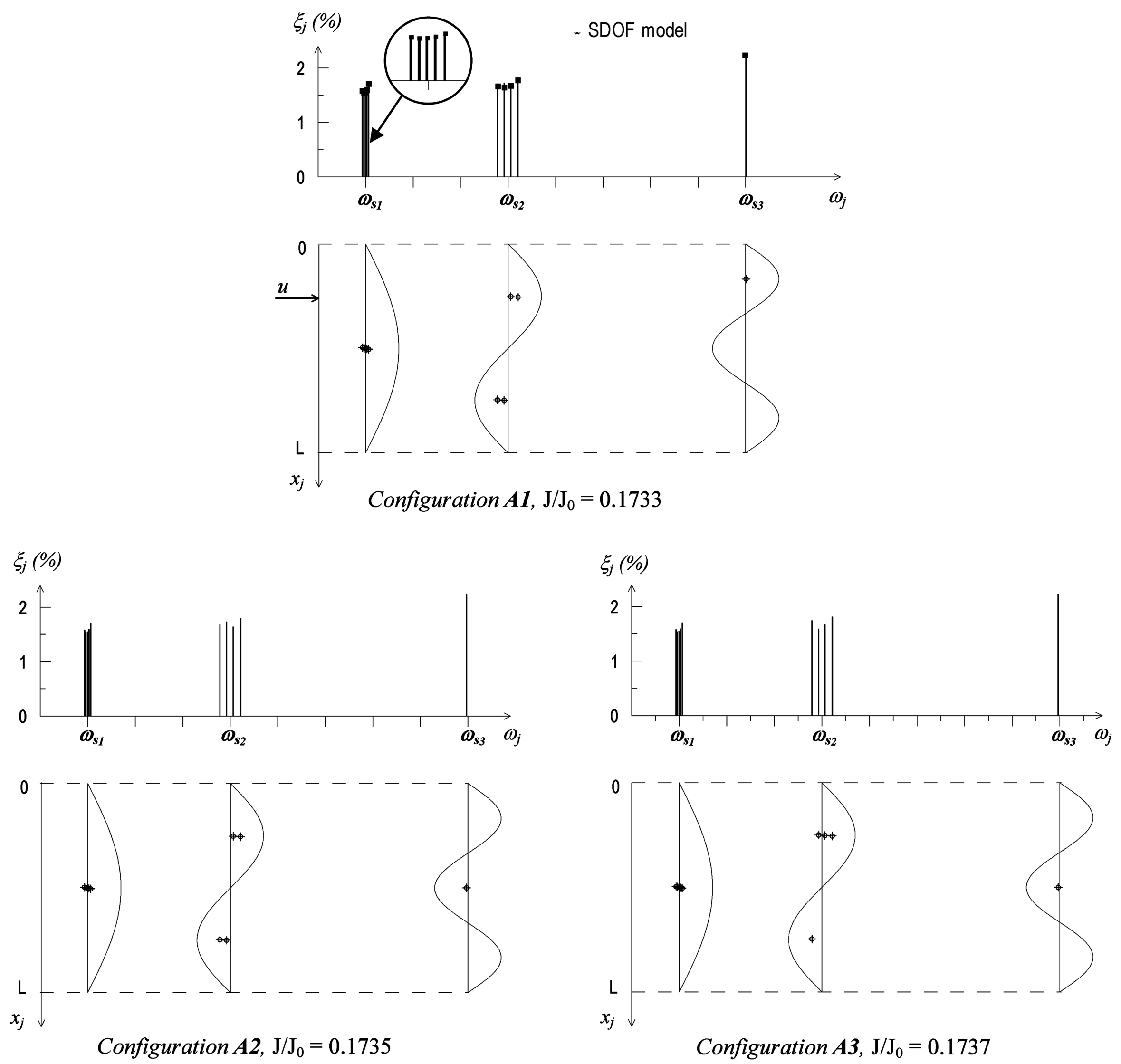


Fig. 2 Three most effective configurations of 10 TMD's in *Group A*

TMD's. Note that, since modal displacements at the anti-nodes of this simple beam are all equal (to unity), different solutions in the same group yield nearly the same performance values. Several groups were identified, but only three groups with lowest performance values are listed in Table 1 as *Groups A*, *B* and *C*. For each group, the smallest obtainable value of performance index J, normalized by that of the uncontrolled beam J_0, is presented in the last column of the table.

From Table 1, it is observed that in the three most effective groups, the highest number of TMD's is allocated for the first mode and the lowest number is for the third. This is quite reasonable considering the percent contribution of each structural vibration mode to the target response quantity. To be specific, when no TMD's are added, the percent contributions of the first,

Table 2 Optimal TMD parameters for a simple beam (*Configuration A1*)

Mode i	TMD j	$\frac{\omega_j}{\sqrt{EI/\rho AL^4}}$	$\gamma_j=\omega_j/\omega_{si}$	ξ_j (%)	x_j/L	SDOF analogy design γ_j	SDOF analogy design ξ_j (%)	SDOF analogy design x_j/L
1	1	9.216	0.934	1.577	0.497	0.936	1.578	0.500
	2	9.524	0.965	1.540	0.498	0.967	1.541	0.500
	3	9.819	0.995	1.549	0.500	0.997	1.551	0.500
	4	10.130	1.026	1.597	0.502	1.029	1.600	0.500
	5	10.493	1.063	1.706	0.504	1.066	1.711	0.500
2	6	37.291	0.945	1.679	0.747	0.947	1.663	0.750
	7	38.667	0.979	1.730	0.749	0.981	1.641	0.750
	8	40.046	1.014	1.635	0.252	1.014	1.676	0.250
	9	41.549	1.052	1.789	0.254	1.052	1.778	0.250
3	10	88.929	1.001	2.236	0.168	0.999	2.236	0.167

$\omega_j = (i\pi)^2\sqrt{EI/\rho AL^4}(i=1, 2, 3)$ are natural frequencies of the simple beam

second, and third modes of the beam to the total vibration energy index are 62%, 31% and 7% respectively. By comparing the smallest values of J/J_0 among these three groups, *Group A* is identified as the best group. Fig. 2 shows the three most effective configurations of TMD's in *Group A*. In the figure, the damping ratios and locations of TMD's are plotted against their natural frequencies. At each TMD frequency, in the upper portion is a bar whose length is proportional to TMD damping ratio, while in the lower portion, TMD location along the span is represented by a circle symbol. The mode shapes of the beam are also depicted at the corresponding natural frequencies for ease of reference. The optimal configuration of TMD's is *Configuration A1* with parameters tabulated in Table 2.

As mentioned earlier, it is generally believed that the concept of SDOF analogy can be used to design TMD's for structures with widely spaced frequencies. Another design scheme is then introduced to examine the validity of this SDOF analogy. In this design scheme, 5, 4 and 1 TMD's are assigned to the first, second, and third vibration modes of the beam respectively, following

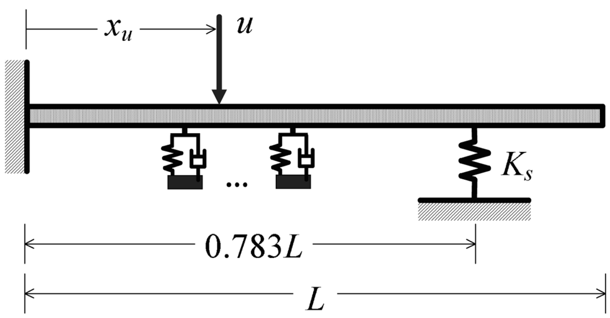


Fig. 3 A spring supported cantilever beam with attached multiple TMD's

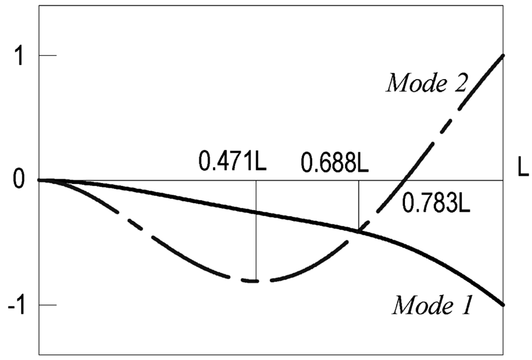


Fig. 4 The first two mode shapes of the spring supported cantilever beam (without TMD's)

Configuration A1. All these TMD's also have identical masses. Their locations are set exactly to the anti-nodes of the controlled modes, according to the SDOF analogy, as shown in the last column of Table 2. Then, the natural frequencies and damping ratios of TMD's for each mode are numerically optimized, neglecting the effects of other modes. The results are compared with those of *Configuration A1* in Table 2 and Fig. 2. Excellent agreement is achieved, confirming that the SDOF analogy is applicable very well in this case.

4. Structures with two closely spaced frequencies

A basic but informative example of designing TMD's for closely-spaced natural frequencies structures is presented in this section. The main structure is a spring-supported cantilever beam depicted in Fig. 3. The beam has length L, mass per unit length ρA, flexural rigidity EI, and the vertical spring has stiffness K_s. This model was employed in the study of Abe and Igusa (1995). It is similar to that proposed by Setareh and Hanson (1992) to represent a one-way slab with a transverse girder. The special feature of the model is that the stiffness and position of the spring can be selected to make the first two natural frequencies of the beam close to each other.

In this example, the spring is located at the node of the second mode shape ($x = 0.783L$) with the stiffness $K_s = 240EI/L^3$, following Abe and Igusa (1995). The first two natural frequencies of the main structure are obtained as $\omega_{s1} = 21.018\sqrt{EI/\rho AL^4}$ and $\omega_{s2} = 22.035\sqrt{EI/\rho AL^4} = 1.048\omega_{s1}$, and the corresponding two mode shapes are shown in Fig. 4. The beam's damping is very small and assumed to be proportional with equal modal damping ratios ξ_{si}=0.5% for all i.

4.1 Optimal design of multiple TMD's

The beam is now subjected to a wide-band random force modeled by white noise at x_u. The total vibration energy of the beam is to be minimized. Since control of the first two modes with closely spaced frequencies is of interest, the effects of higher modes are neglected. Three cases of different loading positions are investigated: (1) $x_u = 0.471L$, (2) $x_u = L$ and (3) $x_u = 0.688L$. While the first loading position corresponds to an anti-node of the second mode shape, the second loading position is at the beam tip, which is the anti-node of both the first and second modes. The last position is selected in

Table 3 Optimal TMD parameters for a spring-supported cantilever beam

Design scheme	TMD j	Load case 1 ($x_u=0.471L$)				Load case 2 ($x_u=L$)				Load case 3 ($x_u=0.688L$)				Two-load case (R) ($x_{u1}=0.471L$ and $x_{u2}=L$)			
		ω_j/ω_{s1}	ξ_j (%)	x_j/L	J/J_0	ω_j/ω_{s1}	ξ_j (%)	x_j/L	J/J_0	ω_j/ω_{s1}	ξ_j (%)	x_j/L	J/J_0	ω_j/ω_{s1}	ξ_j (%)	x_j/L	J_R
A	1	1.019	4.987	0.489	0.206	1.005	8.567	1.000	0.110	1.019	4.297	0.629	0.269	-	-	-	-
B	1	0.991	2.414	0.524	0.188	0.940	7.360	1.000	0.106	1.003	5.337	1.000	0.198	1.012	7.455	1.000	0.166
	2	1.071	3.708	0.455		1.052	5.429	1.000		1.018	3.699	0.532		1.024	4.163	0.507	
C	1	0.952	1.047	0.540	0.154	0.871	2.005	1.000	0.095	0.952	2.031	1.000	0.175	0.925	2.376	1.000	0.149
	2	0.973	1.007	0.518		0.907	1.939	1.000		0.956	1.107	0.568		0.971	2.318	1.000	
	3	0.996	1.110	0.491		0.941	1.916	1.000		0.977	1.012	0.574		0.976	1.235	0.527	
	4	1.018	1.112	0.494		0.974	1.901	1.000		0.999	1.078	0.560		1.002	1.232	0.522	
	5	1.020	2.869	1.000		1.008	1.821	1.000		1.002	2.127	1.000		1.017	2.390	1.000	
	6	1.041	1.162	0.491		1.029	0.459	1.000		1.019	1.055	0.566		1.028	1.265	0.513	
	7	1.064	1.151	0.485		1.052	1.674	1.000		1.042	1.174	0.549		1.056	1.304	0.502	
	8	1.089	1.264	0.487		1.090	1.974	1.000		1.056	2.201	1.000		1.070	2.472	1.000	
	9	1.097	2.334	1.000		1.134	2.155	1.000		1.067	1.298	0.547		1.091	1.449	0.494	
	10	1.128	1.355	0.485		1.190	2.393	1.000		1.108	1.434	0.521		1.127	2.325	1.000	

$\omega_{s1} = 21.018\sqrt{EI/\rho AL^4}$ and $\omega_{s2}=1.048\ \omega_{s1}$; $\xi_{s1}=\xi_{s2}=0.5\%$

Scheme A: use only one TMD, mass ratio = 0.5%

Scheme B: use two TMD's of identical masses, total mass ratio = 0.5%

Scheme C: use 10 TMD's of identical masses, total mass ratio = 0.5%

between where the transverse displacements of the two modes are equal. In addition, to gain further insight into the effects of single and multiple TMD's on this beam, three design schemes with different numbers of TMD's are conducted for each load case: (A) $n=1$, (B) $n=2$ and (C) $n=10$. For all schemes, the same total mass of TMD's of 0.5% beam mass is used. In scheme (B) or (C) where multiple TMD's are used, all TMD's are assigned to have identical masses.

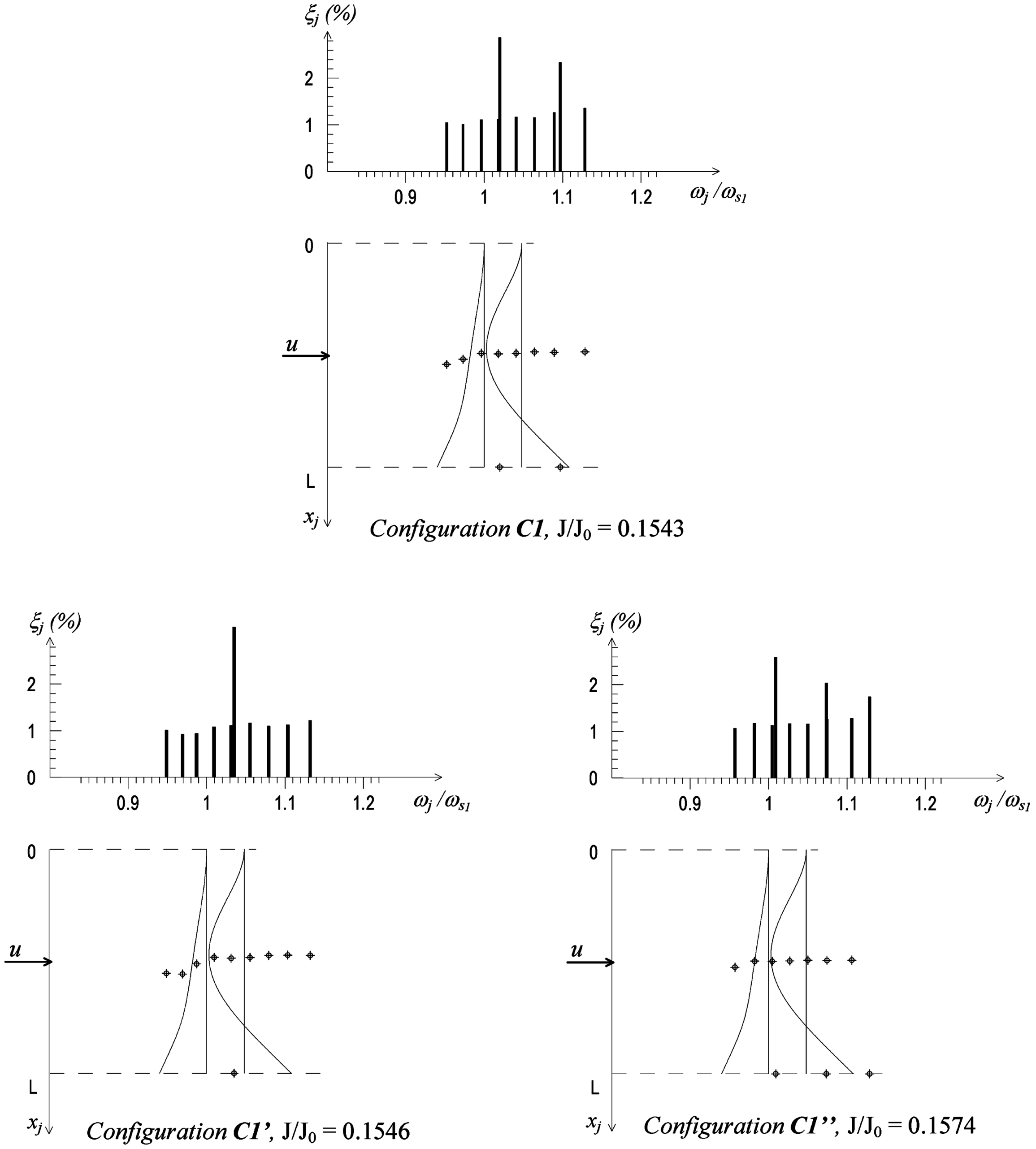


Fig. 5 Three most effective configurations of 10 TMD's for load case 1 ($x_u = 0.471L$)

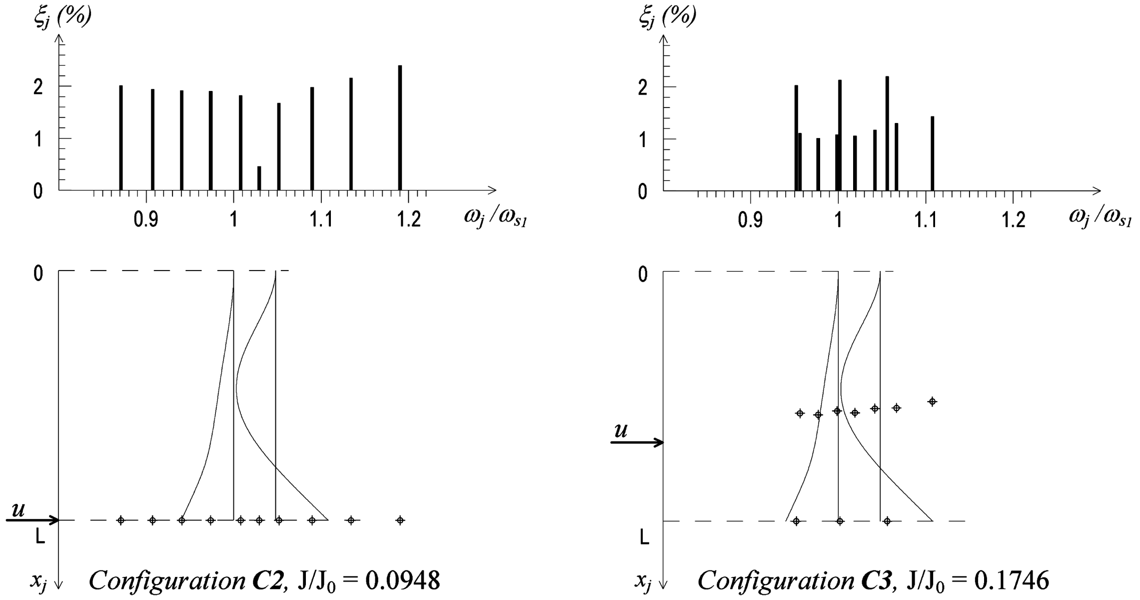


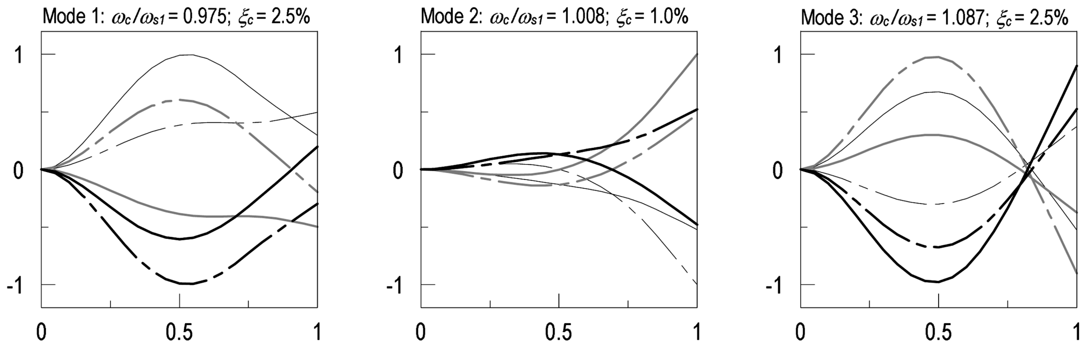
Fig. 6 Optimal configurations of 10 TMD's for load cases (2) and (3)

With random initial values for the design variables, the optimal search for natural frequencies ω_j, damping ratios ξ_j, and locations x_j of TMD's ($j = 1, \cdots, n$) converges rapidly. The optimal mechanical and location parameters of TMD's for the beam are listed in Table 3 together with the resultant total vibration energy J, normalized by that of the beam without TMD's J_0. It is noted that for each load case in scheme (A), only a single optimal solution was obtained. But for schemes (B) and (C), where $n \geq 2$, the optimization process converges to many local minima. These local minima correspond to configurations of TMD's different from each other in both frequency tuning and placement. By comparing their performance index values, the global minimum can be identified. To demonstrate, the three most effective configurations of 10 TMD's (scheme C) for the first load case ($x_u = 0.471L$) are graphically presented in Fig. 5. In the figure, *Configuration C1* is the optimal configuration with lowest performance value, though *Configurations C1'* and *C1''* also yield nearly the same performance. For the other two load cases in scheme (C), the optimal configurations of TMD's are plotted in Fig. 6.

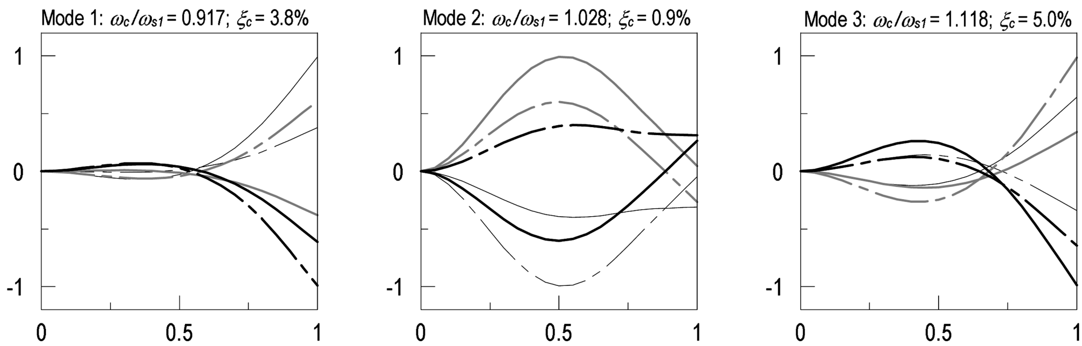
Note that *Configurations C1*, *C1''* and *C3* in Figs. 5-6 have some common features. In these configurations, TMD's are generally divided into two groups, one located at the mid-span and another at the beam tip. Locations of the TMD's at the mid-span are close to, but not exactly at, the anti-node of the second mode. In each group, TMD's have rather uniform frequency spacing and share the same damping level, apparently to achieve multiple tuning effect. Of the two groups, the one with a larger number of TMD's has a closer frequency spacing, a wider frequency range, and a lower damping level. The frequency range of this group covers both the two structural frequencies while that of the other smaller group seems to be restricted around a certain structural frequency. For example, in *Configurations C1* and *C1''*, frequencies of the TMD's in the smaller group at the beam tip are distributed around the second structural frequency ω_{s2}. In comparison, those in the smaller group in *Configuration C3* are closer to ω_{s1}.

The resultant values of performance index in Table 3 indicate that multiple TMD's are very effective in controlling structures with two closely-spaced natural frequencies. With increasing n, the frequency range of TMD's is wider, the damping ratios of TMD's are lower, and the response index is further reduced, i.e., better vibration control is gained. However, comparing the values of J/J_0 in all design schemes, it can be seen that the control of an optimal single TMD ($n=1$) is practically as effective as

(a) Load case 1: $x_u = 0.471L$



(b) Load case 2: $x_u = L$



(c) Load case 3: $x_u = 0.688L$

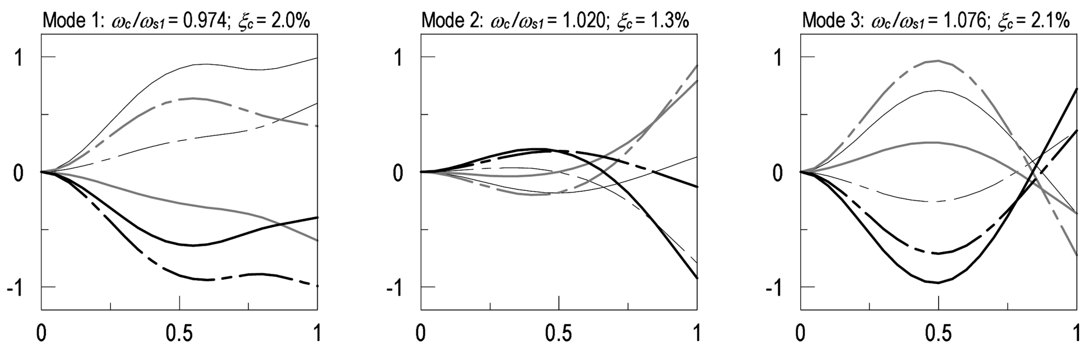


Fig. 7 Illustration of complex modified modes of the beam with an optimal single TMD from Scheme A
-·-·- $\omega_c t = 0$; —— $\omega_c t = \pi/3$; - · - $\omega_c t = 2\pi/3$; —— $\omega_c t = \pi$; - · - $\omega_c t = 4\pi/3$; —— $\omega_c t = 5\pi/3$

that of optimal multiple TMD's. In other words, a single TMD can be made effective in controlling structures of two closely-spaced natural frequencies. This finding is different from Abe and Igusa's conclusion that it is not possible to control two close modes with a single TMD.

To gain more insight into this finding, a detailed modal analysis is performed for the combined system of the beam and a single TMD. For convenience, hereafter the term "original modes" applies for two (real) mode shapes of the main structure without TMD (Fig. 4), whereas the term "modified modes" is referred to three modes newly formed after adding the TMD. Each of the modified modes contains the motion of both beam and TMD, but the TMD motion is not presented here for simplicity. Since the combined system is non-proportionally damped, the modified modes are complex functions, where the motions along the span in each mode differ in phase as well as in amplitude. The change in phase of the beam's response can be illustrated by plotting free vibration of a complex mode at several time instants within a vibration period (Hurty and Rubinstein 1964). In Fig. 7, such illustrations are provided for the modified modes of the beam with an optimal single TMD from scheme (A), for all three design load cases. In this illustration, the modal response is evaluated at every one-sixth of a vibration period. Besides, the amplitudes are not shown to decrease with the motion in order to simplify the picture, but their damping ratios are noted instead.

The mode plots in Fig. 7 reveal that in all load cases, two out of three modified modes have relatively high damping. Accordingly the responses of these modes are well suppressed by the reduction in the dynamic magnification factor. This is a well-known mechanism of vibration control using TMD's. Nonetheless, there is a particular mode with low damping but relatively low amplitude at the loading point. In the figure it is the second mode for all load cases. Normally such a low-damping mode can be excited easily but here it cannot because of the low modal displacement at the loading point. This means that a single TMD can suppress vibration of structures with two closely-spaced natural frequencies by reformulating the original mode shapes into two highly-damped modes and one low-damping mode with low modal displacement at the loading point.

In the work of Abe and Igusa (1995), several simplifying assumptions were adopted. Among these, the most critical one is probably the performance index that they used as the target of the analysis. The index was defined as the sum of the squares of the modal coordinate amplitudes, which did not directly represent any physical response quantity of the main structure. In this work, the optimization is based on a physically meaningful index and no simplification is employed in the analysis. This might be the reason why a different conclusion on the effectiveness of a single TMD in control two close modes is attained.

The low-damping mode with low modal displacement at the loading point can also be found in other design schemes. To demonstrate, the free vibration of modified modes of the beam with two optimal TMD's from scheme (B) are evaluated and plotted in Fig. 8. As shown in the figure, low-damping modes have been formed in load cases (1) and (2), which are the second and third modified modes respectively. It is noticed that the formulation of a low-damping mode occurs if two optimal TMD's stay close to the loading point (see Table 3). For load case (3), the two optimal TMD's locate away from the loading point, and all four modified modes are high-damping modes. Although not shown here, the illustrations of the modified modes of the beam with 10 optimal TMD's from scheme (C) expose similar results, i.e., the low-damping modes can be found in the first two load cases. As a general observation, when the majority of optimal TMD's locate near the loading point, it can make the modal displacement at that point become very small and a low-damping mode may be formed.

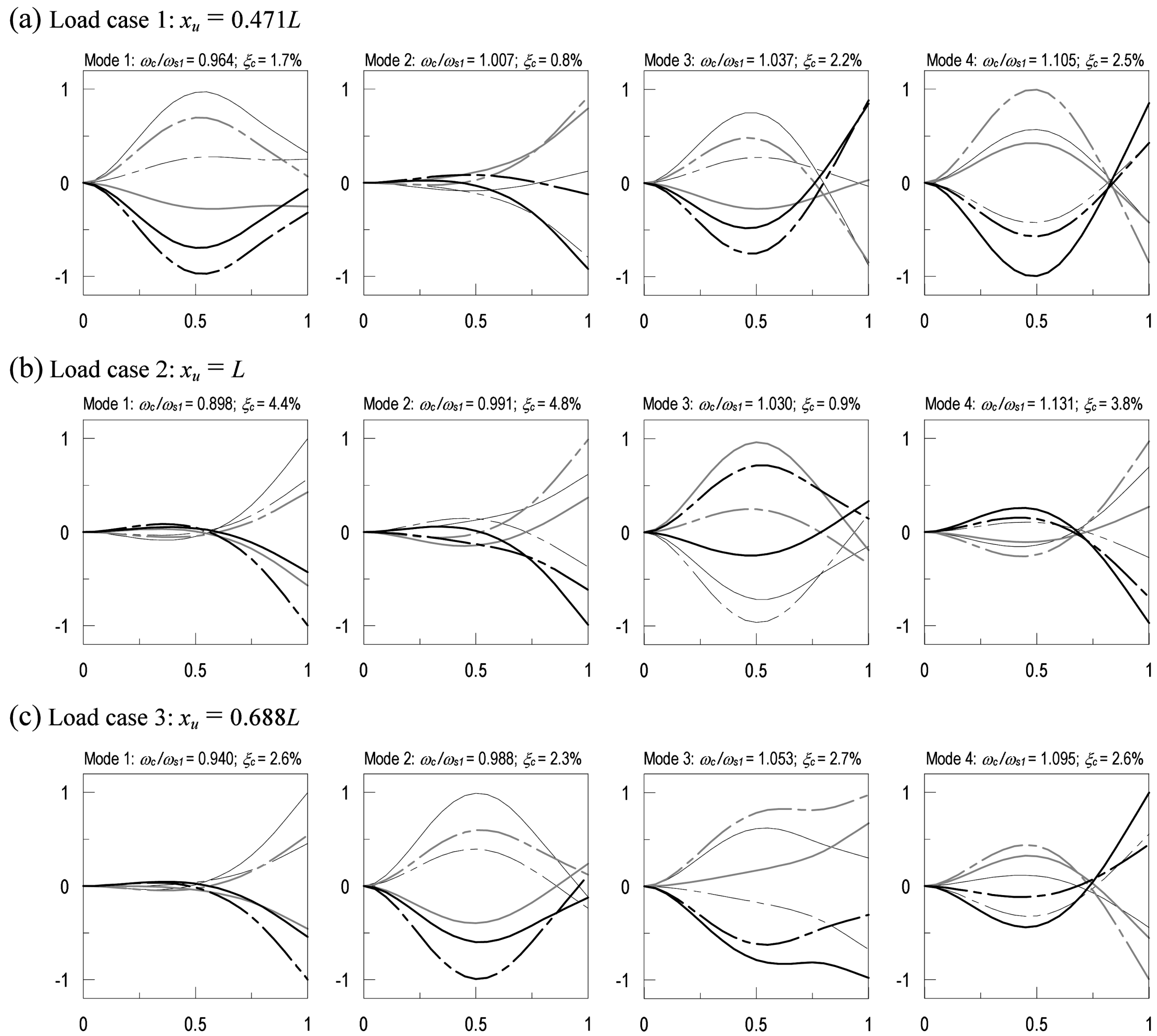


Fig. 8 Illustration of complex modified modes of the beam with two optimal TMD's from Scheme B
-·-·- $\omega_c t = 0$; — $\omega_c t = \pi/3$; – · · $\omega_c t = 2\pi/3$; — $\omega_c t = \pi$; – · · $\omega_c t = 4\pi/3$; — $\omega_c t = 5\pi/3$

4.2. Design for robustness with respect to loading position

The optimal configurations of TMD's obtained so far are for minimizing the beam's total vibration energy under a point load applying at a specific position. However, from the above discussion, it is known that TMD's may suppress the structural response by formulating a low-damping mode with low amplitude at the loading point. In such cases, if the beam is subjected to loading at a position other than designed, the low-damping mode is likely to be much more excited. That is, the control effectiveness of TMD's can be greatly impaired when the loading position changes. This can be seen from Table 4 by comparing, for each optimally designed configuration of TMD's, the results of the normalized total vibration energy of the beam under a white-noise load at three different positions: $x_u = 0.471L$, $0.688L$,

Table 4 Normalized total energy of the beam with different optimal configurations of TMD's

Design scheme	Design loading position	Configuration	J/J_0 for loading at		
			$x_u=0.471L$	$x_u=0.688L$	$x_u=L$
A	$0.471L$	A1	0.206	0.319	0.518
	L	A2	0.531	0.535	0.110
	$0.688L$	A3	0.262	0.269	0.434
B	$0.471L$	B1	0.188	0.322	0.549
	L	B2	0.538	0.529	0.106
	$0.688L$	B3	0.199	0.198	0.144
	$0.471L$ and L	BR	0.194	0.202	0.137
C	$0.471L$	C1	0.154	0.204	0.234
	L	C2	0.618	0.617	0.095
	$0.688L$	C3	0.172	0.175	0.157
	$0.471L$ and L	CR	0.179	0.189	0.120

and L. In this table, an optimal configuration of TMD's is designated by its corresponding design scheme and load case number, e.g., *Configuration B2* denotes the two TMD's (scheme B) optimized for loading at $x_u=L$ (load case 2). Obviously every optimal single TMD (scheme A), while effective for the design load case, is much less effective for the other load cases. This means a single TMD is not robust at all. Likewise *Configurations B1*, *B2*, *C1* and *C2*. The exceptions are *Configurations B3* and *C3*, for which no any low-damping mode is formed, and their control performance is reasonably effective for every loading position.

Note that, once a complex modal analysis of the combined structure-TMD's system is made, one can directly check whether a low-damping mode is formed, or whether the optimized configurations of TMD's like *B3* and *C3* are robust. However, it is not so practical to carry out such type of analysis on a routine basis. A more practical approach is to modify the proposed method in such a way that the numerical optimizer will search for multiple TMD's whose control effectiveness is not so sensitive to the change in loading position, i.e., robustness design. To demonstrate, a new performance index is defined as

$$J_R = \frac{1}{2}\left(\frac{J_1}{J_{01}} + \frac{J_2}{J_{02}}\right) \tag{12}$$

where J_1 is the total vibration energy of the beam as it is equipped by TMD's and subjected to a white-noise force at $x_u=0.471L$; J_2 is that response of the same beam-TMD's system but under loading at $x_u=L$. These indices are normalized by their corresponding total vibration energy values of the beam without using TMD's, J_{01} and J_{02}. The index J_R expresses the average performance of TMD's with two different loading positions. By doing so, the numerical search is likely to converge to a solution equally

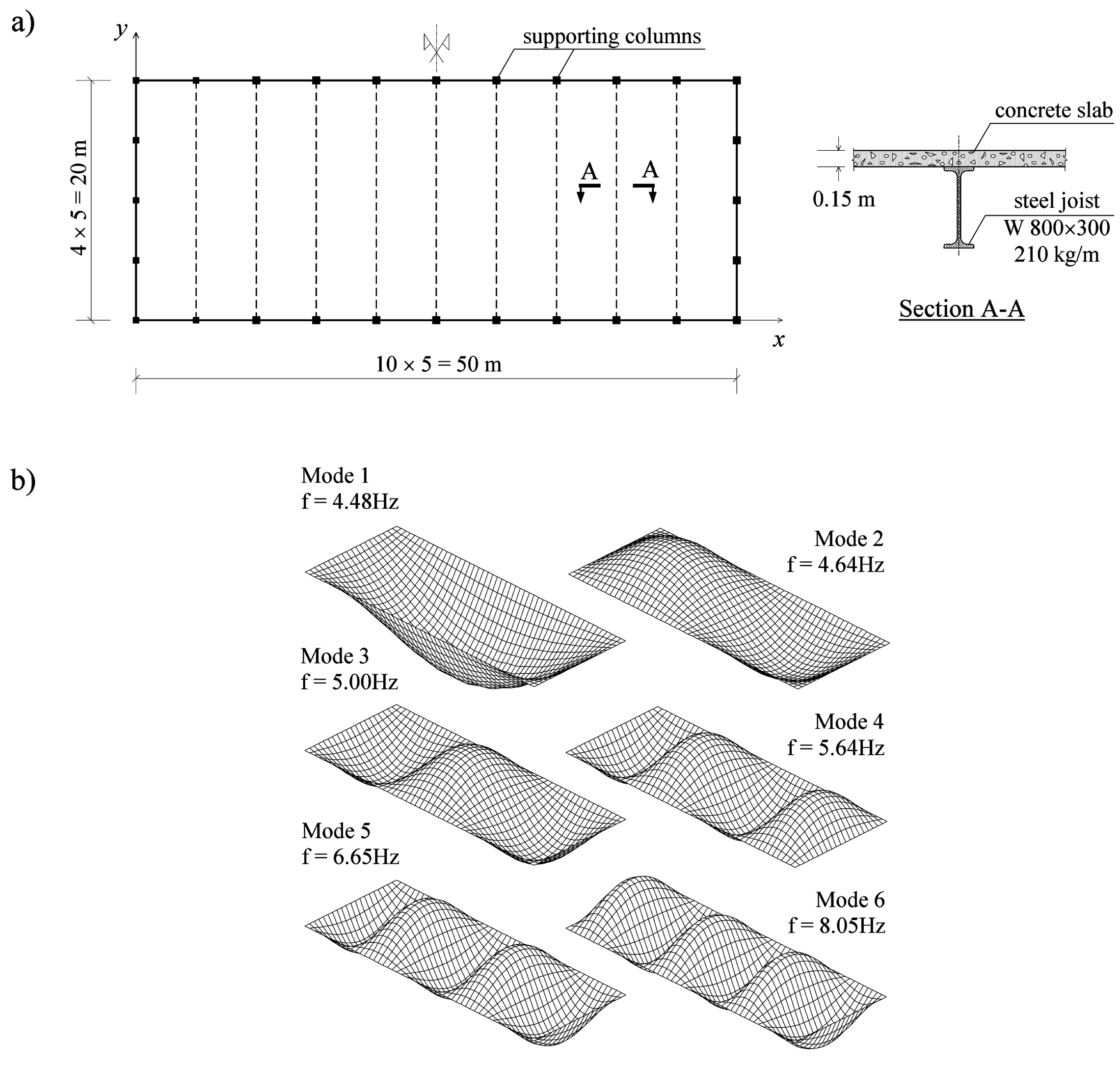


Fig. 9 A badminton floor and its first six natural frequencies and mode shapes

effective for both load cases. This solution is expected to be a set of TMD's that yields no low-damping mode and thus remains effective for other load cases as well.

The robust TMD's that minimize the index in Eq. (12) are *Configurations BR* and *CR* for, respectively, $n=2$ and 10 as shown in the last four columns of Table 3. Their robustness can be verified in Table 4, where the resultant performance index is reasonably low in every loading condition. Therefore, this modified method can be used to search for a configuration of multiple TMD's that are both effective and robust with respect to loading position.

5. Structures with multiple closely spaced frequencies

The investigation is now extended to the case of a realistic structure with many closely-spaced natural frequencies ($m > 2$). It is a 50 m×20 m badminton floor made of a 15 cm thick concrete slab stiffened by a series of steel joist framing members placing at a regular interval of 5 m. Its plan view and section are depicted in Fig. 9(a) where each joist is simply supported by columns at its two ends. This composite floor was reported to have disturbing vibration during normal badminton exercises, even though the floor had been properly designed for strength and deflection requirements. The vibration sometimes makes the users feel uneasy or annoyed, and hence creates a serviceability problem.

The application of multiple TMD's to suppress the vibration of this floor system appears to have several advantages over the use of a single TMD. First, a single TMD may not be effective in this case where the number of closely spaced modes is greater than two. Moreover, multiple TMD's can be constructed in small and light units and thus can be easily integrated into the floor without any special requirements such as additional installation space or local structural strengthening for a large concentrated load.

5.1. Finite element model of the floor system

To make an optimal design of TMD's for this composite floor, it is necessary to first identify modal properties of the structure, e.g., natural frequencies, mode shapes and modal masses. A finite element model is developed for this purpose. A fine mesh size of 0.25 m×0.25 m with total of 16281 grid nodes is created, from which the steel joist and the concrete slab are idealized by beam and plate elements respectively. Using the structural analysis software STRAND7 of G+D Computing Pty Ltd (2002), the modal properties of the floor are readily obtained. Many modes with closely spaced frequencies have been found. The first 6 natural frequencies and mode shapes are presented in Fig. 9(b). Note that, to reflect the real system behavior accurately, this finite element model has already been calibrated by data from on-site vibration measurements. The same damping ratio of 1% is assumed for all modes of the floor for simplicity.

5.2. Formulation of optimal design problem

As mentioned earlier, several on-site vibration measurements were conduced during normal service conditions where many people were playing badminton on the floor. The recorded time histories of the floor vibration show a characteristic of multi-mode, narrow-band, stationary random responses. Their Fourier magnitude spectra show approximate relative contributions from different modes. The actual dynamic excitation resulted from random running, jumping, and skipping of many badminton players is quite complex and difficult to model explicitly. However, it is possible to approximately reproduce similar multi-mode, narrow-band, stationary random responses by using a set of three independent white-noise forces u_1, u_2, and u_3 applying at positions $(x_{u1}, y_{u1}) = (6.25, 10)$, $(x_{u2}, y_{u2}) = (20, 15)$, and $(x_{u3}, y_{u3}) = (43.75, 6.66)$, respectively (the unit is in meter) as shown in Figs. 10 and 11. This set of white-noise forces may be considered as an "equivalent" dynamic excitation to the actual complex dynamic excitation in this case. Also multiple forces are required in order to obtain robust TMD's, according to the discussion in Section 4.2. In addition, as the floor vibration was dominated by lower-frequency modes, the vibration modes of the first 5 lowest structural frequencies are considered in this numerical example, i.e., $m = 5$.

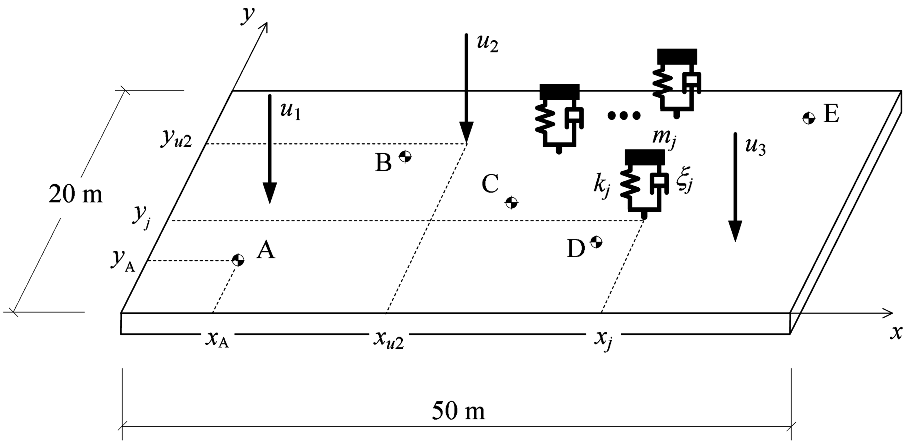


Fig. 10 A diagram of the floor showing the location parameters of multiple TMD's, input forces (u_1-u_3), and selected points (A-E) to represent the overall floor vibration level

Since the target is to reduce the floor vibration level in order to satisfy human comfort requirements, the acceleration response is of particular interest. In this study, the sum of mean square acceleration responses at 5 points A - E (Fig. 10) is to be minimized. The coordinates of these points are A (6.25, 5), B (18.75, 15), C (25, 10), D (33.33, 6.67), and E (43.75, 16.67). They are scattered over the floor so that the vibrations at various points on the floor can be properly represented. The reduction in the sum of mean square acceleration responses then indicates a vibration attenuation of the whole floor area.

For the case where the main structure is subjected to a single white-noise force with spectral intensity S_0, the expression of the mean square acceleration response at a point (x_s, y_s) is given by Hoang and Warnitchai (2005).

$$J_S = tr[\mathbf{R}[\mathbf{AXA}^T + 2\pi S_0(\mathbf{A}+\mathbf{I})\mathbf{bb}^T]] \quad (13)$$

in which $\mathbf{R} = \mathbf{rr}^T$, $\mathbf{r} = \{\mathbf{0}\ \mathbf{s}^T\Phi\}^T$, and $\mathbf{s}$ is an $m+n$-vector: $\mathbf{s} = \{\psi_1(x_s, y_s) \cdots \psi_m(x_s, y_s)\ 0 \cdots 0\}^T$. The other notations follow Eqs. (5-6). It is easy to modify Eq. (13) so that the performance index represents the mean square acceleration response at the same point due to multiple forces. If the number of forces is n_L, then the vector $\mathbf{f}$ in the equation of motion (1) must be replaced by an influence $(m+n) \times n_L$-matrix $\mathbf{F}$. Furthermore, if all these forces are independent white-noises with identical intensity values S_0, the same expression as Eq. (13) is obtained for the mean square acceleration response at a selected point, except that the vector $\mathbf{b}$ is now replaced by a matrix $\mathbf{B} = [\mathbf{0}\ \mathbf{F}^T\Phi]^T$:

$$J_M = tr[\mathbf{R}[\mathbf{AXA}^T + 2\pi S_0(\mathbf{A}+\mathbf{I})\mathbf{BB}^T]] \quad (14)$$

in which $\mathbf{X}$ is the solution of the Lyapunov equation:

$$\mathbf{AX} + \mathbf{XA}^T + 2\pi S_0\mathbf{BB}^T = \mathbf{0} \quad (15)$$

The target performance index for the optimal design of the floor in this section is then expressed as

$$J = J_{MA} + J_{MB} + J_{MC} + J_{MD} + J_{ME} \tag{16}$$

where J_{MA} is the mean square acceleration response at the point A due to three independent white-noises (u_1 - u_3), which is evaluated by Eq. (14) with the matrix **R** corresponding to the coordinates of the point A; the other terms (J_{MB} – J_{ME}) are defined in a similar way.

Table 5 Optimal parameters of 10 TMD's for a badminton floor

TMD j	$\omega_j/2\pi$ (Hz)	ξ_j (%)	x_j (m)	y_j (m)
1	4.413	3.391	18.284	9.861
2	4.538	4.083	41.223	9.999
3	4.713	4.683	8.226	9.928
4	5.107	5.226	21.975	10.134
5	5.183	5.483	42.994	9.885
6	5.551	3.578	31.946	10.000
7	5.802	3.900	6.136	10.043
8	6.483	3.789	5.448	9.910
9	6.680	3.844	44.602	10.131
10	7.119	3.664	5.409	9.858

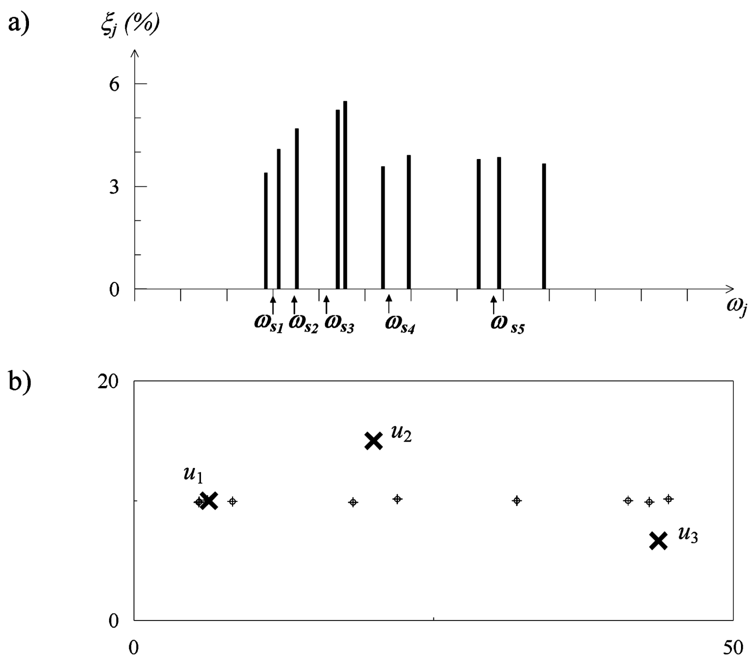


Fig. 11 Optimal configuration of 10 TMD's for the floor: (a) damping ratios vs. frequencies; (b) locations

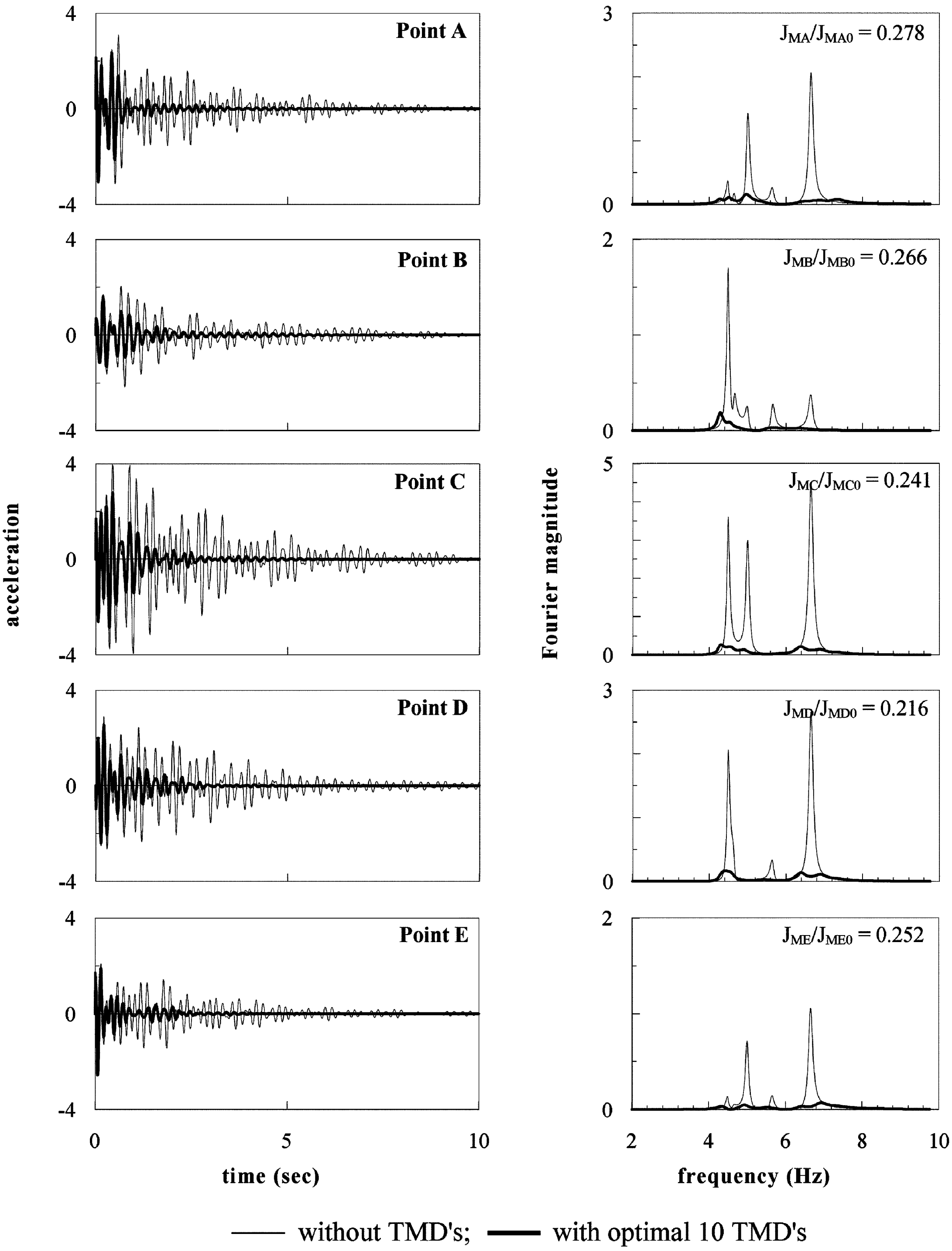


Fig. 12 Floor acceleration responses from initial impulses

A set of 10 TMD's is used where all TMD's have equal masses. The total mass of TMD's is set at 1% of the floor mass. Accordingly there are 40 design variables to be optimized: natural frequencies ω_j, damping ratios ξ_j and locations (x_j, y_j) of TMD's, where $j = 1, \ldots, 10$.

5.3. Numerical optimization

The numerical optimization is then conducted with computer-generated random initial values for design variables. In each iteration of the optimization process, whenever the location variables of TMD's are changed, the corresponding mode-shape values of the main structure at the attachment points of TMD's need to be updated. But at this time, the floor's modes shapes are no longer analytical functions; they are computed as discrete values at grid nodes of the finite element model (Fig. 9b). Consequently, an interpolation is required to find modal displacement at any given point between grid nodes. To do so, for each mode, the discrete data at grid nodes of the mode shape are sorted into a matrix, from which a two-dimensional data interpolation (cubic interpolation method) can be utilized. Because of this interpolation, the optimization process consumes longer time of running to yield an optimal result - approximately 20 hours on a Pentium 4 computer.

With many different sets of initial values for design variables, the optimization process converges to many different local minima. These local minima actually have nearly the same performance values, and the solution with lowest performance value is identified as the global minimum. This globally optimal solution is tabulated in Table 5 and plotted in Fig. 11. Its performance index value J, normalized by that of the floor without TMD's J_0, is as low as 24.8%. The result clearly demonstrates the effectiveness of multiple TMD's in controlling structures with multiple closely spaced frequencies. Furthermore, the normalized mean square of acceleration at each of the five selected points (for example, J_{MA}/J_{MA0} for point A) is also evaluated. The results are 27.8, 26.6, 24.1, 21.6 and 25.2% for points A to E respectively. They are rather uniform although the response levels at these points differ widely. The ratio of mean square responses at these points is $J_{MA} : J_{MB} : J_{MC} : J_{MD} : J_{ME} = 2.22 : 1.40 : 2.43 : 1.79 : 1$. It shows that the obtained optimal TMD's not only effectively reduce the overall vibration level of the floor, but also yield uniform control effectiveness at various points on the floor.

As shown in Fig. 11, the optimal TMD's have their natural frequencies distributed over a range covering all five controlled structural frequencies, and they have rather low damping ratios. Their locations are well distributed along the center line parallel to the x-axis ($y = 10$ m). This is reasonable because $y = 10$ m is the line of maximum modal displacements with respect to the y-axis of all controlled modes (see Fig. 9b). A complex modal analysis on the combined floor-TMD's system is also conducted. All modes of the combined system have damping ratios in the range of 2.2 to 4%; no mode with particularly low damping is found, implying that the obtained optimal TMD's can produce robust performance as expected.

To verify the control effectiveness, the acceleration response of the floor equipped with the optimally designed 10 TMD's is evaluated. The floor is simultaneously excited by three rectangular-wave unit impulses at the designed loading points. The duration of these impulses is 50 ms. The resulted acceleration time histories and their corresponding Fourier magnitude spectra at five points of interest (A-E) are computed and plotted in Fig. 12. The normalized mean square acceleration at each point is also given for reference. Comparing responses of the floor with and without optimal TMD's in Fig. 12, it is obvious that damping in the floor system has been substantially increased and its dynamic response to excitation in the resonant region has been greatly suppressed by the effect of TMD's.

6. Conclusions

In this paper, a new method using a numerical optimizer has been applied to design multiple TMD's for suppressing multi-mode structural response of beams and floor structures. The method shows to be very capable for simultaneous searching of optimal placement and tuning of TMD's for these structures under wide-band loading. Based on several numerical case studies using the method, the following conclusions can be made:

(1) The results from the case of a simple beam with widely-spaced natural frequencies confirm that multiple TMD's can be adequately designed by treating each vibration mode of the main structure as an equivalent SDOF system. The optimal TMD's for controlling each mode have distributed frequencies and are located at the anti-node(s) of the corresponding mode shape.

(2) The results from the case of a cantilever beam with two closely-spaced natural frequencies indicate that multiple TMD's ($n \geq 2$) are very effective in suppressing such type of structure. The obtained optimal TMD's have their natural frequencies distributed over a range covering the two controlled structural frequencies and have rather low damping ratios.

(3) It is possible to design a single TMD to effectively reduce vibration of the cantilever beam with two closely spaced frequencies. Although a single TMD cannot raise the damping level in every vibration mode of the combined structure-TMD system, it can force the mode with low-damping to have a low modal displacement at the loading point. This is a newly identified control mechanism. By this mechanism, however, the control effectiveness of the TMD can be greatly impaired when the loading position changes. This problem can be solved by properly modifying the target performance index to account for two different loading positions.

(4) The results from the case of a large floor structure with five closely-space natural frequencies demonstrate that the proposed method is capable of handling realistic design problems of multiple TMD's for complex structures. In this case, a finite element model is required to compute the natural frequencies, mode shapes and modal masses of the floor. Moreover, the loading is simulated by multiple independent white-noise forces acting at different positions, and acceleration responses at several points are simultaneously controlled. Despite this complexity, the optimal TMD's can be obtained without any major difficulties, and they are shown to be very effective and robust.

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