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A nonlocal quasi-3D theory for bending and free flexural vibration behaviors of functionally graded nanobeams

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Abstract. In this paper, size dependent bending and free flexural vibration behaviors of functionally graded (FG) nanobeams are investigated using a nonlocal quasi-3D theory in which both shear deformation and thickness stretching effects are introduced. The nonlocal elastic behavior is described by the differential constitutive model of Eringen, which enables the present model to become effective in the analysis and design of nanostructures. The present theory incorporates the length scale parameter (nonlocal parameter) which can capture the small scale effect, and furthermore accounts for both shear deformation and thickness stretching effects by virtue of a hyperbolic variation of all displacements through the thickness without using shear correction factor. The material properties of FG nanobeams are assumed to vary through the thickness according to a power law. The neutral surface position for such FG nanobeams is determined and the present theory based on exact neutral surface position is employed here. The governing equations are derived using the principal of minimum total potential energy. The effects of nonlocal parameter, aspect ratio and various material compositions on the static and dynamic responses of the FG nanobeam are discussed in detail. A detailed numerical study is carried out to examine the effect of material gradient index, the nonlocal parameter, the beam aspect ratio on the global response of the FG nanobeam. These findings are important in mechanical design considerations of devices that use carbon nanotubes.

Keywords: nanobeam; nonlocal elasticity theory; bending; vibration; stretching effect; functionally graded materials

1. Introduction

Structural elements such as the nano-scale beams have attracted attention of scientific community in solid-state physics, materials science, and nano-electronics due to their superior mechanical, chemical and electronic properties. Conducting experiments with nano-scale size specimens is both expensive and difficult. Hence, development of appropriate mathematical models for nanostructures is an important issue concerning the application of nanostructures. The nanostructures is modeled into three main categories using atomistic (Ball 2001, Baughman *et al*. 2002), hybrid atomistic-continuum mechanics (Bodily and Sun 2003, Li and Chou 2003a,b) and continuum mechanics (Eringen 1972, Eringen and Edelen 1972). Continuum mechanics approach is less computationally expensive than the former two approaches. Further, it has been found that continuum mechanics results are in good agreement with those obtained from atomistic and hybrid approaches. Further, it has been found that continuum mechanics results are in good agreement with those obtained from atomistic and hybrid approaches. Due to the presence of small scale effects at the nano scale, size-dependent continuum mechanics models such as the strain gradient theory (Nix and Gao 1998), couple stress theory (Hadjesfandiari and Dargush 2011), modified couple stress theory (Asghari and Ahmadian 2010, Ma and Reddy 2008, Reddy 2011), and nonlocal elasticity theory (Eringen 1972, Eringen and Edelen 1972, Eringen 1983) are used. Among these theories, the nonlocal elasticity theory initiated by Eringen (1983) is widely used.

Introduced by Eringen (1983), nonlocal elasticity can successfully account for the scale effect in elasticity and has been shown to effectively simulate many complex phenomena in multi-scale mechanics including lattice dispersion of elastic waves, wave propagation in composites, dislocation mechanics, fracture mechanics and surface tension effects in fluids. Peddieson *et al*. (2003) first applied the nonlocal Eringen elasticity theory (Eringen 1983) to nanotechnology and derived expressions for the static deformations of beam structures based on a simplified nonlocal beam model. Subsequently, based on the nonlocal constitutive relation of Eringen, numerous studies have

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appeared which have developed nonlocal beam models for predicting the responses of nanostructures. These investigations include static analysis (Wang and Liew 2007, Pijaudier-Cabot and Bazant 1987, Lim and Wang 2007, Reddy and Pang 2008), buckling calculations (Zhang *et al*. 2004, Zhang *et al*. 2006, Wang *et al*. 2006, Tounsi *et al*. 2013 a, b, Semmah *et al*. 2014, Zidour *et al*. 2014, Chemi *et al*. 2015), vibration modelling (Yoon *et al*. 2003, Zhang *et al.* 2005 a, b, Rakrak *et al*. 2016), wave propagation simulations (Lu *et al*. 2007, Tounsi *et al.* 2008, Heireche *et al*. 2008, Song *et al*. 2010, Narendar and Gopalakrishnan, 2011) and thermo-mechanical (Mustapha and Zhong, 2010a, Maachou *et al*. 2011, Zidour *et al*. 2012, Bensattalah *et al.* 2016) computations of nanostructures. Recently, Mustapha and Zhong (2010b) investigated the free vibration of an axially-loaded non-prismatic single-walled carbon nanotube embedded in a two-parameter elastic medium with a Bubnov–Galerkin method. Roque *et al.* (2011) used the nonlocal elasticity theory of Eringen to study bending, buckling and free vibration of Timoshenko nanobeams with a meshless numerical method. Reddy (2007) implemented a range of different beam theories including those of Euler–Bernoulli, Timoshenko, Levinson (1981) and Reddy (1984) to simulate bending, buckling and vibration of nonlocal beams. Benguediab *et al.* (2013) proposed a comprehensive nonlocal shear deformation beam theory for bending, buckling and vibration analysis of homogeneous nanobeams founded on Eringen's nonlocal elasticity theory. Berrabah *et al.* (2013) presented a unified nonlocal shear deformation theory to study bending, buckling and free vibration of nanobeams. Recently, Eltaher *et al*. (2016) studied the static stability of nonlocal nanobeams using higher-order beam theories.

With the development of the material technology, functionally graded materials FGM are extensively used. The mechanical and thermal response of such materials with spatial gradients in composition and microstructure is of considerable interest in numerous technological areas such as tribology, optoelectronics, biomechanics, nanotechnology and high temperature technology. In the last few years, a great deal of researches in explanation simple and FGM plates using higher order theories have been presented (Bouderba *et al.* 2013, Tounsi *et al*. 2013c, Ait Amar Meziane *et al*. 2014, Zidi *et al*. 2014, Ait Yahia *et al*. 2015, Attia *et al*. 2015, Ait Atmane *et al*. 2015, Beldjelili *et al*. 2016, Bouderba *et al*. 2016, Boukhari *et al*. 2016, Bousahla *et al*. 2016, Houari *et al*. 2016, Tounsi *et al*. 2016). These studies have tried to demonstrate various behaviors in considering bending, linear vibration, buckling as well as wave propagation. In recent years, the application of FGMs has broadly been spread in micro-and nano-scale devices and systems such as thin films (Fu *et al*. 2003, Lu *et al*. 2011), atomic force microscopes (Rahaeifard *et al*. 2009), micro- and nano-electro-mechanical systems (MEMS and NEMS) (Witvrouw and Mehta 2005, Lee *et al*. 2006). In such applications, size effects have been experimentally observed (Fleck *et al*. 1994, Stolken and Evans 1998, Chong *et al*. 2001, Lam *et al.* 2003), and conventional beam models based on classical continuum theories do no account for such size effects due to lack of material length scale parameters. So far, only a few works have been reported for functionally graded (FG) nanobeams based on the nonlocal elasticity theory. Janghorban and Zare (2011) investigated nonlocal free vibration axially FG nanobeams by using differential quadrature method. Eltaher *et al.* (2012) studied free vibration of FG nanobeam based on the nonlocal Euler–Bernoulli beam theory. Eltaher *et al.* (2014) analyzed the vibration of nonlinear graduation of nano-Timoshenko beam considering the neutral axis position. Belkorissat *et al*. (2015) examined vibration properties of FG nano-plate using a new nonlocal refined four variable model. Larbi Chaht *et al*. (2015) analyzed the bending and buckling response of size-dependent nanoscale FG beams including the thickness stretching effect. Ebrahimi and Salari (2016) examined the thermal loading effects on electro-mechanical vibration behavior of piezoelectrically actuated size-dependent Timoshenko FG nanobeams. Ebrahimi and Barati (2016a) presented an analytical solution for nonlocal buckling characteristics of higher-order inhomogeneous nanosize beams embedded in elastic medium. Ehyaei *et al*. (2016) discussed the nonlocal vibration analysis of FG nanobeams with different boundary conditions. Ebrahimi and Barati (2016a) studied the buckling response of embedded piezo- electro-magnetically actuated nanoscale beams. Bounouara *et al*. (2016) employed a nonlocal zeroth-order shear deformation theory for free vibration of FG nanoscale plates resting on elastic foundation. Ahouel *et al.* (2016) investigated the size-dependent mechanical behavior of FG trigonometric shear deformable nanobeams including neutral surface position concept.

In this paper, size-dependent functionally graded higher order beam model is developed to account for the size effect, so-called ‘‘*stretching effect*’’, and material variation through the thickness of the beam. The axial and transverse displacements are supposed to be hyperbolic variation through the thickness according to the same assumptions considered by Hebali *et al*. (2014) and Belabed *et al*. 2014). The material properties of the FG nanobeam are assumed to vary in the thickness direction. Since, the material properties of FG beam vary through the thickness direction, the neutral plane of such plate may not coincide with its geometric middle plane (Yahoobi and Feraidoon 2010). In addition, Ould Larbi *et al*. (2013), Bouremana et al. (2013), Said *et al*. (2014), Khalfi *et al.* (2014) and Bousahla *et al*. (2014) show that the stretching – bending coupling in the constitutive equations of an FG structures does not exist when the coordinate system is located at the physical neutral surface of the structure. Therefore, the governing equations for the FG beam can be simplified. Based on the present nonlocal shear and normal deformation theory and the exact position of neutral surface together with the Hamilton's principle, the equations of motion of the FG nanobeams are obtained. Analytical solutions for the static bending and free vibration problems are presented for a simply supported beam to bring out the effects of both material length scale parameter and the thickness stretching on the deflection and frequency. Since most nanoscale devices involve beam-like elements that may be functionally graded and undergo moderately large rotations, the newly developed beam models can be used to capture the size effects in functionally graded nanobeams.

2. Mathematical formulations

2.1 Material properties

Consider a uniform FG nanobeam of thickness h, length L, and width b made by mixing two distinct materials (metal and ceramic). The coordinate x is along the longitudinal direction and z is along the thickness direction. For such beams, the neutral axis may not coincide with its geometric mid-axis (Ould Larbi *et al*. 2013, Yahoobi and Feraidoon 2010, Bourada *et al.* 2015, Al-Basyouni *et al*. 2015, Mouaici *et al*. 2016, Bellifa *et al.* 2016) as shown in Fig. 1. Indeed, since in FG beams the condition of mid-axis symmetry does not exist, the stretching and bending equations are coupled. But, if the origin of the coordinate system is suitably selected in the thickness direction of the FG beam so as to be the neutral axis, the analysis of the FG beams can easily be treated with the homogenous isotropic beam theories, because the stretching and bending equations of the beam are not coupled.

Here, two different datum axes are employed for the measurement of z, namely, z_{ms} and z_{ns} measured from the middle surface, and the neutral surface of the beam, respectively (Fig. 1). The volume-fraction of ceramic V_C is expressed based on z_{ms} and z_{ns} coordinates (Fig. 1) as

$$P(z_{ns}) = (P_t - P_b)\left(\frac{z_{ns}+C}{h}+\frac{1}{2}\right)^k + P_b \tag{1}$$

where P_t and P_b are the corresponding material property at the top and bottom surfaces of the nanoscale beam. k is the material distribution parameter which takes the value greater or equal to zero

The position of the neutral axis of the FG nanoscale beam is determined to satisfy the first moment with respect to Young's modulus being zero as follows (Ould Larbi *et al*. 2013)

$$\int_{-h/2}^{h/2} E(z_{ms})(z_{ms}-C)dz_{ms} = 0 \tag{1}$$

Thus, the position of neutral axis can be obtained as

$$C = \frac{\int_{-h/2}^{h/2} E(z_{ms}) z_{ms} dz_{ms}}{\int_{-h/2}^{h/2} E(z_{ms}) dz_{ms}} \tag{3}$$

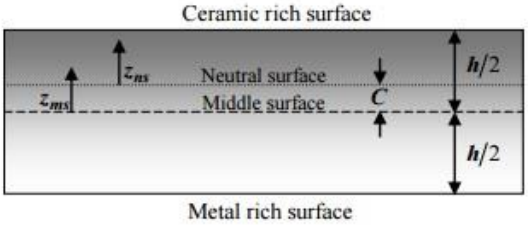

Fig. 1 The position of middle surface and neutral surface for a functionally graded beam

2.2 Kinematics

In order to incorporate both shear deformation and thickness stretching effects, the axial and transverse displacements are supposed to be hyperbolic variation through the thickness (Soldatos 1992, Bessaim *et al*. 2013, Hamidi *et al*. 2015, Meradjah *et al*. 2015, Bennoun *et al*. 2016):

Based on the assumptions made above, the displacement field of the present theory can be obtained as

$$u(x, z_{ns}, t) = u_0(x,t) - z_{ns}\frac{\partial w_b}{\partial x} - f(z_{ns})\frac{\partial w_s}{\partial x} \tag{4a}$$

$$w(x, z_{ns}, t) = w_b(x,t) + w_s(x,t) + w_{st}(x, z_{ns}, t) \tag{4b}$$

where u_0 is the axial displacement along the midplane of the nanoscale beam; w_b, w_s and w_{st} are the bending, shear and stretching components of transverse displacement along the midplane of the beam. Furthermore

$$f(z) = (z_{ns}+C) - h\sinh\left(\frac{z_{ns}+C}{h}\right) + (z_{ns}+C)\cosh\left(\frac{1}{2}\right) \tag{4c}$$

The component due to the stretching effect w_{st} can be given as

$$w_{st}(x, z_{ns}, t) = g(z_{ns})\,\varphi(x,t) \tag{4d}$$

The additional displacement φ accounts for the effect of normal stress is included and $g(z_{ns})$ is given as follows

$$g(z_{ns}) = 1 - f'(z_{ns}) \tag{4e}$$

The nonzero strains of the considered beam theory are

$$\varepsilon_x = \varepsilon_x^0 + z_{ns}\,k_x^b + f(z_{ns})\,k_x^s,\quad \gamma_{xz} = g(z_{ns})\,\gamma_{xz}^0 \text{ and } \varepsilon_z = g'(z_{ns})\,\varepsilon_z^0 \tag{5}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x},\quad k_x^b = -\frac{\partial^2 w_b}{\partial x^2},\quad k_x^s = -\frac{\partial^2 w_s}{\partial x^2}, \tag{6}$$

$$\gamma_{xz}^{0} = \frac{\partial w_s}{\partial x} + \frac{\partial \varphi}{\partial x}, \quad \varepsilon_z^{0} = \varphi$$

2.3 Nonlocal theory and constitutive relations

Behavior of materials at the nanoscale is different from those of their bulk counterparts. In the theory of nonlocal elasticity Eringen (1983), the stress at a reference point x is supposed to be a functional of the strain field at every point in the body. For example, in the non – local elasticity, the uniaxial constitutive law is written as (Eringen 1983)

$$\sigma_x - \mu \frac{d^2 \sigma_x}{dx^2} = Q_{11}\varepsilon_x + Q_{13}\varepsilon_z \tag{7a}$$

$$\tau_{xz} - \mu \frac{d^2 \tau_{xz}}{dx^2} = Q_{55}\gamma_{xz} \tag{7b}$$

$$\sigma_z - \mu \frac{d^2 \sigma_z}{dx^2} = Q_{13}\varepsilon_x + Q_{33}\varepsilon_z \tag{7c}$$

The Q_{ij} expressions in terms of engineering constants are

$$Q_{11}(z_{ns}) = Q_{33}(z_{ns}) = \frac{E(z_{ns})}{1-\nu^2}, \qquad Q_{13}(z_{ns}) = \nu\, Q_{11}(z_{ns}), \quad Q_{55}(z_{ns}) = \frac{E(z_{ns})}{2(1+\nu)} \tag{7d}$$

and $\mu = (e_0 a)^2$ is a nonlocal coefficient revealing the nanoscale influence on the behaviour of nanoscale beams, e_0 is a constant appropriate to each material and a is an internal characteristic length. In general, a conservative estimate of the nonlocal parameter is $e_0 a < 2.0$ nm for a single wall carbon nanotube (Wang 2005, Benzair *et al*. 2008, Heireche *et al*. 2008 a, b, c, Tounsi *et al*. 2008, Tounsi *et al*. 2013 b, c, d, Berrabah *et al.* 2013, Benguediab *et al*. 2014 a, b, Zidour *et al*. 2014, Semmah *et al*. 2014).

2.4 Equations of motion

In this section, the equations of motion are determined by employing Hamilton's principle as (Reddy 2002, Draiche *et al*. 2014, Nedri *et al*. 2014, Mahi *et al*. 2015, Bourada *et al*. 2016)

$$\int_{t_1}^{t_2} (\delta U + \delta V - \delta K)\, dt = 0 \tag{8}$$

where t is the time; t_1 and t_2 are the initial and end time, respectively; δU is the virtual variation of the strain energy; δV is the variation of work carried out by the applied forces; and δK is the virtual variation of the kinetic energy. The variation of the strain energy of the beam can be expressed as

$$\begin{aligned} \delta U &= \int_0^L \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} (\sigma_x \delta\varepsilon_x + \sigma_z \delta\varepsilon_z + \tau_{xz}\delta\gamma_{xz})\, dz_{ns}\, dx \\ &= \int_0^L \left(N \frac{d\delta u_0}{dx} + N_z \delta\varphi - M_b \frac{d^2 \delta w_b}{dx^2} - M_s \frac{d^2 \delta w_s}{dx^2} + Q\left[\frac{d\delta w_s}{dx} + \frac{d\delta\varphi}{dx} \right] \right) dx \end{aligned} \tag{9}$$

where N, M_b, M_s, N_z and Q are the stress resultants defined as

$$(N, M_b, M_s) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} (1, z_{ns}, f(z_{ns}))\, \sigma_x\, dz_{ns},$$
$$N_z = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \sigma_z g'(z_{ns})\, dz_{ns}, \text{ and } \quad Q = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \tau_{xz} g(z_{ns})\, dz_{ns} \tag{10}$$

The variation of work carried out by externally transverse loads q can be expressed as

$$\delta V = -\int_0^L q\,\delta(w_b + w_s)\, dx \tag{11}$$

The variation of the kinetic energy can be expressed as

$$\begin{aligned} \delta K &= \int_0^L \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \rho(z_{ns})[\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w}]\, dz_{ns}\, dx \\ &= \int_0^L \Big\{ I_0[\dot{u}_0 \delta\dot{u}_0 + (\dot{w}_b + \dot{w}_s)(\delta\dot{w}_b + \delta\dot{w}_s)] + J_0[(\dot{w}_b + \dot{w}_s)\delta\dot{\varphi} + \dot{\varphi}\,\delta(\dot{w}_b + \dot{w}_s)] \\ &\quad - I_1\left(\dot{u}_0 \frac{d\delta\dot{w}_b}{dx} + \frac{d\dot{w}_b}{dx}\delta\dot{u}_0 \right) + I_2\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\dot{w}_b}{dx} \right) - J_1\left(\dot{u}_0 \frac{d\delta\dot{w}_s}{dx} + \frac{d\dot{w}_s}{dx}\delta\dot{u}_0 \right) \\ &\quad + K_2\left(\frac{d\dot{w}_s}{dx}\frac{d\delta\dot{w}_s}{dx} \right) + J_2\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\dot{w}_s}{dx} + \frac{d\dot{w}_s}{dx}\frac{d\delta\dot{w}_b}{dx} \right) + K_0\dot{\varphi}\,\delta\dot{\varphi} \Big\} dx \end{aligned} \tag{12}$$

where dot-superscript convention denotes the differentiation with respect to the time variable t; and (I_i, J_i, K_i) are mass inertias expressed by

$$(I_0, I_1, I_2) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} (1, z_{ns}, z_{ns}^2)\rho(z_{ns})\, dz_{ns} \tag{13a}$$

$$(J_0, J_1, J_2) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} (g, f, z_{ns} f)\rho(z_{ns})\, dz_{ns} \tag{13b}$$

$$(K_0, K_2) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} (g^2, f^2)\rho(z_{ns})\, dz_{ns} \tag{13c}$$

Substituting the relations for δU, δV, and δK from Eqs. (9), (11), and (12) into Eq. (8) and integrating by parts, and collecting the coefficients of δu_0, δw_b, δw_s and $\delta \varphi$, the following equations of motion of the FG beam are found

$$\delta u_0: \ \frac{dN}{dx} = I_0 \ddot{u}_0 - I_1 \frac{d\ddot{w}_b}{dx} - J_1 \frac{d\ddot{w}_s}{dx} \tag{14a}$$

$$\delta w_b: \ \frac{d^2 M_b}{dx^2} + q = I_0(\ddot{w}_b + \ddot{w}_s) + J_0 \ddot{\varphi} + I_1 \frac{d\ddot{u}_0}{dx} - I_2 \frac{d^2 \ddot{w}_b}{dx^2} - J_2 \frac{d^2 \ddot{w}_s}{dx^2} \tag{14b}$$

$$\delta w_s: \ \frac{d^2 M_s}{dx^2} + \frac{dQ}{\partial x} + q = I_0(\ddot{w}_b + \ddot{w}_s) + J_0 \ddot{\varphi} + J_1 \frac{d\ddot{u}_0}{dx} - J_2 \frac{d^2 \ddot{w}_b}{dx^2} - K_2 \frac{d^2 \ddot{w}_s}{dx^2} \tag{14c}$$

$$\delta \varphi: \ \frac{dQ}{dx} - N_z = J_0(\ddot{w}_b + \ddot{w}_s) + K_0 \ddot{\varphi} \tag{14d}$$

By virtue of Eqs. (5), (7), and (10), the *force–strain* and the *moment–strain relations* of the present nonlocal beam theory can be obtained as follows

$$N - \mu \frac{d^2 N}{dx^2} = A_{11} \frac{du_0}{dx} - B_{11}^s \frac{d^2 w_s}{dx^2} + X_{13} \varphi \tag{15a}$$

$$M_b - \mu \frac{d^2 M_b}{dx^2} = -D_{11} \frac{d^2 w_b}{dx^2} - D_{11}^s \frac{d^2 w_s}{dx^2} + Y_{13} \varphi \tag{15b}$$

$$M_s - \mu \frac{d^2 M_s}{dx^2} = B_{11}^s \frac{du_0}{dx} - D_{11}^s \frac{d^2 w_b}{dx^2} - H_{11}^s \frac{d^2 w_s}{dx^2} + Y_{13}^s \varphi \tag{15c}$$

$$Q - \mu \frac{d^2 Q}{dx^2} = A_{55}^s \left(\frac{dw_s}{dx} + \frac{d\varphi}{dx} \right) \tag{15d}$$

$$N_z - \mu \frac{d^2 N_z}{dx^2} = X_{13} \frac{du_0}{dx} - Y_{13} \frac{d^2 w_b}{dx^2} - Y_{13}^s \frac{d^2 w_s}{dx^2} + Z_{33} \varphi \tag{15e}$$

where

$$\left(A_{11}, D_{11}, B_{11}^s, D_{11}^s, H_{11}^s \right) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} Q_{11} \left(1, z_{ns}^2, f(z_{ns}), z_{ns}\, f(z_{ns}), f^2(z_{ns}) \right) dz_{ns} \tag{16a}$$

and

$$A_{55}^s = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} Q_{55} [g(z_{ns})]^2 dz_{ns},$$

$$\left[X_{13}, Y_{13}, Y_{13}^s \right] = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} Q_{13} [1, z_{ns}, f(z_{ns})] g'(z_{ns}) dz_{ns}, \tag{16b}$$

$$Z_{33} = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} Q_{33} [g'(z_{ns})]^2 dz_{ns},$$

By using Eqs. (15) and (14), the nonlocal governing equations can be written in terms of displacements (u_0, w_b, w_s, φ) as

$$\begin{aligned} A_{11} \frac{d^2 u_0}{dx^2} - B_{11}^s \frac{d^3 w_s}{dx^3} + X_{13} \frac{d\varphi}{dx} = I_0 \left(\ddot{u}_0 - \mu \frac{d^2 \ddot{u}_0}{dx^2} \right) \\ - I_1 \left(\frac{d\ddot{w}_b}{dx} - \mu \frac{d^3 \ddot{w}_b}{dx''} \right) - J_1 \left(\frac{d\ddot{w}_s}{dx} - \mu \frac{d^3 \ddot{w}_s}{dx''} \right) \end{aligned} \tag{17a}$$

$$\begin{aligned} - D_{11} \frac{d^4 w_b}{dx^4} - D_{11}^s \frac{d^4 w_s}{dx^4} + Y_{13} \frac{d^2 \varphi}{dx^2} + q - \mu \frac{d^2 q}{dx^2} = I_0 \left((\ddot{w}_b + \ddot{w}_s) - \mu \frac{d^2 (\ddot{w}_b + \ddot{w}_s)}{dx^2} \right) + \\ J_0 \left(\ddot{\varphi} - \mu \frac{d^2 \ddot{\varphi}}{dx^2} \right) + I_1 \left(\frac{d\ddot{u}_0}{dx} - \mu \frac{d^3 \ddot{u}_0}{dx^3} \right) - I_2 \left(\frac{d^2 \ddot{w}_b}{dx^2} - \mu \frac{d^4 \ddot{w}_b}{dx^4} \right) - J_2 \left(\frac{d^2 \ddot{w}_s}{dx^2} - \mu \frac{d^4 \ddot{w}_s}{dx^4} \right) \end{aligned} \tag{17b}$$

$$\begin{aligned} B_{11}^s \frac{d^3 u_0}{dx^3} - D_{11}^s \frac{d^4 w_b}{dx^4} - H_{11}^s \frac{d^4 w_s}{dx^4} + A_{55}^s \frac{d^2 w_s}{dx^2} + \left(A_{55}^s + Y_{13}^s \right) \frac{d^2 \varphi}{dx^2} + q - \mu \frac{d^2 q}{dx^2} = \\ I_0 \left((\ddot{w}_b + \ddot{w}_s) - \mu \frac{d^2 (\ddot{w}_b + \ddot{w}_s)}{dx^2} \right) + J_0 \left(\ddot{\varphi} - \mu \frac{d^2 \ddot{\varphi}}{dx^2} \right) + J_1 \left(\frac{d\ddot{u}_0}{dx} - \mu \frac{d^3 \ddot{u}_0}{dx^3} \right) - \\ J_2 \left(\frac{d^2 \ddot{w}_b}{dx^2} - \mu \frac{d^4 \ddot{w}_b}{dx^4} \right) - K_2 \left(\frac{d^2 \ddot{w}_s}{dx^2} - \mu \frac{d^4 \ddot{w}_s}{dx^4} \right) \end{aligned} \tag{17c}$$

$$\begin{aligned} - X_{13} \frac{du_0}{dx} + Y_{13} \frac{d^2 w_b}{dx^2} + \left(A_{55}^s + Y_{13}^s \right) \frac{\partial^2 w_s}{\partial x^2} + A_{55}^s \frac{\partial^2 \varphi}{\partial x^2} - Z_{33} \varphi = J_0 \left((\ddot{w}_b + \ddot{w}_s) - \mu \frac{d^2 (\ddot{w}_b + \ddot{w}_s)}{dx^2} \right) \\ + K_0 \left(\ddot{\varphi} - \mu \frac{d^2 \ddot{\varphi}}{dx^2} \right) \end{aligned} \tag{17d}$$

The equations of motion of local beam theory can be deduced from Eq. (17) by setting the nonlocal parameter μ equal to zero.

3. Analytical solution

The above equations of motion are analytically solved for bending and free vibration problems. The Navier solution technique is employed to obtain the analytical solutions for a simply supported FG nanobeam. The solution is assumed to be of the form

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \\ \varphi \end{Bmatrix} == \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\lambda\, x)\, e^{i \omega t} \\ W_{bm} \sin(\lambda\, x)\, e^{i \omega t} \\ W_{sm} \sin(\lambda\, x)\, e^{i \omega t} \\ \Phi_{stm} \sin(\lambda\, x)\, e^{i \omega t} \end{Bmatrix} \tag{18}$$

where U_m, W_{bm}, W_{sm} and Φ_{stm} are arbitrary parameters to be determined, ω is the eigenfrequency associated with m th eigenmode, and $\lambda = m\pi / L$. The transverse load q is also expanded in the Fourier sine series as

$$q(x) = \sum_{n=1}^{\infty} Q_n \sin \lambda\, x, \quad Q_n = \frac{2}{L} \int_0^L q(x) \sin(\lambda\, x)\, dx \tag{19}$$

The Fourier coefficients Q_n associated with some typical loads are given

$$Q_n = q_0, \quad n = 1 \quad \text{for sinusoidal load} \tag{20a}$$

$$Q_n = \frac{4 q_0}{n\pi}, \quad n = 1,3,5..... \quad \text{for uniform load} \tag{20b}$$

$$Q_n = \frac{2q_0}{L}\sin\frac{n\pi}{2}, \quad n = 1,2,3.... \text{ for point load } Q_0 \text{ at the midspan,} \tag{20c}$$

Substituting Eqs. (18) and (19) into Eq. (17), the analytical solutions can be obtained by

$$\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} - \alpha\omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{12} & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ 0 & m_{24} & m_{34} & m_{44} \end{bmatrix}\right)\begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \\ \Phi_{stm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \alpha Q_n \\ \alpha Q_n \\ 0 \end{Bmatrix} \tag{21}$$

where

$$a_{11} = A_{11}\lambda^2,\ a_{12} = 0,\ a_{13} = -B_{11}^s\lambda^3,$$
$$a_{14} = -X_{13}\lambda,\ a_{22} = D_{11}\lambda^4,\ a_{23} = D_{11}^s\lambda^4,$$
$$a_{24} = Y_{13}\lambda^2,\ a_{33} = A_{55}\lambda^2 + H_{11}\lambda^4,$$
$$a_{34} = \lambda^2 Y_{13}^s + A_{55}\lambda^2,\ a_{44} = \lambda^2 A_{55}^s + Z_{33},$$
$$\alpha = 1 + \mu\lambda^2,$$
$$m_{11} = -I_0,\ m_{12} = \lambda I_1,\ m_{13} = \lambda J_1, \tag{22}$$
$$m_{22} = -\left(I_0 + \lambda^2 I_2\right),$$
$$m_{23} = -\left(I_0 + \lambda^2 J_2\right), m_{24} = m_{34} = -J_0,$$
$$m_{33} = -\left(I_0 + \lambda^2 K_2\right),$$
$$m_{44} = -K_0$$

4. Numerical results

Numerical results presented in this section demonstrate the influence of the thickness stretching, nonlocal parameter, the material distribution parameter, and slenderness ratio on deflections and frequencies of FG nanobeams.

In the following investigation, two FG nanobeams are examined. The first FG nanobeam has the following material properties: $E_t = 0.25$ TPa , $E_b = 1$ TPa , $\nu_t = \nu_b = 0.3$ (Zemri *et al.* 2015). The second FG nanobeam is composed of steel and alumina (Al_2O_3). The bottom surface of the beam is pure steel, whereas the top surface of the beam is pure alumina. The material properties are as follows: $E_t = 390$ GPa , $E_b = 210$ GPa , $\rho_t = 3960$ kg/m^3 , $\rho_b = 7800$ kg/m^3 , $\nu_t = \nu_b = 0.3$ (Eltaher *et al.* 2012). For convenience, the following nondimensionalizations are employed

- $\bar{w} = 100 w \dfrac{E_t I}{q_0 L^4}$ for uniform load;
- $\bar{\omega} = \omega L^2 \sqrt{\dfrac{\rho_t A}{E_t I}}$ frequency parameter;

The results of reference are illustrated and discussed, to evaluate accuracy of deflection and frequencies predicted by the present theoretical formulation, the non-dimensional deflections and natural frequencies of simply supported FG nanobeam with various nonlocal scale parameters previously analyzed by Navier method are reexamined. Table 1 compares the non-dimensional maximum deflections $\bar{w}$ predicted by the present nonlocal quasi-3D theory and the results presented by Zemri *et al.* (2015) which have been obtained by a refined nonlocal shear deformation theory with different nonlocal scale parameter, material distribution parameter, and slenderness ratio. It should be pointed out that the thickness stretching effect is neglected in analytical formulation presented by Zemri *et al.* (2015). Consequently, it is observed from Table 1 that the present model without the thickness stretching effect ($\varepsilon_z = 0$), provides identical results to those of Zemri *et al.* (2015) for all values of thickness ratio, *L*/*h*, material distribution parameter k and nonlocal scale parameter ($e_0 a$). However, the results of the present model with the thickness stretching effect ($\varepsilon_z \neq 0$) demonstrate that the inclusion of the thickness stretching effect leads to a reduction in the magnitudes of deflection of FG nanobeams. Thus, with the thickness stretching effect incorporated, FG nano-scale beams exhibit greater stiffness, and this characteristic is particularly important in applications.

Table 2 documents the values for the computed non-dimensional frequencies. The present computations are benchmarked with the earlier results of Zemri *et al.* (2015) and good correlation is observed with the present model without the thickness stretching effect ($\varepsilon_z = 0$). The results obtained by utilizing the present neutral axis-based model with the thickness stretching effect ($\varepsilon_z \neq 0$) show that the inclusion of the thickness stretching effect manifests in an enhancement in the frequencies. According to this table, frequencies diminish with increasing nonlocal scale parameter ($e_0 a$). In addition, the increase of the material distribution parameter k leads to a decrease of frequencies.

Figs. 2 and 3 demonstrate the influence of slenderness ratio on the non-dimensional deflection and the frequency of the FG nanobeam. The local and nonlocal results are provided for $e_0 a$= 0 and $e_0 a$= 1 nm, respectively. The material distribution parameter is assumed to be constant i.e., k=1. In these examples, the aspect ratio varies from *L*/*h*=10 to *L*/*h*=50. It is found that deflections predicted by the nonlocal theory exceed in magnitude those calculated with the classical theory ($e_0 a = 0$). On the other hand, the nonlocal solution of the frequency is lower in magnitude than the ferquency due to the nonlocality effects. Also, it can be observed that the inclusion of the thickness stretching effect leads to a significant reduction in nanobeam deflection and an increase in frequency values for FG nanobeams. These results effectively prove that the incorporation of nonlocal scale parameter softens the nanobeam, whereas the incorporation of thickness stretching effect makes it stiffer. As such both nonlocality and thickness stretching influences exert a significant effect on nanobeam structural performance.

Table 1 Dimensionless transverse deflections ($\overline{w}$) of the FG nanobeam for uniform load

L/h	k	Nonlocal scale parameter, $e_0a(nm)$														
		0			0.5			1			1.5			2		
		Ref[a]	Present $\varepsilon_{zz}=0$	Present $\varepsilon_{zz}\neq 0$	Ref[a]	Present $\varepsilon_{zz}=0$	Present $\varepsilon_{zz}\neq 0$	Ref[a]	Present $\varepsilon_{zz}=0$	Present $\varepsilon_{zz}\neq 0$	Ref[a]	Present $\varepsilon_{zz}=0$	Present $\varepsilon_{zz}\neq 0$	Ref[a]	Present $\varepsilon_{zz}=0$	Present $\varepsilon_{zz}\neq 0$
10	0	5.3383	5.3383	5.3269	5.4659	5.4659	5.4543	5.8487	5.8487	5.8364	6.4867	6.4867	6.4732	7.3799	7.3799	7.3649
	0.3	3.2181	3.2181	3.1871	3.2951	3.2950	3.2632	3.5258	3.5258	3.4918	3.9104	3.9104	3.8728	4.4488	4.4488	4.4062
	1	2.4194	2.4194	2.3893	2.4773	2.4773	2.4464	2.6509	2.6508	2.6179	2.9401	2.9401	2.9037	3.3452	3.3451	3.3037
	3	1.9234	1.9234	1.9086	1.9694	1.9694	1.9543	2.1074	2.1075	2.0913	2.3375	2.3375	2.3197	2.6595	2.6595	2.6394
	10	1.5790	1.5790	1.5738	1.6168	1.6168	1.6115	1.7301	1.7301	1.7244	1.9189	1.9189	1.9127	2.1831	2.1832	2.1762
30	0	5.2228	5.2228	5.2215	5.2367	5.2367	5.2354	5.2785	5.2785	5.2771	5.3481	5.3481	5.3467	5.4455	5.4455	5.4452
	0.3	3.1475	3.1475	3.1219	3.1559	3.1559	3.1302	3.1811	3.1811	3.1552	3.2230	3.2230	3.1968	3.2818	3.2818	3.2550
	1	2.3732	2.3732	2.3471	2.3795	2.3795	2.3534	2.3985	2.3985	2.3721	2.4302	2.4301	2.4034	2.4744	2.4744	2.4472
	3	1.8892	1.8892	1.8780	1.8943	1.8943	1.8830	1.9094	1.9094	1.8980	1.9346	1.9346	1.9231	1.9698	1.9698	1.9581
	10	1.5488	1.5488	1.5467	1.5529	1.5529	1.5508	1.5653	1.5653	1.5632	1.5860	1.5860	1.5838	1.6149	1.6149	1.6127
100	0	5.2096	5.2096	5.2095	5.2109	5.2109	5.2107	5.2146	5.2146	5.2145	5.2209	5.2209	5.2207	5.2296	5.2296	5.2295
	0.3	3.1395	3.1395	3.1145	3.1403	3.1402	3.1152	3.1425	3.1425	3.1174	3.1463	3.1463	3.1212	3.1515	3.1516	3.1264
	1	2.36794	2.3679	2.3423	2.3685	2.3685	2.3429	2.3702	2.3702	2.3445	2.3731	2.3731	2.3474	2.3770	2.3770	2.3513
	3	1.8854	1.8854	1.8745	1.8858	1.8858	1.8750	1.8872	1.8872	1.8763	1.8894	1.8894	1.8786	1.8926	1.8926	1.8817
	10	1.5454	1.5454	1.5436	1.5458	1.5457	1.5440	1.5469	1.5469	1.5451	1.5487	1.5487	1.5469	1.5513	1.5513	1.5495

[a] Taken from Zemri *et al*. (2015)

The effect of both nonlocal scale parameter and slenderness ratio on the non-dimensional deflection and the frequency of the FG nanobeam, are illustrated in Figs. 4 and 5. The results in these figures are obtained by employing the present nonlocal quasi-3D theory. The material distribution parameter is assumed to be constant (i.e., k=1). These results demonstrate that the nanobeam has a nonlinear behaviour under the effect of the nonlocality, especially for small slenderness ratio. It can be concluded that FG nanobeam responses are slenderness ratio-dependent based on nonlocal elasticity.

To examine the effect of the material distribution parameter k on the bending and vibration responses of FG nanobeams, the transverse deflection $\overline{w}$, and frequency $\overline{\omega}$, respectively are plotted in Figs. 6 and7, respectively. It can be observed that increasing the material distribution parameter k , leads to a decrease in deflections and frequencies.

5. Conclusions

Bending and vibrational response of the FG nanoscale beams are studied on the basis of nonlocal elasticity formulation in conjunction with Navier analytical procedure. Eringen's theory of nonlocal elasticity together with a quasi-3D hyperbolic theory are employed to model the nanoscale beam. The position of neutral surface is found and the nonlocal hyperbolic shear and normal deformation theory based on neutral surface is adopted to determine the equations of motion of FG nanoscale beams. Exactitude of the results is examined via available data in the literature. It is concluded that the scale parameter, the material distribution parameter, and slenderness ratio play important roles in bending and dynamic response of FG nanoscale beams.

Table 2 Dimensionless fundamental frequency ($\bar{\omega}$) of the FG nanobeam

L/h	k	Nonlocal parameter, $e_0a(nm)$: 0			0.5			1			1.5			2		
		Ref[(a)]	Present $\varepsilon_{zz}=0$	Present $\varepsilon_{zz}\neq0$	Ref[(a)]	Present $\varepsilon_{zz}=0$	Present $\varepsilon_{zz}\neq0$	Ref[(a)]	Present $\varepsilon_{zz}=0$	Present $\varepsilon_{zz}\neq0$	Ref[(a)]	Present $\varepsilon_{zz}=0$	Present $\varepsilon_{zz}\neq0$	Ref[(a)]	Present $\varepsilon_{zz}=0$	Present $\varepsilon_{zz}\neq0$
10	0	9.7075	9.7075	9.6837	9.5899	9.5899	9.5664	9.2612	9.2612	9.2385	8.7813	8.7813	8.7598	8.2197	8.2197	8.1995
	0.3	8.1709	8.1709	8.1557	8.0719	8.0719	8.0569	7.7952	7.2704	7.7807	7.3913	7.3913	7.3775	6.9185	6.9185	6.9057
	1	6.8814	6.8814	6.8744	6.7981	6.7981	6.7911	6.5651	6.5651	6.5583	6.2249	6.2249	6.2185	5.8267	5.8267	5.8208
	3	6.0755	6.0755	6.0663	6.0019	6.0019	5.9929	5.7962	5.7962	5.7875	5.4959	5.4958	5.4876	5.1443	5.1444	5.1366
	10	5.5768	5.5768	5.5624	5.5092	5.5092	5.4950	5.3204	5.3204	5.3067	5.0447	5.0447	5.0317	4.7221	4.7221	4.7099
30	0	9.8511	9.8511	9.8506	9.8376	9.8376	9.8371	9.7975	9.7975	9.7970	9.7318	9.7318	9.7313	9.6419	9.6419	9.6414
	0.3	8.2902	8.2902	8.2929	8.2788	8.2789	8.2816	8.2451	8.2451	8.2478	8.1898	8.1898	8.1925	8.1141	8.1141	8.1168
	1	6.9832	6.9833	6.9923	6.9737	6.9737	6.9828	6.9452	6.9453	6.9543	6.8987	6.8987	6.9076	6.8349	6.8350	6.8439
	3	6.1712	6.1712	6.1795	6.1627	6.1627	6.1710	6.1376	6.1376	6.1458	6.0964	6.0964	6.1046	6.0401	6.0401	6.0482
	10	5.6655	5.6655	5.6676	5.6578	5.6578	5.6599	5.6347	5.6347	5.6368	5.5969	5.5969	5.5990	5.5452	5.5452	5.5473
100	0	9.8679	9.8679	9.8680	9.8667	9.8667	9.8668	9.8631	9.8631	9.8631	9.8570	9.8570	9.8571	9.8485	9.8485	9.8486
	0.3	8.3042	8.3042	8.3073	8.3032	8.3032	8.3063	8.3001	8.3001	8.3032	8.2950	8.2950	8.2981	8.2878	8.2878	8.2910
	1	6.9952	6.9952	7.0047	6.9943	6.9943	7.0038	6.9917	6.9917	7.0012	6.9874	6.9874	6.9969	6.9814	6.9814	6.9909
	3	6.1824	6.1824	6.1912	6.1817	6.1817	6.1905	6.1794	6.1794	6.1882	6.1756	6.1756	6.1844	6.1703	6.1703	6.1790
	10	5.6759	5.6759	5.6786	5.6752	5.6752	5.6779	5.6731	5.6731	5.6758	5.6697	5.6697	5.6723	5.6648	5.6648	5.6674

[(a)] Taken from Zemri *et al*. (2015)

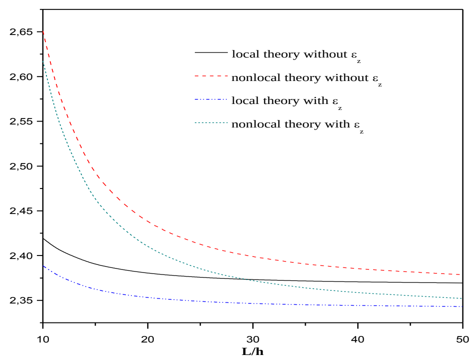

Fig. 2 Effect of the aspect ratio on dimensionless deflection ($\bar{w}$) for uniform load with $k=1$ and $e_0a=1$ nm

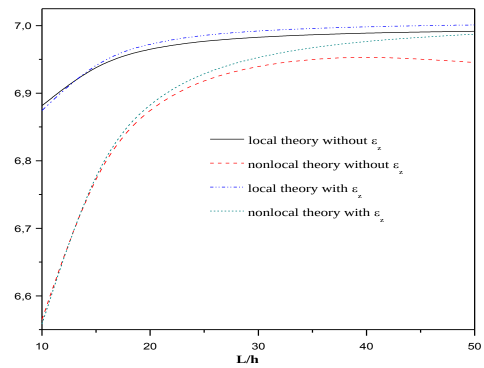

Fig. 3 Effect of the aspect ratio on dimensionless frequency ($\bar{\omega}$) with $k=1$ and $e_0a=1$ nm

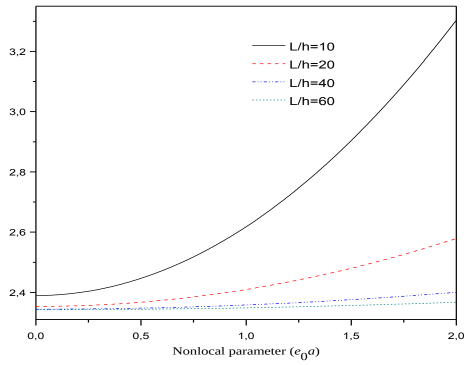

Fig. 4 Effect of nonlocal parameter on dimensionless deflection ($\bar{w}$) for uniform load with $k=1$

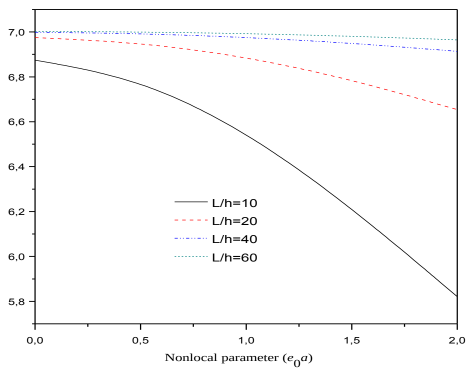

Fig. 5 Effect of nonlocal parameter on dimensionless frequency ($\bar{\omega}$) with $k=1$

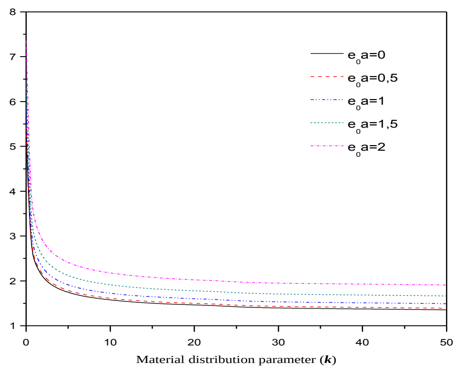

Fig. 6 Effect of the material distribution parameter on dimensionless deflection ($\bar{w}$) for uniform load with $L/h=10$

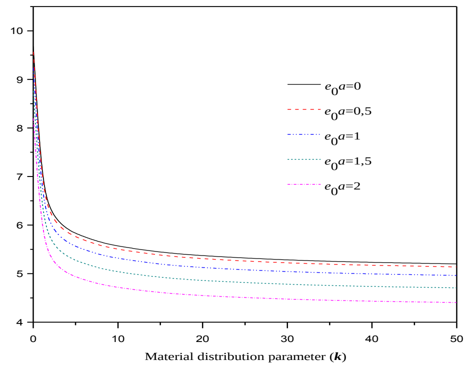

Fig. 7 Effect of the material distribution parameter on dimensionless frequency ($\bar{\omega}$) with $L/h=10$

References

Ahouel, M., Houari, M.S.A., Adda Bedia, E.A. and Tounsi, A. (2016), "Size-dependent mechanical behavior of functionally graded trigonometric shear deformable nanobeams including neutral surface position concept", *Steel Compos. Struct.*, **20**(5), 963-981.

Ait Amar Meziane, M., Abdelaziz, H.H. and Tounsi, A. (2014), "An efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions", *J. Sandw. Struct. Mater.*, **16**(3), 293-318.

Ait Atmane, H., Tounsi, A., Bernard, F. and Mahmoud, S.R. (2015), "A computational shear displacement model for vibrational analysis of functionally graded beams with porosities", *Steel Compos. Struct.*, **19**(2), 369-384.

Ait Yahia, S., Ait Atmane, H., Houari, M.S.A. and Tounsi, A. (2015), "Wave propagation in functionally graded plates with porosities using various higher-order shear deformation plate theories", *Struct. Eng. Mech.*, **53**(6), 1143-1165.

Al-Basyouni, K.S., Tounsi, A. and Mahmoud, S.R. (2015), "Size dependent bending and vibration analysis of functionally graded micro beams based on modified couple stress theory and neutral surface position", *Compos. Struct.*, **125**, 621-630.

Asghari, M., Kahrobaiyan, M.H. and Ahmadian, M.T. (2010), "A nonlinear Timoshenko beam formulation based on the modified couple stress theory", *Int. J. Eng. Sci.*, **48**, 1749-1761.

Attia, A., Tounsi, A., Adda Bedia, E.A. and Mahmoud, S.R. (2015), "Free vibration analysis of functionally graded plates with temperature-dependent properties using various four variable refined plate theories", *Steel Compos. Struct.*, **18**(1), 187-212.

Ball, P. (2001), "Roll up for the revolution", *Nature* (London), **414**, 142-144.

Baughman, RH, Zakhidov, AA. and de Heer, WA. (2002), "Carbon nanotubes the route towards applications", *Science*, **297**, 787-792.

Belabed, Z., Houari, M.S.A., Tounsi, A., Mahmoud, S.R. and Anwar Bég, O. (2014), "An efficient and simple higher order shear and normal deformation theory for functionally graded material (FGM) plates", *Compos. : Part B*, **60**, 274-283.

Beldjelili, Y., Tounsi, A. and Mahmoud, S.R. (2016), "Hygro-thermo-mechanical bending of S-FGM plates resting on variable elastic foundations using a four-variable trigonometric plate theory", *Smart Struct. Syst.*, **18**(4). 755-786.

Belkorissat, I., Houari, M.S.A., Tounsi, A., Adda Bedia, E.A. and Mahmoud, S.R. (2015), "On vibration properties of functionally graded nano-plate using a new nonlocal refined four variable model", *Steel Compos. Struct.*, **18**(4), 1063-1081.

Benguediab, S., Semmah, A., Larbi Chaht, F. Mouaz, S. and Tounsi, A. (2014a), "An investigation on the characteristics of bending, buckling and vibration of nanobeams via nonlocal beam theory", *Int. J. Comput. Meth.*, **11**(6), 1350085.

Benguediab, S., Tounsi, A., Zidour, M. and Semmah, A. (2014b), "Chirality and scale rffects on mechanical buckling properties of zigzag double-walled carbon nanotubes", *Compos. : Part B*, **57**, 21-24.

Bellifa, H., Benrahou, K.H., Hadji, L., Houari, M.S.A. and Tounsi, A. (2016), "Bending and free vibration analysis of functionally graded plates using a simple shear deformation theory and the concept the neutral surface position", *J. Braz. Soc. Mech. Sci. Eng.*, **38**(1), 265-275.

Bennoun, M., Houari, M.S.A. and Tounsi, A. (2016), "A novel five variable refined plate theory for vibration analysis of functionally graded sandwich plates", *Mech. Adv. Mater. Struct.*, **23**(4), 423-431.

Bensattalah, T., Daouadji, T.H., Zidour, M., Tounsi, A. and Adda Bedia, E.A. (2016), "Investigation of thermal and chirality effects on vibration of single-walled carbon nanotubes embedded in a polymeric matrix using nonlocal elasticity theories", *Mech. Compos. Mater.*, [In press].

Benzair, A., Tounsi, A., Besseghier, A., Heireche, H., Moulay, N., and Boumia, L. (2008), "The thermal effect on vibration of single-walled carbon nanotubes using nonlocal Timoshenko beam theory", *J. Phys, D: Appl, Phys.*, **41**, 225404.

Berrabah, H.M., Tounsi, A., Semmah, A. and Adda Bedia, E.A. (2013), "Comparison of various refined nonlocal beam theories for bending, vibration and buckling analysis of nanobeams", *Struct. Eng. Mech.*, **48**(3), 351-365.

Bessaim, A., Houari, M.S.A., Tounsi, A., Mahmoud, S.R. and Adda Bedia, E.A. (2013), "A new higher-order shear and normal deformation theory for the static and free vibration analysis of sandwich plates with functionally graded isotropic face sheets", *J. Sandw. Struct. Mater.*, **15**(6), 671-703.

Bodily, B.H. and Sun, C.T. (2003), "Structural and equivalent continuum properties of single walled carbon nanotubes", *Int. J. Mater. Prod. Technol.*, **18**, 381-397.

Bouderba, B., Houari, M.S.A. and Tounsi, A. (2013), "Thermomechanical bending response of FGM thick plates resting on Winkler–Pasternak elastic foundations", *Steel Compos. Struct.*, **14**(1), 85-104.

Bouderba, B., Houari, M.S.A. and Tounsi, A. and Mahmoud, S.R. (2016), "Thermal stability of functionally graded sandwich plates using a simple shear deformation theory", *Struct. Eng. Mech.*, **58**(3), 397-422.

Boukhari, A., Ait Atmane, H., Tounsi, A., Adda Bedia, E.A. and Mahmoud, S.R. (2016), "An efficient shear deformation theory for wave propagation of functionally graded material plates", *Struct. Eng. Mech.*, **57**(5), 837-859.

Bounouara, F., Benrahou, K.H., Belkorissat, I. and Tounsi, A. (2016), "A nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation", *Steel Compos. Struct.*, **20**(2), 227-249.

Bourada, M., Kaci, A., Houari, M.S.A. and Tounsi, A. (2015), "A new simple shear and normal deformations theory for functionally graded beams", *Steel Compos. Struct.*, **18**(2), 409-423.

Bourada, F., Amara, K. and Tounsi, A. (2016), "Buckling analysis of isotropic and orthotropic plates using a novel four variable refined plate theory ", *Steel Compos. Struct.*, **21**(6). 1287-1306.

Bouremana, M., Houari, M.S.A., Tounsi, A., Kaci, A. and Adda Bedia, E.A. (2013), "A new first shear deformation beam theory based on neutral surface position for functionally graded beams", *Steel Compos. Struct.*, **15**(5), 467-479.

Bousahla, A.A., Houari, M.S.A., Tounsi, A. and Adda Bedia, E.A., (2014), "A novel higher order shear and normal deformation theory based on neutral surface position for bending analysis of advanced composite plates", *Int. J. Comput. Meth.*, **11**(6), 1350082.

Bousahla, A.A., Benyoucef, S. Tounsi, A. and Mahmoud, S.R. (2016), "On thermal stability of plates with functionally graded coefficient of thermal expansion", *Struct. Eng. Mech.*, **60**(2), 313-335.

Chemi, A., Heireche, H., Zidour, M., Rakrak, K. and Bousahla, A.A. (2015), "Critical buckling load of chiral double-walled carbon nanotube using non-local theory elasticity", *Adv. Nano Res.*, **3**(4), 193-206.

Chong, ACM, Yang, F, Lam, DCC, Tong, P. (2001), "Torsion and bending of micron-scaled structures", *J Mater Res*, **16(04)**, 1052–1058.

Draiche, K., Tounsi, A. and Khalfi, Y. (2014), "A trigonometric four variable plate theory for free vibration of rectangular composite plates with patch mass", *Steel Compos. Struct.*, **17**(1), 69-81.

Ebrahimi, F. and Salari, E. (2016), "Thermal loading effects on electro-mechanical vibration behavior of piezoelectrically actuated inhomogeneous size-dependent Timoshenko nanobeams", *Adv. Nano Res.*, **4**(3), 197-228.

Ebrahimi, F. and Barati, R. (2016a), "Analytical solution for nonlocal buckling characteristics of higher-order inhomogeneous nanosize beams embedded in elastic medium", *Adv. Nano Res.*, **4**(3), 229 -249.

Ebrahimi, F. and Barati, R. (2016b), "An exact solution for buckling analysis of embedded piezo- electro-magnetically actuated nanoscale beams", *Adv. Nano Res.*, **4**(2), 65-84.

Ehyaei, J., Ebrahimi, F. and Salari, E. (2016), "Nonlocal vibration analysis of FG nano beams with different boundary conditions", *Adv. Nano Res.*, **4**(2), 85-111.

Eltaher, M.A., Khater, M.E., Park, S., Abdel-Rahman, E. and Yavuz, M. (2016), "On the static stability of nonlocal nanobeams using higher-order beam theories", *Adv. Nano Res.*, **4**(1), 51-64.

Eltaher, M.A., Emam, S.A. and Mahmoud, F.F. (2012), "Free vibration analysis of functionally graded size-dependent nanobeams", *Appl Math. Comput.*, **218**, 7406-7420.

Eltaher, M. A., Abdelrahman, A. A., Al-Nabawy, A., Khater, M. and Mansour, A. (2014), "Vibration of nonlinear graduation of nano-Timoshenko beam considering the neutral axis position", *Appl. Math. Comput.*, **235**, 512-529.

Eringen, AC. (1972), "Nonlocal polar elastic continua", *Int. J. Eng. Sci.*, **10**, 1-16.

Eringen, A.C. and Edelen, D.G.B. (1972), "On nonlocal elasticity", *Int. J. Eng. Sci.*, **10**, 233-248.

Eringen, A.C. (1983), "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves", *J. Appl. Phys.*, **54,** 4703-4710.

Fleck, N.A, Muller, G.M., Ashby, M.F. and Hutchinson, JW. (1994), "Strain gradient plasticity: theory and experiment", *Acta. Metall. Mater.*, **42**(2), 475-487.

Fu, Y., Du, H. and Zhang, S. (2003), "Functionally graded TiN/TiNi shape memory alloy films", *Mater. Lett*, **57**(20), 2995-2999.

Hadjesfandiari, A.R. and Dargush, G.F. (2011), "Couple stress theory for solids", *Int. J. Solids Struct.*, **48**, 2496-2510.

Hamidi, A., Houari, M.S.A., Mahmoud, S.R. and Tounsi, A. (2015), "A sinusoidal plate theory with 5-unknowns and stretching effect for thermomechanical bending of functionally graded sandwich plates", *Steel Compos. Struct.*, **18**(1), 235-253.

Hebali, H., Tounsi, A., Houari, M.S.A., Bessaim, A. and Adda

Bedia, E.A. (2014), "New quasi-3D hyperbolic shear deformation theory for the static and free vibration analysis of functionally graded plates", *J. Eng. Mech. - ASCE*, **140**, 374-383.

Heireche, H., Tounsi, A., Benzair, A., Maachou, M. and Adda Bedia, EA. (2008), "Sound wave propagation in single-walled carbon nanotubes using nonlocal elasticity", *Physica E.*, **40**, 2791-2799.

Houari, M.S.A., Tounsi, A., Bessaim, A. and Mahmoud, S.R. (2016), "A new simple three -unknown sinusoidal shear deformation theory for functionally graded plates", *Steel Compos. Struct.*, (Accepted).

Janghorban, M. and Zare, A. (2011), "Free vibration analysis of functionally graded carbon nanotubes with variable thickness by differential quadrature method", *Physica E*, **43**, 1602-1604.

Khalfi, Y., Houari, M.S.A. and Tounsi, A. (2014), "A refined and simple shear deformation theory for thermal buckling of solar functionally graded plates on elastic foundation", *Int. J. Comput. Meth.*, **11**(5), 135007.

Lam, D.C.C., Yang, F., Chong, A.C.M., Wang, J. and Tong, P. (2003), "Experiments and theory in strain gradient elasticity", *J. Mech. Phys. Solids*, **51**(8), 1477–1508.

Larbi Chaht, F., Kaci, A., Houari, M.S.A., Tounsi, A., Anwar Bég, O. and Mahmoud, S.R. (2015), "Bending and buckling analyses of functionally graded material (FGM) size-dependent nanoscale beams including the thickness stretching effect", *Steel Compos. Struct.*, **18**(2), 425-442.

Lee, Z., Ophus, C., Fischer, L., Nelson-Fitzpatrick, N., Westra, K.L., Evoy, S. *et al*. (2006), "Metallic NEMS components fabricated from nanocomposite Al–Mo films", *Nanotechnology*, **17**(12), 3063-3070.

Levinson, M. (1981), "A new rectangular beam theory", *J. Sound Vib.*, **74**, 81.

Li, C. and Chou, T.W. (2003a), "A structural mechanics approach for the analysis of carbon nanotubes", *Int. J. Solids Struct.*, **40**, 2487-2499.

Li, C, Chou, TW. (2003b), "Single-walled carbon nanotubes as ultrahigh frequency nanomechanical resonators", *Phys Rev B*, **68**, 073405.

Lim, C.W. and Wang, C.M. (2007), "Exact variational nonlocal stress modeling with asymptotic higher-order strain gradients for nanobeams", *J. Appl. Phys.*, **101**, 054312.

Lu, P., Lee, H.P., Lu, C. and Zhang, P.Q. (2007), "Application of nonlocal beam models for carbon nanotubes", *Int. J. Solids Struct.*, **44**, 5289-5300.

Lu, C., Wu, D. and Chen, W. (2011), "Non-linear responses of nano-scale FGM films including the effects of surface energies", *IEEE T. Nanotechnol*, **10**(6), 1321-1327.

Maachou, M., Zidour, M., Baghdadi, H., Ziane, N. and Tounsi, A. (2011), "A nonlocal Levinson beam model for free vibration analysis of zigzag single-walled carbon nanotubes including thermal effects", *Solid State Commun.*, **151**, 1467-1471.

Ma, H.M., Gao, X.L. and Reddy, J.N. (2008), "A microstructure-dependent Timoshenko beam model based on a modified couple stress theory", *J. Mech. Phys. Solids*, **56**, 3379-3391.

Mahi, A., Adda Bedia, E.A. and Tounsi, A. (2015), "A new hyperbolic shear deformation theory for bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates", *Appl. Math. Model.*, **39**, 2489-2508.

Meradjah, M., Kaci, A., Houari, M.S.A., Tounsi, A. and Mahmoud, S.R. (2015), "A new higher order shear and normal deformation theory for functionally graded beams", *Steel Compos. Struct.*, **18**(3), 793-809.

Mouaici, F., Benyoucef, S., Ait Atmane, H. and Tounsi, A. (2016), "Effect of porosity on vibrational characteristics of non-homogeneous plates using hyperbolic shear deformation theory", *Wind Struct.*, **22**(4), 429-454.

Mustapha, K.B. and Zhong, Z.W. (2010a), "The thermo-mechanical vibration of a single-walled carbon nanotube studied using the Bubnov–Galerkin method", *Physica E.*, **43**, 375-381.

Mustapha, K.B. and Zhong, Z.W. (2010b), "Free transverse vibration of an axially loaded non-prismatic single-walled carbon nanotube embedded in a two-parameter elastic medium", *Comput. Mater. Sci.*, **50**, 742-751.

Narendar, S. and Gopalakrishnan, S. (2011), "Nonlocal wave propagation in rotating nanotube", *Results Phys.*, **1**, 17-25.

Nedri, K., El Meiche, N. and Tounsi, A. (2014), "Free vibration analysis of laminated composite plates resting on elastic foundations by using a refined hyperbolic shear deformation theory", *Mech. Compos. Mater*, **49**(6), 641-650.

Nix, W.D. and Gao, H. (1998), "Indentation size effects in crystalline materials: A law for strain gradient plasticity", *J. Mech. Phys. Solids*, **46**, 411-425.

Ould Larbi, L., Kaci, A., Houari, M.S.A. and Tounsi, A. (2013), "An efficient shear deformation beam theory based on neutral surface position for bending and free vibration of functionally graded beams", *Mech. Based Des. Struct. Mach.*, **41**, 421-433.

Peddieson, J., Buchanan, G.R. and McNitt, R.P. (2003), "Application of nonlocal continuum models to nanotechnology", *Int. J. Eng. Sci.*, **41** (2003) 305-312.

Pijaudier-Cabot, G. and Bazant, ZP. (1987), "Nonlocal damage theory", *J. Eng. Mech.- ASCE*, **113**, 1512-1533.

Rahaeifard, M., Kahrobaiyan, M.H. and Ahmadian, M.T. (2009), "Sensitivity analysis of atomic force microscope cantilever made of functionally graded materials", *Proceedings of the 3rd International conference on micro- and nanosystems.* DETC2009-86254:539-544.

Rakrak, K., Zidour, M., Heireche, H., Bousahla, A.A. and Chemi, A. (2016), "Free vibration analysis of chiral double-walled carbon nanotube using non-local elasticity theory", *Adv. Nano Res.*, **4**(1), 31-44.

Reddy, J.N. (2011), "Microstructure-dependent couple stress theories of functionally graded beams", *J. Mech. Phys. Solids*, **59**, 2382-2399.

Reddy, J.N. and Pang, S.D. (2008), "Nonlocal continuum theories of beams for the analysis of carbon nanotubes", *J. Appl. Phys.*, **103**, 023511.

Reddy, J.N. (2007), "Nonlocal theories for bending, buckling and vibration of beams", *Int. J. Eng. Sci.*, **45**, 288.

Reddy, J.N. (2002), "*Energy principles and variational methods in applied mechanics*", John Wiley & Sons Inc.

Reddy, J.N. (1984), "A simple higher-order theory for laminated composite plates", *J. Appl. Mech.- ASME*, **51**, 745.

Roque, C.M.C., Ferreira, A.J.M. and Reddy, J.N. (2011), "Analysis of Timoshenko nanobeams with a nonlocal formulation and meshless method", *Int. J. Eng. Sci.*, **49**, 976-984.

Said, A., Ameur, M., Bousahla, A.A. and Tounsi, A. (2014), "A new simple hyperbolic shear deformation theory for functionally graded plates resting on Winkler–Pasternak elastic foundations", *Int. J. Comput. Met.*, **11**(6), 1350098.

Semmah, A., Tounsi, A., Zidour, M., Heireche, H. and Naceri, M. (2015), "Effect of chirality on critical buckling temperature of a zigzag single-walled carbon nanotubes using nonlocal continuum theory", *Fullerenes, Nanotubes Carbon Nanostructures*, **23**, 518-522.

Soldatos, K.P. (1992), "A transverse shear deformation theory for homogeneous monoclinic plates", *Acta Mech*, **94**, 195–200.

Song, J., Shen, J. and Li, X.F. (2010), "Effects of initial axial stress on waves propagating in carbon nanotubes using a generalized nonlocal model", *Comput .Mater Sci.*, **49**, 518–523.

Stolken, J.S. and Evans, A.G. (1998), "A microbend test method for measuring the plasticity length scale", *Acta Mater.*, **46**(14), 5109-5115.

Tounsi, A, Semmah, A. and Bousahla, A.A. (2013a), "Thermal buckling behavior of nanobeams using an efficient higher-order nonlocal beam theory", *J. Nanomech. Micromech. - ASCE*, **3**, 37-42.

Tounsi, A., Benguediab, S., Adda Bedia, E.A., Semmah, A. and Zidour, M. (2013b), "Nonlocal effects on thermal buckling properties of double-walled carbon nanotubes", *Adv. Nano Res.*, **1**(1), 1-11.

Tounsi, A., Houari, M.S.A., Benyoucef, S. and Adda Bedia, E.A. (2013c), "A refined trigonometric shear deformation theory for thermoelastic bending of functionally graded sandwich plates", *Aerosp. Sci. Technol*., **24**, 209-220.

Tounsi, A., Heireche, H., Berrabah, H.M., Benzair, A. and Boumia, L. (2008), "Effect of small size on wave propagation in double-walled carbon nanotubes under temperature field", *J. Appl. Phys.*, **104**, 104301.

Tounsi, A., Houari, M.S.A. and Bessaim, A. (2016), "A new 3-unknowns non-polynomial plate theory for buckling and vibration of functionally graded sandwich plate", *Struct. Eng. Mech.*, (Accepted).

Wang, Q. and Liew, K.M. (2007), "Application of nonlocal continuum mechanics to static analysis of micro- and nano-structures", *Phys. Lett. A*, **363**, 236-242.

Wang, Q., Varadan, V.K. and Quek, S.T. (2006), "Small scale effect on elastic buckling of carbon nanotubes with nonlocal continuum models", *Phys. Lett. A***, 357**, 130-135.

Witvrouw, A. and Mehta, A. (2005), "The use of functionally graded poly-SiGe layers for MEMS applications", *Mater. Sci. Forum.*, **492**, 55-60.

Yaghoobi, H. and Feraidoon, A. (2010), "Influence of neutral surface position on deflection of functionally graded beam under uniformly distributed load", *World Appl. Sci. J.*, **10**(3), 337-341.

Yoon, J., Ru, C.Q. and Mioduchowski, A. (2003), "Vibration of an embedded multiwall carbon nanotube", *Compos. Sci. Technol*., **63**, 1533-1542.

Zemri, A., Houari, M.S.A., Bousahla, A.A. and Tounsi, A. (2015), "A mechanical response of functionally graded nanoscale beam: An assessment of a refined nonlocal shear deformation theory beam theory", *Struct. Eng. Mech.*, **54**(4), 693-710.

Zhang, Y.Q., Liu, G.R. and Wang, J.S. (2004), "Small-scale effects on buckling of multiwalled carbon nanotubes under axial compression", *Phys. Rev. B.*, **70**, 205430.

Zhang, Y, Liu, G, Han, X. (2005a), "Transverse vibrations of double-walled carbon nanotubes under compressive axial load", *Phys Lett A*., **340**, 258–266.

Zhang, Y.Q., Liu, G.R. and Xie, X.Y. (2005b), "Free transverse vibrations of double-walled carbon nanotubes using a theory of nonlocal elasticity", *Phys . Rev. B.*, **71**, 195404.

Zhang, Y.Q., Liu, G.R. and Han, X. (2006), "Effect of small length scale on elastic buckling of multi-walled carbon nanotubes under radial pressure", *Phys. Lett. A*, **349**, 370-376.

Zidi, M., Tounsi, A., Houari, M.S.A., Adda Bedia, E.A. and Anwar Bég, O. (2014), "Bending analysis of FGM plates under hygro-thermo-mechanical loading using a four variable refined plate theory", *Aerosp. Sci. Technol.,* **34**, 24-34.

Zidour, M., Benrahou, K.H., Semmah, A., Naceri, M., Belhadj, H.A. and Bakhti, K. (2012), "The thermal effect on vibration of zigzag single walled carbon nanotubes using nonlocal Timoshenko beam theory", *Comput. Mater. Sci.*, **51**, 252-260.

Zidour, M., Daouadji, T.H., Benrahou, K.H., Tounsi, A., Adda Bedia, E.A. and Hadji, L. (2014), "Buckling analysis of chiral single-walled carbon nanotubes by using the nonlocal Timoshenko beam theory", *Mech. Compos. Mater.*, **50**(1), 95-104.

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