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A measuring system for determination of a cantilever beam support moment

Askold P. Loktionov*

Department of Mechanics and Engineering, Southwest State University, 50 let Octobrya, 94, Kursk, 305040, Russia

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Abstract. This investigation is aimed to develop a model of experimental-computation determination of a support moment of a cantilever beam loaded with concentrated force at its end including the optimal choice of coordinates of deflection data points and parameters of transformation of deflection data in case of insufficient accuracy of the assignment of initial parameters (support settlement, angle of rotation of the bearing section) and cantilever beam length. The influence of distribution and characteristics of sensors on the cantilever beam on the accuracy of determining the support moment which improves in the course of transition from the uniform distribution of sensors to optimal non-uniform distribution is shown. On the basis of the theory of inverse problems the method of transformation reduction at numerical differentiation of deflection functions has been studied. For engineering evaluation formulae of uncertainty estimate to determine a support moment of a cantilever beam at predetermined uncertainty of measurements using sensors have been obtained.

Keywords: cantilever beam; support moment; inverse problem; numerical differentiation; approximation; sensor; measurement uncertainty; condition number

1. Introduction

The health conditions of in-service civil infrastructures can be evaluated by employing structural health monitoring technology. A reliable health evaluation result depends heavily on the quality of the data collected from the structural monitoring sensor network. Thus, the problem of use in diagnosing sensor has received considerable attention in recent years (Dmitrienko *et al*. 2011, Huang *et al*. 2016). Optimal placement of sensors is an integral component in the development of effective structural monitoring of building structures (Zhou *et al*. 2015a, b, Yi *et al*. 2015). Theoretical and computational issues arising in the selection of the optimum topology sensor network for estimating coverage area with sensor placement in building structural monitoring discussed (Haque *et al*. 2015). Yi *et al.* (2015) proposed an algorithm, which combines the artificial fish swarm algorithm with the monkey algorithm, as a strategy for the optimal placement of a predefined number of sensors. Lü *et al*. (2015) explored the three-point method that uses a grating eddy current absolute position sensor for bridge deflection estimation. Real spatial positions of the measuring points along the span axis are directly used as relative reference points of each other rather than using any other auxiliary static reference points for measuring devices in a conventional method. Dung and Sasaki (2016) investigated the effect of sensor location attached on a cantilever beam. Ozbey *et al*. (2016) studied experimentally measured relative displacements and deformations in a simply supported beam. Loktionov (2013) studied the use of methods of approximation and reduction of measurements to determine the load on the cantilever beam. Reduction of measurements allows determining a support moment of a cantilever beam, loaded with bending moment, if initial values of the problem are not assigned. Determination of the bending moment realized Lagrange approximation of the second derivative of the deflection and numerical differentiation of information-measuring system. The deflection is measured at points on a grid of approximation. Chekushkin *et al*. (2015) examined polynomial methods to create functional dependencies in data-measuring systems. Cheney and Kincaid (2013) described interpolation and numerical differentiation of functions without input errors on the set of roots of Chebyshev. The initial conditions in the calculation scheme may be dependent on the external load or to be unknown. In solving the inverse problem, the sensor signals are used to calculate initial conditions in the calculation scheme. The procedure of the solving of an inverse problem of structural analysis with measurement of the beam curvature and subsequent calculation of beam characteristics, in particular, of bending moments in a beam was described by Liew and Choo (2004). The authors highlighted the low accuracy of determination of the curvature near beam ends by means of the method of polynomial approximation using measurement aggregation of six sensors.

The need to eliminate the influence of the coordinates of an external force, the deflection of the support section and the angle of rotation of the support section is of great importance in the problem of experimental and theoretical determination of the support moment of cantilever beam. The purpose of this work is to improve methods of polynomial approximation of not measured directly

*Corresponding author
E-mail: loapa@mail.ru

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characteristics of a measured object using an information and measurement system of numerical differentiation. The goal of the investigation is to develop a unified formalism of measurement and criteria of estimation of the effectiveness of the solving of a measurement reduction problem.

A method of experimental and computation determination of internal forces (shear and bending moment) in case of cross bending of a beam, loaded with concentrated force is proposed in the paper. The method is based on the reduction of measurements using Lagrange approximation in numerical differentiation. The procedure of reduction of measurements lies in the reaching of limit optimum relationships in measurement aggregation of sensors according to the class of transformation, number and position of sensors on a beam. The application significance of the investigation is in the increase of accuracy of determination of beam target characteristics.

2. Reduction of measurements by means of Lagrange approximation of numerical differentiation

The procedure of reduction of measurements of forces acting in a mechanical structure was described by Loktinov (2007). Optimization of characteristics of measurement aggregation of sensors and their arrangement in structures were studied. Under reduction of measurement we understand formalism allowing obtaining more precise description of non-observable system “analyzed object - environment” using the results of measurements of the observable system “measured object - environment-information and measurement system” (Mordvinov 2010, Zelenkova and Skripka 2015). The information and measurement system consists of measuring and calculation components. The measuring component includes transforming aggregation of sensors and a communication channel. Characteristics of a measured object unlike the analyzed object are distorted by the interaction with the information and measurement system, in particular, with its measurement component and in some cases with the environment. The algorithm implemented by the calculation component of the information and measurement system, derives from the output signal of the communication channel the most precise required values of target characteristics of the analyzed object not available for direct examination.

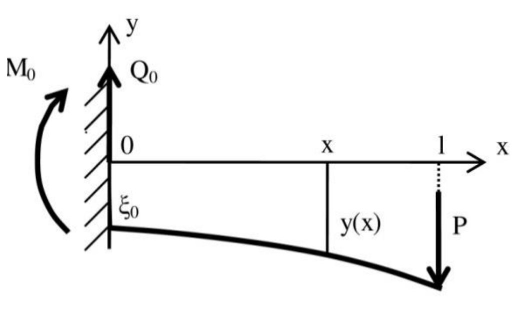

Fig. 1 Deflection Curve of Beam

The system was examined as is shown in Fig. 1 as applied to a cantilever beam.

The specific features of inverse problems of structural mechanics with numerical differentiation and use of measurement reduction without loss of generality are examined using the example of a mechanical structure load-bearing element - a cantilever beam with length l and permanent flexural rigidity EI, loaded with concentrated load P at its end. Here beam deflections are measured, target characteristic of the load-bearing element - support moment M_0 is determined.

Select the following system elements - the measured object (cantilever beam), the external environment (the external load on the beam, the coordinate of the external force, the conditions of fixing the beam), the measurement component (sensors), a computer component, the object under study (the bending moment and shear force acting in the beam). According to the results of measurements of indirect manifestations of the object being analyzed, in particular deflections, can be obtained more accurate values of the desired bending moment and shear force.

Function of deflection $y(x)$ is assigned using the results of joint measurements - finite values of deflections $y^*(x_i)$ on the longitudinal registration section $[x_a, x_b]$ of the beam at data points x_i. Minimum number of data points depends on the characteristics of the cantilever beam, in particular, on the accuracy of its length l assignment and initial parameters - support settlement ξ_0 and angle of rotation of the support section ξ_1.

The effectiveness of measurement reduction is connected with the optimization of sensors measuring aggregation. Let us study two classes of deflection data transformation and consequently, of a sensors. The used sensors can have similar characteristics. Class of p-transformation of deflection data, i.e. linear transformation with a uniform continuous norm of absolute uncertainty in measurement, similar reduced uncertainty of measurements ε_p of sensor readings in all data points x_i (i=1, 2, … n) and similar limits of measurements y_p, equal to the upper limit sup $|y^*(x_i)|$ for a compact set of real numbers $\{y^*(x_1), \ldots, y^*(x_n)\}$. The following relations are performed in p-transformation

$$\|\Delta y(x_i)\| = \Delta_m(y(x_i)) = \varepsilon_p y_p \geq \Delta(y(x_i)) = |y^*(x_i) - y(x_i)|, \quad (1)$$

where $\Delta_m(y(x_i))$ is the upper limit of absolute values of deflection measurement uncertainty.

However, sensors can have different characteristics. Let us consider a special case - class of pi-transformation, which is different from p-transformation class by measurements limits $y_{pi}=y(x_i)$. The following relations are performed in pi-transformation

$$\Delta_m(y(x_i)) = \varepsilon_p y_{pi} \geq \Delta(y(x_i)). \quad (2)$$

Restoring not directly measured target characteristics of the element of a structural unit we consider the class of inverse problems, presented by the function

$$f(x) = y''(x), \qquad x \in \mathrm{I} \quad (3)$$

where I=[0, l] is the section of problem-solving assignment,

$y(x)$ is the solution of the initial problem with differential equation

$$y'''(x) = f_o(x) \quad (4)$$

and $f_o(x)=P/EI$ is the external load in Eq. (4).

Remark 1 To take account Saint-Venant's principle, sensors are placed in section $[x_a, x_b]$ outside the area of application of forces and fixing of the beam in a general case at the non-uniform grid of approximation

$$0.05\, l = x_a \le x_i \le x_b = 0.95\, l. \quad (5)$$

After deformation Bernoulli beam cross section is flat and normal to the beam axis deformed (Gere and Timoshenko 1997, Kuchumov 2011). The initial problem (4) for the Bernoulli beam has particular solutions

$$y(x_i) = \xi_0 + \xi_1 x_i + \xi_2 \frac{x_i^2}{2} + \xi_3 \frac{x_i^3}{6} \quad (6)$$

for initial conditions

$$y_0 = \xi_0,\ \ y_0' = \xi_1,\ y_0'' = \xi_2\,,$$

where ξ_0, ξ_1, ξ_2, ξ_3 are the initial parameters of function $y(x)$, $\xi_2 = \dfrac{M_0}{EI}$, $\xi_3 = \dfrac{Q_0}{EI}$, $\xi_2 = -\,l\xi_3$, E is the Young modulus of beam material, I is the moment of inertia and $Q_0=P$.

Along with the target not directly measured characteristics of object M_0 to the deflections $y^*(x_i)$ a set of other arguments are transformed, characterizing influence quantities including those relevant to the external environment. By reduction of measurements we obtain approximation to function $f(x)$ - function $f^*(x)$ with minimum value of the level of its uncertainty.

Range $\{y_i^*\}$ forms an observation space - a finite-dimensional coordinate Euclidian space of measured data points in Eq. (5), which allows calculating the value of the support moment M_0 through function $f(x)$ in Eq. (3) when x=0 using Lagrange one-dimensional approximation of the first degree

$$\xi_2 = y''(0) \approx \sum_{i=1}^{n} L_i y^*(x_i)\,, \quad (7)$$

where L_i are the Lagrange coefficients and n is the number of parameters of approximation (number of sensors in the measurement aggregation).

Approximation polynomial going through the observed values of the function in approximation grid points (at the points of sensors location) is in some cases impractical. Considering the uncertainty in values $y^*(x_i)$ approximation polynomial repeats this uncertainty. However, it is possible to select such a polynomial whose curve goes near these points. The concept of proximity is specified by the type of approximation. The type of metrics is as a rule determined by the experiment type. Mean-square approximation is quite acceptable to process observation data, it smoothes some inaccuracy of function $y(x)$, takes into account uncertainty of measurement at nodal grid points. The effectiveness of mean-square approximation decreases in case of small quantity of observed data. In this paper more rigid conditions are set, i.e., a uniform continuous norm of absolute uncertainty of measurements.

3. Estimation of effectiveness of measurements reduction

Let us consider the procedure of using reduction of measurements for Lagrange approximation of numerical differentiation in Eq. (7) in terms and notions of the proposed method. Criteria for assessing the effectiveness of solving the task - the absolute (ν_Δ) conditioning number of the task and the relative (ν_δ) condition number of the reduction problem. We use the following relations

$$\Delta_m(f) \le \nu_\Delta \sup \Delta_m(y), \quad (8)$$

$$\delta_m(f) \le \nu_\delta\ \varepsilon_p, \quad (9)$$

where $\Delta_m(f)$, $\delta_m(f)$, $\Delta_m(y)$ and $\delta_m(y)$ are respectively upper limits of absolute and relative values of uncertainty of solution and deflection measurements.

Limit optimum relationships in measurement aggregation of sensors according to the class of transformation, number and location of sensors on the beam are obtained by regularization of aggregation and obtaining of minimum condition number of the problem of a measurement reduction - the size of the structural uncertainty of the solution of problem.

The objective function in the problem of reduction with estimating the effectiveness of Eq. (8) has the form min ν_Δ=min $\Delta_m(f)/\Delta_m(y)$, and for Eq. (9) min ν_δ=min $\delta_m(f)/\varepsilon_p$ with boundary constraints in accordance with the Eq. (5). There $\Delta_m(f)$, $\Delta_m(y)$, $\delta_m(f)$ the function of controlled variables x_i.

Regularization is in choice of the class of transformation of sensors and a communication channel signal and a type of distribution of approximation grid nodal points. There is no single strategy of selection of optimum nodal points of the grid ensuring process convergence of the obtaining of measure of structural error of the problem of measurements reduction for all continuous at section $[x_a, x_b]$ functions. Some particular results were described by Loktionov (2013). In particular, for p-transformation of deflection $y^*(x_i)$ the problem of regularization according to Eq. (7) is reduced to the problem of regularization of a compact set of grid nodal points in the implementation of the objective function (Lebesgue constant of the second type)

$$\Lambda_{n2} = \min \sum_{i=0}^{n} |L_i|\,. \quad (10)$$

Consider the optimization problem for a function Eq. (10) with boundary constraints in accordance with the Eq. (5).

Loktionov (2013) announced a theorem: For function approximation using Eq. (3) by Lagrange one-dimensional approximation for uniform approximation norm, equality of the number of approximation parameters n in Eq. (7) and increased by a degree m function $y(x)$, performing relations in Eq. (1), optimum nodal points of approximation formula to obtain minimum absolute number of uncertainty of the problem of reduction are points of Chebyshev alternance of the order m+1. Here *pi-transformation* method uses successive extreme points of Chebyshev polynomials. Deviations points are equal and alternate in sign. Accuracy of approximation by this method above interpolation method Cheney and Kincaid (2013) on the set of roots of Chebyshev.

When pi-approximation transducers have various approximation norm and this theorem does not apply to the totality of the transducers.

Chebyshev polynomial of degree m at section $[x_a, x_b]$ has Chebyshev alternance of order m+1– a set of points m+1, where polynomial takes a value with maximum modulo with sequential alteration of signs.

The Chebyshev points are obtained by taking equally-spaced points on a semicircle and projecting them down onto the horizontal axis (Cheney and Kincaid 2013). Constructing points Chebyshev alternance shown in Fig. 2.

In particular, for m=1 two points

$$x_1 = x_a,\ x_2 = x_b, \tag{11}$$

for m = 2 three points

$$x_1 = x_a,\quad x_2 = \frac{x_a + x_b}{2},\ x_3 = x_b, \tag{12}$$

for m = 3 four points

$$x_1 = x_a,\quad x_2 = \frac{3x_a + x_b}{4},\quad x_3 = \frac{x_a + 3x_b}{4},\ x_4 = x_b. \tag{13}$$

The special features of inverse problems of numerical differentiation and procedure of using measurements reduction without loss of generality are examined using the example of function $y(x)$ in Eq. (6). In the model of a beam Timoshenko cross-sections remain flat, but not perpendicular to the deformed axis of the beam (Gere and Timoshenko 1997, Kuchumov 2011). The results obtained in this paper are suitable for the model of the beam Timoshenko, which takes into account the effect of shear deformation on deflection of the beam.

$$y(x) = \xi_0 + \xi_1 x - k\frac{Q}{GF}x + \xi_2\frac{x^2}{2} + \frac{\xi_3 x^3}{6},$$

where k is the coefficient of shear, G is the material shear modulus, F is the beam cross-section area.

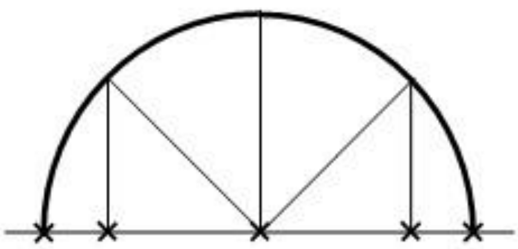

Fig. 2 Approximation with Chebyshev points (m=4)

4 Problem with low accuracy of assignment of support settlement, angle of rotation and beam length

Four deflection data $y(x_i)$ are enough to determine the initial parameter ξ_2 by Eqs. (6) and (7), if accuracy of cantilever beam parameters assignment l (coordinates of areas of force application P), ξ_0 and ξ_1 is insufficient. In order for approximation polynomial according to Eq. (7) at n=4 to implement at numerical differentiation by measurement and compution system a formula of the second derivative of the function of deflection at the point with coordinate x=0 and not to be the function of length l and initial data ξ_0, ξ_1, according to the method of undetermined coefficients, it is necessary to meet the following conditions

$$\sum_{i=1}^{4} L_i = 0,\quad \sum_{i=1}^{4} L_i x_i = 0,\quad \sum_{i=1}^{4} L_i x_i^2 = 2,\quad \sum_{i=1}^{4} L_i x_i^3 = 0. \tag{14}$$

Lagrange coefficients satisfy the conditions of Eq. (14)

$$L_1 = 2\frac{x_2 + x_3 + x_4}{(x_2 - x_1)(x_3 - x_1)(x_4 - x_1)},\quad L_2 = -2\frac{x_1 + x_3 + x_4}{(x_2 - x_1)(x_3 - x_2)(x_4 - x_2)},$$
$$L_3 = 2\frac{x_1 + x_2 + x_4}{(x_3 - x_1)(x_3 - x_2)(x_4 - x_3)},\quad L_4 = -2\frac{x_1 + x_2 + x_3}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}. \tag{15}$$

The approximate solution as per Eq. (7) satisfies the perturbed initial data - deflection data $y^*(x_i)$, the level of errors of initial data within uncertainty of each deflection point. The integrated absolute uncertainty of value ξ_2 for all 4 data points equals to

$$\Delta\xi_2 = \sum_{i=1}^{4} L_i \Delta y(x_i). \tag{16}$$

Relative uncertainty of value ξ_2 equals to

$$\delta\xi_2 = \frac{1}{|\xi_2|}\sum_{i=1}^{4} L_i \Delta y(x_i). \tag{17}$$

Let us compare the uncertainty of special results of solving the problem of determination of a beam support moment using values of absolute and relative measure of structural uncertainty - minimum absolute and relative condition number of the reduction problem.

The absolute condition number of the reduction problem ν_Δ and relative condition number of the reduction problem ν_δ are determined by relations as per Eqs. (8)-(9)

$$\Delta_m(\xi_2) \le \nu_\Delta \sup \Delta_m(y(x_i)), \tag{18}$$

$$\delta_m(\xi_2) \le \nu_\delta\, \varepsilon_p, \tag{19}$$

where $\Delta_m(\xi_2)$ and $\delta_m(\xi_2)$ are respectively the limit of absolute and relative values of uncertainty of the result of the problem.

4.1 Non-uniform distribution of data points

4.1.1 P-transformation of deflection data $y(x_i)$

Taking into account y_p=$y(x_3)$ of Eqs. (1), (7), (16) and

(18) we obtain an absolute condition number of the reduction problem

$$v_{\Delta} = \sum_{i=1}^{4} \left| L_i \right| . \qquad (20)$$

Regularization result as per Eqs. (10), (15) and (20) - is the minimum absolute conditioning number of $132/l^2$ at distribution of deflection coordinates by Eq. (13).

Remark 2. Determining relative condition number of the reduction problem let us introduce at $[x_a, x_b]$ relations

$$\left|\xi_0\right| + \left|\xi_1 x_i\right| << \left|y(x_i)\right| ,$$

and from Eq. (6) we obtain approximate equation

$$\xi_2 \approx 2y(x_i)\frac{1}{x_i^2(1 - x_i/3l)} . \qquad (21)$$

Then the accuracy of determination of relative condition number of the reduction problem is determined by the accuracy of Eq. (21).

Taking into account Eq. (21) at $x_i = x_4$, Eqs. (1), (9) and (19) we obtain relative condition number of the problem

$$v_{\delta} = \frac{x_4^2(1 - x_4/3l)}{2} \sum_{i=1}^{4} \left| L_i \right| . \qquad (22)$$

The result of regularization of Eqs. (15) and (22) is coordinate distribution of deflection data in Eq. (13), and minimum relative conditioning number of the problem is 40.6.

4.1.2 Pi-transformation of deflection data $y(x_i)$

Taking into account relations of Eqs. (2), (16) and $y_{pi} = y(x_i)$ we obtain

$$\Delta_m(\xi_2) = \varepsilon_p \sum_{i=1}^{4} \left| L_i y(x_i) \right| . \qquad (23)$$

Substitution of Eq. (21) in Eq. (23) yields the upper limit of the absolute uncertainty of problem-solving results

$$\Delta_m(\xi_2) = \frac{\varepsilon_p y(x_4)}{x_4^2\left(1 - \dfrac{x_4}{3l}\right)} \sum_{i=1}^{4} \left| L_i x_i^2 \left(1 - \frac{x_i}{3l}\right) \right| .$$

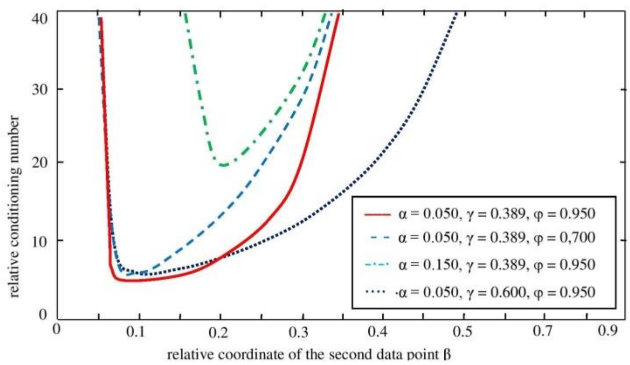

Fig. 3 Influence on data points coordinate accuracy

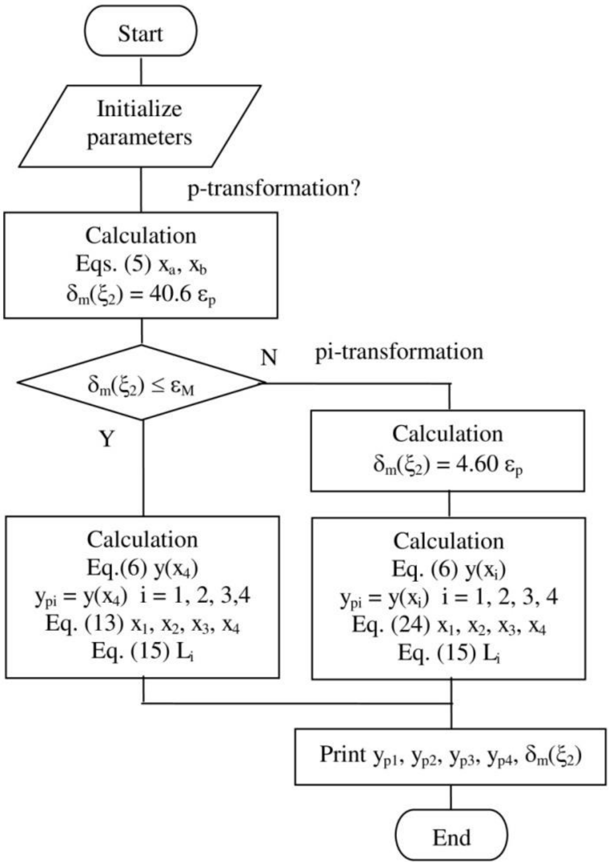

Fig. 4 Algorithm - the choice and placement of transducers in section 4.1

Then, using Eq. (18) we obtain absolute condition number of the reduction problem of determination ξ_2

$$v_{\Delta} = \frac{1}{x_4^2\left(1 - \dfrac{x_4}{3l}\right)} \sum_{i=1}^{4} \left| L_i x_i^2 \left(1 - \frac{x_i}{3l}\right) \right| .$$

The result of regularization of the problem by the numerical method is the coordinate distribution of deflection data points

$$x_1 = x_a,\ x_2 = 0.083l,\ x_3 = 0.389l,\ x_4 = x_b, \qquad (24)$$

minimum absolute conditioning number is $14.9/l^2$.

Taking into account Eqs. (2), (17), (19) and (21) we obtain relative condition number of the reduction problem

$$v_{\delta} = \frac{1}{2} \sum_{i=1}^{4} \left| L_i x_i^2 \left(1 - \frac{x_i}{3l}\right) \right| .$$

The result of the regularization of the problem by numerical method is coordinate distribution of the data points in Eq. (24), minimum relative condition number is 4.60.

In case of no regularization, in particular when $x_2 < 0.051l$ or $x_2 > 0.35l$ relative condition number of the reduction problem increases more than 3 times (it becomes more than 16) as shown in Fig. 3, where $\alpha = x_1/l$, $\beta = x_2/l$, $\gamma = x_3/l$, $\varphi = x_4/l$.

Selection and placement of transducers in section 4.1 is shown in Fig. 4. It is necessary to enter the value of l, ε_p, approximate values of the initial parameters of the function $y(x)$, ε_M - the expected measurement uncertainty of the initial parameter ξ_2.

4.2 Uniform distribution of deflection data points

The expectedly worst results were obtained at regularization only of length and location of the section with uniform distribution of deflectometers along the beam

$$x_1 = x_a,\ x_2 = (2\,x_a + x_b)/3,\ x_3 = (x_a + 2\,x_b)/3,\ x_4 = x_b.$$

At p-transformation of the deflection data, minimum absolute conditioning number is $148/l^2$, minimum relative conditioning number is 45.7, and at pi-transformation they are $50.6/l^2$ and 15.6 respectively.

5. Problem with low accuracy of assignment of support settlement and angle of rotation, and known value of beam length

Three deflection data $y(x_i)$ are enough to determine the initial parameter ξ_2 by Eq. (6), if accuracy of assignment of parameters ξ_0, ξ_1 is insufficient and beam length assignment l is sufficient. In order for approximation polynomial as per Eq. (7) at n=3 to implement at numerical differentiation by measurement and computation system a formula of the second derivative of the function of deflection at the point with coordinate x=0 and not to be a function of initial data ξ_0, ξ_1, according to the method of undetermined coefficients it is necessary to meet the following conditions

$$\sum_{i=1}^{3} L_i = 0, \quad \sum_{i=1}^{3} L_i x_i = 0, \quad \sum_{i=1}^{3} L_i x_i^2 \left(1 - \frac{x_i}{3l}\right) = 2\,. \tag{25}$$

Lagrange coefficients satisfy the conditions of Eq. (25)

$$L_1 = \frac{2}{(x_2 - x_1)(x_3 - x_1)K_M}, \quad L_2 = -\frac{2}{(x_2 - x_1)(x_3 - x_2)K_M}, \quad L_3 = \frac{2}{(x_3 - x_2)(x_3 - x_1)K_M} \tag{26}$$

where $K_M = 1 - \dfrac{x_1 + x_2 + x_3}{3l}$.

According to the method shown in Section 2 the following results were obtained

5.1 Non-uniform distribution of data points

5.1.1 P-transformation of deflection data $y(x_i)$

Regularization of the absolute conditioning number

$$v_\Delta = \sum_{i=1}^{3} |L_i|\,.$$

gives minimum absolute condition number of $38.7/l^2$ at distribution of deflection data points

$$x_1 = x_a,\ x_2 = 0.4365l,\ x_3 = x_b.$$

Relative condition number is

$$v_\delta = \frac{1}{2} x_3^2 \left(1 - \frac{x_3}{3l}\right) \sum_{i=1}^{3} |L_i|\,. \tag{27}$$

Substituting Eq. (26) in Eq. (27) yields the condition number of the reduction problem

$$v_\delta = \frac{2x_{3l}^2}{(x_{3l} - x_{2l})(x_{2l} - x_{1l})K_M} \left(1 - \frac{x_i}{3l}\right). \tag{28}$$

Function by Eq. (28) at $[x_a, x_b]$ takes on minimum value of 11.2 at data points distribution

$$x_1 = x_a,\ x_2 = 0.3072l,\ x_3 = 0.6015l.$$

In case of no optimization, in particular, at $0.1l > x_1$, $x_2 > 0.42l$, $0.5l > x_3$ or at $0.1l > x_1$, $x_2 > 0.6l$, $0.65l > x_3$ relative condition number of the reduction problem increases more than by three times (it becomes more than 30) as shown in Fig. 5 (function F1 x_1/l=0,050, x_3/l=0,650, for the function F2 x_1/l=0,050, x_3/l=0,500).

5.1.2 Pi-transformation of deflection data $y(x_i)$

Regularization of the absolute condition number of the reduction problem

$$v_\Delta = \frac{1}{x_3^2\left(1 - \dfrac{x_3}{3l}\right)} \sum_{i=1}^{3} \left| L_i x_i^2 \left(1 - \frac{x_i}{3l}\right) \right|$$

gives minimum absolute condition number $5.57/l^2$ at distribution of data points

$$x_1 = x_a,\ x_2 = 0.09435l,\ x_3 = x_b. \tag{29}$$

Relative condition number is

$$v_\delta = \frac{1}{2} \sum_{i=1}^{3} \left| L_i x_i^2 \left(1 - \frac{x_i}{3l}\right) \right|. \tag{30}$$

Substituting Eq. (26) in Eq. (30) yields the condition number of the reduction problem

$$v_\delta = 1 + \frac{2x_2^2}{(x_3 - x_2)(x_2 - x_1)K_M} \left(1 - \frac{x_2}{3l}\right). \tag{31}$$

Function as per Eq. (31) when $[x_a, x_b]$ takes the least value at data points distribution as per Eq. (29).

Substituting Eq. (29) in Eq. (31) yields the minimum relative condition number 1.72.

In case of no regularization, in particular when $x_1 > 0.3l$ or $x_2 > 0.7l$ relative condition number increases more than 7 times (it becomes more than 12) as shown in Fig. 5 (function F3 x_1/l=0,051, x_3/l=0,950, for the function F4 x_1/l 0,300, x_3/l=0,950).

5.2 Uniform distribution of data points

The expectedly worst results were obtained in case of regularization of only length and location of the section with uniform distribution of sensors along the beam. At p-transformation of deflection data minimum absolute condition number of the reduction problem $39.5/l^2$ was

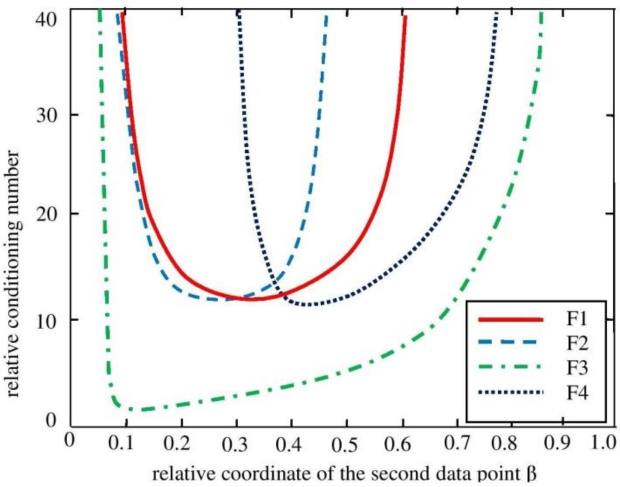

Fig. 5 Influence on data points coordinate accuracy

obtained at distribution of data points as per Eq. (12), minimum relative condition number is 11.3 at data points distribution

$$x_1 = x_a,\ x_2 = 0.3179l,\ x_3 = 0.5857l.$$

At *pi*-transformation minimum absolute condition number of the reduction problem $16.6/l^2$ was obtained at distribution of data points as per Eq. (12) and minimum relative condition number is 4.69 at data points distribution

$$x_1 = x_a,\ x_2 = 0.326l,\ x_3 = 0.602l.$$

6. Problem with known settlement and angle of rotation of support section and low accuracy of the assignment of beam length

Two deflection data $y(x_i)$ are enough to determine the initial parameter ξ_2 as per Eq. (6), if accuracy of length l assignment is not sufficient. In order for approximation binomial

$$\xi_2 = \sum_{i=1}^{2} L_i\left(y^*(x_i) - \xi_0 - \xi_1 x_i\right)$$

to implement at numerical differentiation by the measurement - computation system the formula of second derivative of the function of deflection at the point with coordinate x=0 and not to be the function of length l, as per the method of undetermined coefficients it is necessary to meet the following conditions

$$\sum_{i=1}^{2} L_i x_i^2 = 2, \qquad \sum_{i=1}^{2} L_i x_i^3 = 0. \tag{32}$$

Lagrange coefficients satisfy the conditions of Eq. (32)

$$L_1 = 2\frac{x_2}{(x_2 - x_1)x_1^2}, \quad L_2 = -2\frac{x_1}{(x_2 - x_1)x_2^2}. \tag{33}$$

By the method specified in Section 2 the following results are obtained.

6.1 P-transformation of deflection data $y(x_i)$

Absolute conditioning number has the form of

$$v_\Delta = \sum_{i=1}^{2} |L_i|. \tag{34}$$

Substituting Eq. (33) in Eq. (34) yields the absolute conditioning number of the problem

$$v_\Delta = 2\frac{x_1^3 + x_2^3}{(x_2 - x_1)x_1^2 x_2^2}. \tag{35}$$

Function as per Eq. (35) at $[x_a, x_b]$ takes the least value when x_2=x_b. Then, from the equation $(v_\Delta)'_{x_i} = 0$ we obtain cubic equation

$$x_1^3 + 3x_b^2 x_1 - 2x_b^3 = 0$$

with Cardano solution

$$x_1 = \left(\sqrt[3]{1+\sqrt{2}} + \sqrt[3]{1-\sqrt{2}}\right)x_b \approx 0.596x_b \approx 0.566l. \tag{36}$$

Substitution of x_2=x_b and Eq. (36) in Eqs. (33)-(34) yields the minimum absolute condition number of the reduction problem of $18.7/l^2$.

In case of no optimization, in particular, for coordinate x_1 when $0.9>x_1>0.14$ absolute condition number increases more than by 4.6 times (it becomes more than $87l^2$) as shown in Fig. 6, where $V=l^2 v_\Delta$ is the reduced condition number of the reduction problem.

Relative condition number of the reduction problem has the following form

$$v_\delta = \frac{x_2^2}{2}\left(1 - \frac{x_2}{3l}\right)\sum_{i=1}^{2} |L_i|. \tag{37}$$

Substitution of Eq. (33) in Eq. (37) yields the relative condition number of the reduction problem

$$v_\delta = \left(1 - \frac{x_2}{3l}\right)\frac{x_1^3 + x_2^3}{(x_2 - x_1)x_1^2}. \tag{38}$$

Regularization of Eq. (38) yields the minimum relative condition number of the reduction problem of 5.77 at the distribution of data points $x_2 = x_b$ and by Eq. (36).

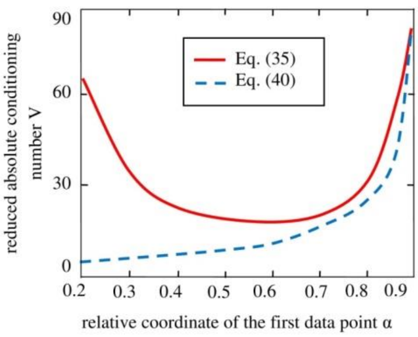

Fig. 6 Influence on accuracy of the first data point coordinate

6.2 Pi-transformation of deflection data $y(x_i)$

Absolute condition number of the reduction problem has the following form

$$v_\Delta = \frac{1}{x_2^2\left(1-\frac{x_2}{3l}\right)}\sum_{i=1}^{2}\left|L_i x_i^2\left(1-\frac{x_i}{3l}\right)\right|. \quad (39)$$

Substitution of Eq. (33) in Eq. (39) yields the absolute condition number of the reduction problem

$$v_\Delta = \frac{2}{\left(1-\frac{x_2}{3l}\right)(x_2-x_1)x_2^2}\left(x_2+x_1-\frac{2x_2x_1}{3l}\right). \quad (40)$$

Regularization gives minimum absolute condition number of the reduction problem of $3.49/l^2$ at distribution of data points by Eq. (11).

In case there is no optimization, in particular, for coordinate x_1 when $x_1>0.7$ the absolute condition number of the reduction problem increases more than by 4.5 times (it becomes more than $15.6/l^2$) as shown in Fig. 6.

Regularization of the relative conditioning number

$$v_\delta = \frac{1}{2}\sum_{i=1}^{2}\left|L_i x_i^2\left(1-\frac{x_i}{3l}\right)\right|$$

gives minimum relative condition number of the reduction problem 3.40 at data points distribution by Eq. (11).

7. Comparison of the results of the solutions of the problem

In the problem in Sub-section 5.1 sensors with accuracy class 0.5 (the reduced uncertainty of measurements is 0.5%) are applied. The values of uncertainty of signal transmission in the communication channel and method of measurement are not taken into account.

In case of *p*-transformation, minimum relative condition number of the reduction problem is 11.2. For maximum pre-calculated deflection, sensors with upper limit of measurement of 2 mm and absolute uncertainty of measurement of 0.01 mm were chosen. The reduced uncertainty of determination of the support moment can amount 0.5%×11.2=5.6%.

In case of *pi*-transformation, relative condition number of the reduction problem is 1.72. According to pre-calculated deflection for optimum coordinate distribution of data points by Eq. (29) sensors with different upper limits of measurements (2 mm and less) were selected. Reduced measurement error of determination of support moment 5%×1.72=0.86% can be reached. The result is - the second variant is 6.5 times better that the first one.

Liew and Choo (2004) studied approximation with a uniform grid and excessive number of data points - six, seven. The increase of the number of data points for the set class of accuracy of sensors can lead to inadmissible values of the class of accuracy of the measurement system on the whole and requires preliminary numerical estimate. Approximation by Eq. (7) with a uniform grid at the distribution of sensors along the beam in case of slight redundancy (n=5) is

$$y''(0) = \frac{1}{24h^2}\left[70y^*(x_1) - 208y^*(x_2) + 228y^*(x_3) - 112y^*(x_4) + 22y^*(x_5)\right],$$

where $h=l/4$ has a minimum absolute condition number of the reduction problem of $427/l^2$. This value corresponds to an unacceptable in practice class of accuracy of at least 20 of a measurement system for determination of support moment of a cantilever beam with sensors of high class of accuracy of 0.5.

8. Conclusions

There were obtained solutions of the problem of experimental-computation determination of the support moment of a cantilever beam loaded with concentrated force at its end including optimum choice of coordinates of distribution of sensors along the beam and data transformation parameters at insufficient accuracy of initial parameters and length of cantilever beam assignment.

- Suitable for engineering evaluation formulae of estimation of error of determination of a support moment of a cantilever beam at specified sensors inaccuracy were obtained.
- The proposed method depending on the required level of uncertainty of determination of sought function with some known restrictions for the level of experimental base allows taking decision about the choice of the type of approximation with uniform distribution of sensors along the beam or using regularization to increase the accuracy, and about the use of comparably simple aggregation of p-transformation sensors or use of more complicated for the implementation of aggregation of pi-transformation sensors to decrease the value of minimum condition number of the reduction problem.

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