Smart Structures and Systems, *Vol. 15, No. 3 (2015) 717-733*
DOI: http://dx.doi.org/10.12989/sss.2015.15.3.717

Assessment of modal parameters considering measurement and modeling errors

Qindan Huang*[1], Paolo Gardoni[2] and Stefan Hurlebaus[3]

[1]Department of Civil Engineering, The University of Akron, Akron, OH 44325-3905, USA
[2]Department of Civil and Environmental Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801-2352, USA
[3]Zachry Department of Civil Engineering, Texas A&M University, College Station, TX 77843-3136, USA

(Received October 25, 2014, Revised January 17, 2015, Accepted February 3, 2015)

Abstract. Modal parameters of a structure are commonly used quantities for system identification and damage detection. With a limited number of studies on the statistics assessment of modal parameters, this paper presents procedures to properly account for the uncertainties present in the process of extracting modal parameters. Particularly, this paper focuses on how to deal with the measurement error in an ambient vibration test and the modeling error resulting from a modal parameter extraction process. A bootstrap approach is adopted, when an ensemble of a limited number of noised time-history response recordings is available. To estimate the modeling error associated with the extraction process, a model prediction expansion approach is adopted where the modeling error is considered as an "adjustment" to the prediction obtained from the extraction process. The proposed procedures can be further incorporated into the probabilistic analysis of applications where the modal parameters are used. This study considers the effects of the measurement and modeling errors and can provide guidance in allocating resources to improve the estimation accuracy of the modal data. As an illustration, the proposed procedures are applied to extract the modal data of a damaged beam, and the extracted modal data are used to detect potential damage locations using a damage detection method. It is shown that the variability in the modal parameters can be considered to be quite low due to the measurement and modeling errors; however, this low variability has a significant impact on the damage detection results for the studied beam.

Keywords: modal parameters; measurement error; modeling error; bootstrap; sample size

1. Introduction

Modal parameters (e.g., modal frequency and mode shape) are used to describe the characteristics of a structure and have been commonly used in system identification to validate and/or update computer simulated models (Doebling and Farrar 2001, Ching *et al*. 2006). They are also often adopted in vibration-based nondestructive testing (NDT) methods for damage detection, where typically the modal parameters of a damaged system are compared with those of the corresponding undamaged system (Doebling *et al*. 1998). The results of the damage detection are useful for estimating the reliability of existing structures and help schedule condition-based

*Corresponding author, Assistant Professor, E-mail: qhuang@uakron.edu

http://www.techno-press.org/?journal=sss&subpage=8 ISSN: 1738-1584 (Print), 1738-1991 (Online)

maintenance (Ceracolo *et al*. 2009, Zio 2009). To effectively use the modal parameters in these applications, it is critical to know the statistics of the modal parameter estimation.

Modal data are usually extracted from time-history vibration responses obtained from a vibration test. The uncertainties in the modal data mainly come from two sources: one is the vibration test and the other is the modal parameter extraction (identification) process. The epistemic uncertainty in the measurement process in which the structural dynamic responses are recorded is called measurement error. The epistemic uncertainty inherent in the extraction process that reflects the inexactness and assumptions in the process is called modeling error. Currently, studies that assess the uncertainties in the modal parameters are still limited in number, especially in the area of considering the modeling error in the modal parameter extraction process. A typical approach is to assume that modal parameters consist of deterministic quantities and additive random errors with zero means (Liu 1995, Papadopoulos and Garcia 1998, Xia *et al*. 2002, Pothisriri and Hjelmstad 2003). In this approach, regardless of the inaccuracy of the assumption, the contributions to the uncertainties from the measurement and the extraction processes are not clear. Thus such approach is not able to provide the necessary guidance when allocating resources to improve the accuracy of the modal data estimate.

This paper presents a procedure that can be used to assess the modal data extracted from dynamic responses obtained from ambient vibrations considering the effect of the measurement and modeling errors. Particularly, when an ensemble of a limited number of noised time-history response recordings is available, a bootstrap approach is used to estimate the statistics of the modal data. To estimate the modeling error associated with the extraction process, a model prediction expansion approach is adopted where the modeling error is considered as an “adjustment” to the prediction obtained from the extraction process. The modeling error considered in this study is due to a specific modal data extraction process, namely, Time Domain Decomposition (TDD) method (Kim *et al*. 2005). However, the proposed procedure can be extended to other extraction methods. This study is especially useful when the modal parameters are further applied to system identification, damage detection, and other applications, where the uncertainties from various sources should be considered. Moreover, the modeling error is estimated separately and it can help provide guidance in allocating resources to improve the accuracy of the modal data estimate.

In the following, first a brief review of the uncertainty types and the methods for error modeling is given. Next, the procedure to consider the measurement and modeling errors is described. Lastly, in a numerical example, the modal parameters of a damaged two-span, continuous beam are extracted considering measurement and modeling errors following the proposed procedure, and the extracted modal data are used to detect potential damage locations using a damage detection method.

2. Uncertainty types and modeling

To appropriately account for the uncertainties, different types of uncertainties need to be understood. This section first describes the characteristics of different uncertainties associated with the process of extracting modal parameters from the responses. The uncertainty classification follows Gardoni *et al*. (2002). Next a review of the available approaches for handling and propagating those uncertainties is presented.

2.1 Measurement error

Measurement error reflects the difference between the recorded responses and the true responses. It may vary with sensor type, layout, installation of equipment, and environment. In numerical studies, the measurement error is generally modeled by adding random errors into the time-history response signals and the error is often simulated as a random Gaussian noise with zero mean and a specific standard deviation. To provide reasonable upper and lower bounds on the measurement error, absolute error and proportional error are commonly used (Banan *et al*. 1994). For the absolute error, the standard deviation of the measurement error is a fixed value. For the proportional error, the standard deviation of the measurement error is proportional to the value of the amplitude of the true response.

There are two common methods to propagate the measurement error to the modal data. One is the perturbation method or sensitivity method, which requires finding the sensitivity of the measurement error to the modal data. In this respect, Longman and Jung (1987) developed a framework for the eigenrealization algorithm (ERA); Peterson *et al*. (1996) used the perturbation analysis for the fast ERA; and Arici and Mosalam (2005) provided a formulation of sensitivity for the observer Kalman filter identification-ERA with direction correlations method. If the sensitivity can be expressed analytically, the perturbation approach is efficient. However, the sensitivity analysis typically uses a first order approximation that may not be accurate, and the computation required to obtain the sensitivity is not an easy task (Doebling and Farrar 2001). The other approach is the sampling method, where a set of modal data is extracted from a set of time-history responses that are contaminated by the measurement error. Note that using the sampling method the variability in the sets of modal parameters reflects the effect from both the measurement error and the modeling error associated with the modal data extraction process. The sampling method is straightforward but requires a good number of time-history responses.

2.2 Modeling error

Modeling error reflects the error due to model inaccuracy. The sources of the error can come from the abstractions, assumptions, and approximations used in the modeling process. There are two types of modeling errors: parameter error and model error. Parameter error refers to the epistemic uncertainties in the model parameters (e.g., stiffness and mass in a finite element model, FEM) in a given model due to the lack of knowledge of the parameter values. The uncertainties in the model parameters are parameter errors. Model error is associated with the model itself, reflecting the fact that there might not be a perfect numerical model to represent a real-world system and/or from the missing portion that is not included in the model. An example is using an elastic-perfectly-plastic stress-strain relationship to model material behavior.

Oberkampf *et al*. (2002) developed a general framework to deal with modeling error in computational simulations associated with the numerical solution of a set of partial differential equations. Reinert and Apostolakis (2006) gave a review of approaches to handle modeling errors, including model set expansion and model prediction expansion. Model set expansion combines different models to produce a meta-model for a real system, incorporating the advantages of various models. Model prediction expansion, on the other hand, applies an “adjustment” directly to the prediction outcome from one model. In this study, since only one specific extraction method is adopted to extract the modal data, using model prediction expansion is appropriate to estimate the modeling error associated with the extraction process.

3. Estimation of measurement and modelling errors in the extracted modal parameters

3.1 Processes for obtaining modal parameters

Two processes are usually involved in obtaining the modal parameters. One is to record dynamic responses from a vibration test that can be ambient, and the other is to extract the modal parameters from the responses. The ambient vibration test is an attractive option because the excitation can be wind, traffic loading, and any other convenient mechanical exciter and only the output needs to be recorded. When an ambient vibration test is used, an output-only modal parameter identification method is needed to extract the modal parameters. In this paper, an ambient extraction technique called Time Domain Decomposition (TDD) method (Kim *et al*. 2005) is used. For the sake of completeness, a brief review of the TDD method is given next.

3.2 Review of Time Domain Decomposition (TDD) method

Acceleration responses are the commonly recorded quantities in a vibration test. If there are p acceleration sensors and n modes are considered, the acceleration output can be written as

$$\ddot{\mathbf{y}}(t) = \sum_{i=1}^{\infty} \ddot{c}_i(t)\boldsymbol{\varphi}_i = \sum_{i=1}^{n} \ddot{c}_i(t)\boldsymbol{\varphi}_i + \boldsymbol{\varepsilon}_t(t) \tag{1}$$

where $\ddot{\mathbf{y}}(t) = [\ddot{y}_1(t),...,\ddot{y}_p(t)]^T$, $\ddot{c}_i(t) = i$th mode contribution factor at time t, $\boldsymbol{\varphi}_i = i$th modal shape, and $\boldsymbol{\varepsilon}_t(t)$ = truncation error. A mode-isolated response, $\ddot{\mathbf{y}}(t)$, is obtained by filtering the acceleration response using a digital band-pass filter that is designed by estimating the frequency bandwidth for each mode. The filtered acceleration $\ddot{\mathbf{y}}^{(i)}(t)$ for the ith mode can be expressed as

$$\ddot{\mathbf{y}}^{(i)}(t) = \ddot{c}_i(t)\boldsymbol{\varphi}_i + \sum_{j=1}^{p-1} \ddot{d}_j(t)\boldsymbol{\psi}_j \tag{2}$$

where the second term refers to the error due to the truncation and the filtering, $\boldsymbol{\psi}_j$ = orthogonal noise base, and $\ddot{d}_j(t) = j$th mode contribution factor to the total error. The dimension of the mode-isolated acceleration vector, $\ddot{\mathbf{y}}^{(i)}(t)$, at the sample, t, is $p\times 1$, and it contains the modal space and orthogonal noise space. As shown in Eq. (2), the dimension of the modal space is only one (i.e., the ith mode shape vector $\boldsymbol{\varphi}_i$) and the dimension of the noise space is $p-1$.

With a total time sample N, and assuming that the orthogonal bases in the modal space consists of $\ddot{\mathbf{c}}_i = [\ddot{c}_i(1),...,\ddot{c}_i(N)]$ and the error space consists of $\ddot{\mathbf{d}}_j = [\ddot{d}_j(1),...,\ddot{d}_j(N)]$, a cross-correlation $\mathbf{E}_i = \mathbf{H}_i\mathbf{H}_i^T$ of $\ddot{\mathbf{y}}^{(i)}(t)$, where $\mathbf{H}_i = [\ddot{\mathbf{y}}^{(i)}(1)\ \ddot{\mathbf{y}}^{(i)}(2)\ \dots\ \ddot{\mathbf{y}}^{(i)}(N)]$, can be calculated as

$$\mathbf{E}_i = \boldsymbol{\varphi}_i q_i \boldsymbol{\varphi}_i^T + \sum_{j=1}^{p-1} \boldsymbol{\psi}_j \sigma_j \boldsymbol{\psi}_j^T \tag{3}$$

where $q_i = \ddot{\mathbf{c}}_i^T\ddot{\mathbf{c}}_i$ and $\sigma_j = \ddot{\mathbf{d}}_j^T\ddot{\mathbf{d}}_j$. Since the energy contributed by the noises is relatively small, then it is appropriate to assume that $q_i > \sigma_1 > \ldots > \sigma_{p-1}$. Thus, by conducting singular value decomposition on

$\mathbf{E}_i$, the first singular vector in the singular vector matrix is the extracted modal shape $\hat{\boldsymbol{\varphi}}_i$ for the ith mode. With the obtained $\hat{\boldsymbol{\varphi}}_i$, the contribution factor can be calculated as

$$\ddot{\mathbf{c}}_i^T = \frac{\hat{\varphi}_i^T \mathbf{H}_i}{\hat{\varphi}_i \hat{\varphi}_i^T} \tag{4}$$

The signal, $\ddot{\mathbf{c}}_i$, contains the ith modal behavior of the acceleration for the entire set of p signals. Therefore, the auto-spectrum of $\ddot{\mathbf{c}}_i$, contains one peak, at which the frequency is the extracted modal frequency $\hat{f}_i$.

3.3 Measurement error in an ambient vibration test

Measurement error in an ambient vibration test can have many causes. The primary sources are listed in the following and the examples of influencing factors are given in the parentheses:

- electromagnetic noises in the sensor (sensor type, signal conditioning type)
- data acquisition and discretization (data-acquisition system, sampling rate)
- data processing and testing economy (signal processing system)

With p sensors, one set of time-history responses $\ddot{\mathbf{y}}_1(t) = [\ddot{y}_{1,1}(t),\ldots, \ddot{y}_{p,1}(t)]^T$ is measured in one recording. It is assumed that these p series of time-history responses have the same measurement noise level. When such measurement is repeated m times, m sets of such time-history responses, $\{\ddot{\mathbf{y}}_1(t),\ddot{\mathbf{y}}_2(t),\ldots,\ddot{\mathbf{y}}_m(t)\}$, are obtained and each set of time-history responses is assumed to have the same level of measurement error. Then, for each set of responses, the TDD method is applied to obtain one set of modal data, thus m sets of modal data are generated. If m is large enough, the statistics of the modal data can be assessed. If m is small (e.g., if the vibration testing is conducted by applying a pulse excitation in order to excite out the modal frequencies of interest, then the number of the vibration testing will be limited), a bootstrap method (Efron 1982) is proposed in this paper to generate new sets of responses based on the recorded ones. The basic idea of the bootstrap method is to randomly generate new responses using the original list of responses. The new sets of responses have the same level of measurement error as the recorded ones.

Note that in an ambient vibration test, the excitation is not recorded and it varies for each recording; thus, the Bootstrap method cannot directly be applied to the responses in the time domain. In the frequency domain, the response amplitude varies with frequencies. However, it is legitimate to assume that the amplitude is constant in a narrow bandwidth. In the TDD, the mode-isolated responses are obtained using a frequency bandwidth that can be considered to be narrow. Additionally, as the normalized mode-isolated responses reflect the normalized mode shapes at the sensor locations, they are independent of the excitation. Thus, a Bootstrap method can be applied to the normalized mode-isolated responses in the frequency domain at each sensor node. This principle can be applied to other output only modal data exaction method such as frequency domain decomposition (Brinker *et al.* 2000).

Fig. 1 shows the flowchart of the proposed procedures for estimating the statistics of the modal data considering measurement and modeling errors, where p acceleration sensors are used and m sets of dynamic responses are collected. In the proposed procedure, firstly the measured responses, $\{\ddot{\mathbf{y}}_1(t),\ddot{\mathbf{y}}_2(t),\ldots,\ddot{\mathbf{y}}_m(t)\}$, are transferred to the frequency domain and normalized, and then are filtered using a digital band-pass filter to obtain the mode-isolated responses in the frequency domain,

$\{\ddot{\mathbf{Y}}_1^{(i)}(\omega),\ddot{\mathbf{Y}}_2^{(i)}(\omega),\ldots,\ddot{\mathbf{Y}}_m^{(i)}(\omega)\}$. Next, the bootstrap technique is applied to $\{\ddot{\mathbf{Y}}_1^{(i)}(\omega),\ddot{\mathbf{Y}}_2^{(i)}(\omega),\ldots,\ddot{\mathbf{Y}}_m^{(i)}(\omega)\}$ to obtain a bootstrap sample $\ddot{\mathbf{Y}}_b^{(i)}(\omega)$. This new bootstrap sample $\ddot{\mathbf{Y}}_b^{(i)}(\omega)$ is then transferred back to the time domain, $\ddot{\mathbf{y}}_b^{(i)}(t)$. For each bootstrap sample, the TDD method is applied to extract a set of modal data. Thus, a pool of modal data is obtained. The procedures can be repeated to obtain more bootstrap samples. As shown in Fig. 1, the proposed procedure involves the extraction process (i.e., the TDD method); therefore, the variability in the pool of modal data shows the influence of the measurement and modeling errors. The advantages of this proposed procedure are: 1) it does not require knowing the measurement noise level in the responses, and 2) the excitation for the vibration can be any ambient conditions as noted previously.

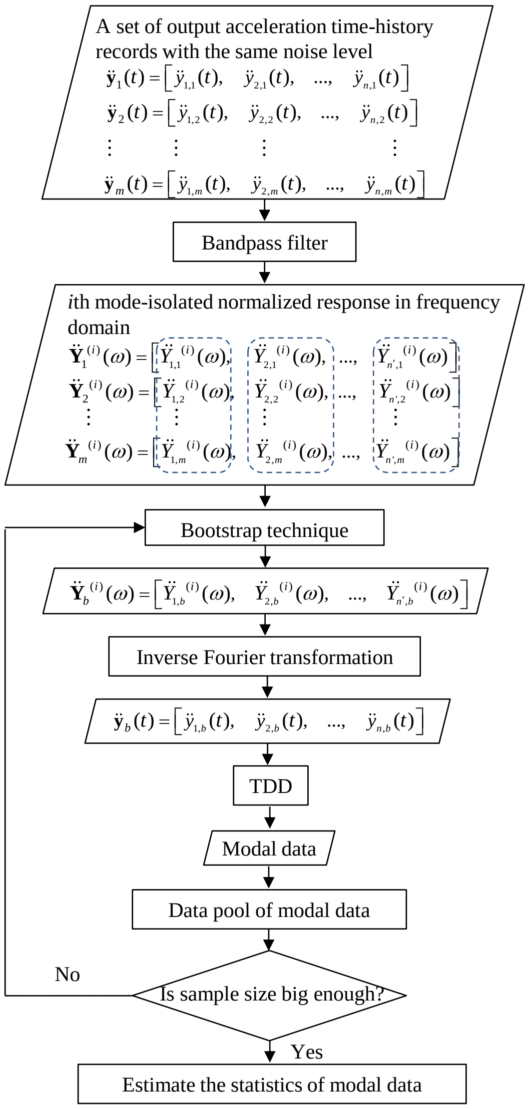


Fig. 1 Flowchart of the proposed procedure to propagate the measurement and modeling errors to the modal data

3.4 Modeling error due to the Time Domain Decomposition (TDD) method

Using the proposed procedure in Fig. 1, the statistics of the modal data reflect the influences from both measurement and modeling errors. However, the effect only due to the modeling error associated with the extraction process sometimes needs to be known. For example, in system identification, the value of the modeling error is important for updating the structural models using the extracted modal parameters if a Bayesian probabilistic framework is applied (Beck and Au 2002, Huang *et al*. 2010). In this study, the modeling error due to TDD alone is investigated. With the use of the TDD method, the modeling error arises mainly from the following sources with the examples of influencing factors given in the parentheses:

- truncating non-dominant modes (excitation type, sensor layout, sampling rate)
- model isolation process (filter width, response length)
- singular value decomposition (noise orthogonal assumption)
- system nonlinearity (excitation type and amplitude)

As discussed previously, it is appropriate to apply model prediction expansion to estimate the modeling error in the TDD method. Specifically, the additive model prediction expansion is adopted here. The relation between the real modal data in the *i*th mode (modal frequencies, f_i, and mode shape, $\boldsymbol{\varphi}_i$) and the extracted modal data ($\hat{f}_i$ and $\hat{\boldsymbol{\varphi}}_i$) can be formulated as follows

$$f_i = \hat{f}_i + e_{f_i} \tag{5}$$

$$\boldsymbol{\varphi}_i = \hat{\boldsymbol{\varphi}}_i + \mathbf{e}_{\boldsymbol{\varphi}_i} \tag{6}$$

where e_{fi} and $\mathbf{e}_{\boldsymbol{\varphi} i}$ are the modeling errors in modal frequency and mode shape due to the TDD process. They can be considered as “adjustments” to the predictions. To assess the “adjustments”, e_{fi} and $\mathbf{e}_{\boldsymbol{\varphi} i}$, one can directly compare the extracted modal data from TDD, $\hat{f}_i$ and $\hat{\boldsymbol{\varphi}}_i$, to the real values, f_i and $\boldsymbol{\varphi}_i$.

In system identification or vibration-based NDT, a preliminary FEM usually needs to be constructed and validated by calibrating the unknown structural parameters (such as stiffness and/or mass) using the extracted modal data. The model used for constructing the preliminary FEM is assumed to be correct. Therefore, for given a set values of the unknown structural parameters, the modal parameters obtained from the modal analysis of the preliminary FEM are considered as the actual values, f_i and $\boldsymbol{\varphi}_i$, for this set of structural parameters. If the modeling error in the constructing the FEM needs to be considered, one can follow the methodology and specific implementation proposed by Haukaas and Gardoni (2011).

As the structural parameters are unknown, we can generate a group of n_e FEMs, where n_e is the number of FEMs in the group, constructed in the same way as the preliminary FEM but with different combinations of structural parameter values that are randomly drawn from the parameter ranges. Note that for setting the parameter ranges, the best guess (predictive values, $\mathbf{x}_0$) need to be made based on the engineering judgment and then one can set $[\mathbf{x}_0/k, k\mathbf{x}_0]$ (where $k > 1$) as the parameter ranges. The larger k is chosen, the bigger the range will be. To effectively span the space of the parameters given a specific number of combinations, Latin hypercube sampling (McKay *et al.* 1979) can be used, where each parameter is approximately sampled uniformly from its range. For each FEM, a modal analysis is conducted to find f_i and $\boldsymbol{\varphi}_i$ that are the true modal parameters for the corresponding FEM, while an excitation can be applied to the FEM to obtain the vibration responses, from which $\hat{f}_i$ and $\hat{\boldsymbol{\varphi}}_i$ can be extracted through TDD. Thus, n_e sets of f_i, $\boldsymbol{\varphi}_i$, $\hat{f}_i$, and $\hat{\boldsymbol{\varphi}}_i$ are obtained; thus e_{fi} and $\mathbf{e}_{\varphi i}$ can be assessed through Eqs. (5) and (6). Note that the estimation of e_{fi} and $\mathbf{e}_{\varphi i}$ are based on the FEMs only and it does not need any measurements from a real structure after the preliminary FEM has been developed.

Using the difference between $f_i$and $\hat{f}_i$, to estimate e_{fi} is straightforward. Given certain influencing factors, the mean, μ_{efi}, and standard deviation, σ_{efi}, of e_{fi} can be estimated from the differences, $f_i - \hat{f}_i$. The differences between $\boldsymbol{\varphi}_i$ and $\hat{\boldsymbol{\varphi}}_i$ for the ith mode are vectors, and can be considered a linear combination of mode shapes of other modes. This is because in the frequency domain the modes are overlapped due to the existence of damping, which makes it impossible to obtain a pure mode-isolated response in the TDD method. Especially, a mode shape is influenced most by the nearest mode shape. A linear regression model is proposed to assess the mode shape error, $\boldsymbol{e}_{\varphi i}$, for the ith mode, and it is written as follows

$$\mathbf{e}_{\varphi_i} = \sum_{j=1}^{N_e} \left[\alpha_{j,i} + \beta_{j,i} \left(f_j - f_i \right)^{\theta_{j,i}} \right] \boldsymbol{\varphi}_j \left(1 - \delta_{ij}\right) + \sigma_{\varphi_i} \boldsymbol{\varepsilon} \tag{7}$$

where N_e = number of modes used, $\alpha_{j,i}$, $\beta_{j,i}$, and $\theta_{j,i}$= model parameters for the ith mode, δ_{ij} = Kronecker delta (i.e., $\delta_{ij} = 1$, for $i = j$; $\delta_{ij} = 0$, for $i \neq j$), $\sigma_{\varphi i}$ = standard deviation of $\mathbf{e}_{\varphi i}$, and $\boldsymbol{\varepsilon}$ = a vector of standard normal random variables. The model parameters can be estimated by the Bayesian updating rule (Box and Tiao 1992). One could use locally uniform distributions for the model parameters as the non-informative priors in the Bayesian updating rule. Note that the further a mode is from the ith mode, the less contribution that mode has to $\mathbf{e}_{\varphi i}$. Moreover, the closeness of two modes can be measured by the difference between the corresponding modal frequencies; thus, the term $\beta_{j,i}(f_j - f_i)^{\theta j,i}$ is included in Eq. (7). A stepwise deletion procedure (Gardoni *et al*. 2002) can be also used to eliminate the unimportant terms in Eq. (7) to obtain an accurate but parsimonious model. Fig. 2 summarizes the steps for accessing e_{fi} and $\mathbf{e}_{\varphi i}$, where the modal data extraction process can be general.

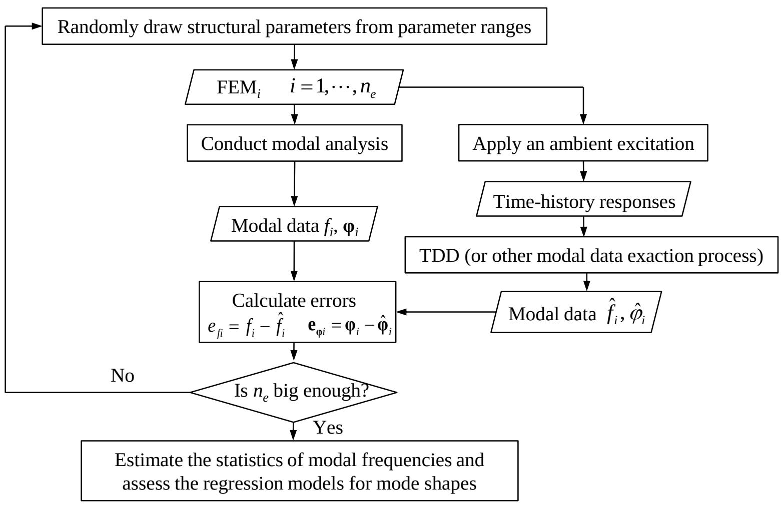


Fig. 2 Flowchart of the proposed procedure to estimate the modeling error in TDD

3.5 Considerations on the sample size for the bootstrap method

In the flowchart shown in Fig. 1, the sample size for the bootstrap method needs to be determined. Efron (1987) suggested 1000 samples for confidence intervals. However, given available computing power and time, it is reasonable to use as many samples as actually needed based on a convergence criterion. The sample size, n_e, in the flowchart in Fig. 2 is determined by the maximum of two sample sizes: a sample size, n_f, for determining the statistical interferences of e_{fi}, and a sample size, n_φ, for assessing the regression models of $\mathbf{e}_{\varphi i}$. Although e_{fi} is unknown, by setting predefined confidence intervals, n_f can be estimated. Assuming that e_{fi} follows a normal distribution, with $100(1-\alpha)\%$ confidence, where α refers to the significance level, the following relationships hold (Ang and Tang 2006)

$$\hat{\mu}_{ef} - t_{\alpha/2}\frac{\hat{S}_{ef}}{\sqrt{n_{\mu f}}} \leq \mu_{ef} \leq \hat{\mu}_{ef} + t_{1-\alpha/2}\frac{\hat{S}_{ef}}{\sqrt{n_{\mu f}}} \tag{8}$$

$$\frac{n_{\sigma f}}{\chi^2_{1-\alpha/2}}\hat{S}_{ef}^{\ 2} \leq \sigma_{ef}^{\ 2} \leq \frac{n_{\sigma f}}{\chi^2_{\alpha/2}}\hat{S}_{ef}^{\ 2} \tag{9}$$

where μ_{ef} = mean of e_{fi}, $\hat{\mu}_{ef}$ = sample mean, σ_{ef} = standard deviation of e_{fi}, $\hat{S}_{ef}$ = sample standard deviation, $t_{\alpha/2}$ = $t_{1-\alpha/2}$ = $(1-\alpha/2)$th quantile of the student's t distribution with degree of $n_{\mu f} - 1$, and $\chi^2_{1-\alpha/2}$=$(1-\alpha/2)$th and $\chi^2_{\alpha/2}$ = $\alpha/2$th quantiles of the χ^2 distribution with degree of $n_{\sigma f} - 1$. When the confidence intervals are given for μ_{ef} and σ_{ef}, then $n_{\mu f}$ and $n_{\sigma f}$ can be calculated through Eqs. (8) and (9) and the sample size n_f is determined as $n_f = \max(n_{\mu f}, n_{\sigma f})$.

The sample size, n_φ, needs to be determined for Eq. (7) to ensure a sufficient statistical power to detect the significant effects. Following Cohen (1988), four input parameters are needed: (1) a significance confidence level, α, (2) a target power level of a F-test, γ, (3) an effect size, ES, and (4) the number of predictors used in the linear regression model, p_e. The power level γ refers to the probability that one rejects the null hypothesis while the alternative hypothesis is true. Type III F-test is used in the procedure, where the null hypothesis states that all coefficients of predictors of interest are zero. To calculate the power of the F-test, the F-distribution and the non-central F-distribution probability density function are used and they are respectively expressed as

$$p(x) = \frac{1}{B\left(\frac{v}{2},\frac{u}{2}\right)}\left(\frac{ux}{ux+v}\right)^{\frac{u}{2}}\left(1-\frac{ux}{ux+v}\right)^{\frac{v}{2}}x^{-1} \tag{10}$$

$$p(g) = \sum_{k=0}^{\infty}\frac{e^{-\lambda/2}(\lambda/2)^k}{B\left(\frac{v}{2},\frac{u}{2}+k\right)k!}\left(\frac{u}{v}\right)^{\frac{u}{2}+k}\left(\frac{v}{u+vg}\right)^{\frac{v+u}{2}+k}g^{\frac{v}{2}-1+k} \tag{11}$$

where $p(\cdot)$ = probability density function, $B(\cdot)$ = Beta function, u = number of predictors or numerator degrees of freedom, v = denominator degrees of freedom, λ = non-centrality parameter, and g = critical value of F-distribution. Note that as no intercept is included in Eq. (7), the sample size is $u + v$; otherwise, the sample size would be $u + v + 1$. The effect size, ES, describes the degree to which the null hypothesis is false: the larger the effect size indicates the greater degree to which the alternative hypothesis is manifested. Conventionally, ES of 0.02, 0.15, and 0.35 are considered as small, medium, and large, respectively. With the four input values, the sample size, n_φ, can be calculated by the following steps:

i) Set a significance confidence level α, a desired power level γ, and an effect size ES, and set an initial value $v = p_e$.

ii) Calculate the $(1-\alpha/2)$th quantiles of the F-distribution g through Eq. (10), with the given values of α, $u(= p_e)$, and v.

iii) Compute $\lambda = ES(u + v)$.

iv) With the values of g,u $(= p_e)$,v, and λ, estimate the corresponding value of the non-central F-distribution through Eq. (11).

v) Compute the power by calculating the cumulative area under the standard normal curve from zero to the value of the non-central F-value estimated from the previous step.

vi) If the power computed from step v) is less than the desired power level γ, increase v value and repeat step ii) through step v).

vii) Determine the sample size $n_\varphi = u + v$.

Then, the sample size, n_e, in the flowchart in Fig. 2 is determined as $n_e = \max(n_f, n_\varphi)$.

4. Illustration

This section illustrates the proposed procedures using a numerical example considering a two-span continuous aluminum beam on elastic supports. Modal data is extracted from an ambient vibration test, and then a damage detection method, called Damage Index Method (DIM) developed by Stubbs and Kim (1996) is applied using the extracted modal data to study the effect of the measurement and modeling errors on the damage detection. This example beam has been previously analyzed by Stubbs and Kim (1996). The FEM of the beam is built in OpenSees (Mckenna and Fenves 2002) using 50 elastic beam-column elements with elastic springs for the supports, as shown in Fig. 3, where E_b refers to the flexural stiffness of the beam and K_1 and K_2 are the spring stiffness. Table 1 summarizes the structural properties for the FEM. A damage scenario is simulated by introducing a reduction of 10% of the flexural stiffness in the beam Element 16. The first four modal frequencies of the damaged structure obtained from the modal analysis are 36.56 Hz, 55.11 Hz, and 141.47 Hz, 158.72 Hz, respectively. For the purpose of the illustration, the stiffness properties of the study structure, E_b, K_1, and K_2 are assumed to be unknown. A preliminary FEM is constructed in the same way as the study structure but with different values of E_b, K_1, and K_2 and assuming that all the elements have the same flexural stiffness. The values of E_b, K_1, and K_2 in the preliminary FEM can be arbitrary since they are unknown.

Next, a vibration test is conducted to collect the dynamic responses, from which the modal parameters are extracted. The proposed procedures are applied to estimate the influence of measurement and modeling errors on the modal data. Finally, DIM is applied to identify the damage location using the extracted modal parameters to investigate the influence of measurement and modeling errors on the damage detection.

A vibration test is conducted by collecting the dynamic responses of the element nodes (a total of 51 nodes) under a force excitation. To avoid the issue of not being able to observe some modes due to a narrow-band excitation, a pulse loading that contains a wide range of frequencies is applied. The recording time for each node response measurement is 4 second with a sampling rate

1000 Hz. As the recording time is one of the influencing factors of the modeling error, the modeling error changes when the recording time varies, as shown later. With the 4 second recording time, there are 51 x 4 x 1000 = 204,000 response points obtained and they are noise-free since in this example they are the responses obtained from a numerical analysis.

Measurement noise is simulated using a proportional error and is added to the noise-free responses. Three different noises levels are considered and they are 0.5%, 1.0%, and 2.0%. Specifically, the measurement noises that are added to the 204,000 response points are assumed to follow independent Gaussian distributions with zero mean and specific standard deviations that are proportional to the amplitudes of the responses. The standard deviations are calculated by multiplying the amplitudes of the responses with the measurement noise level (i.e., based on a 0.5%, 1.0%, and 2.0% coefficient of variation). For each noise level, the procedures of adding the simulated measurement noise to the noise-free responses are repeated 10 times (m = 10 is used in this study just to illustrate a case when a limited number of records is available). Thus 10 sets of responses at each noise level are generated and they can be considered as the recorded responses, $\{\ddot{\mathbf{y}}1(t),\ddot{\mathbf{y}}2(t),\ldots,\ddot{\mathbf{y}}m(t)\}$, obtained from the vibration testing. With the recorded responses, the statistics of the modal parameters considering the measurement and modeling errors can be estimated following the flowchart shown in Fig. 1. Particularly, 1000 bootstrap samples (suggested by Efron 1987) are generated for each measurement noise level. Note that the measurement noise level does not need to be known when using the flowchart in Fig. 1.

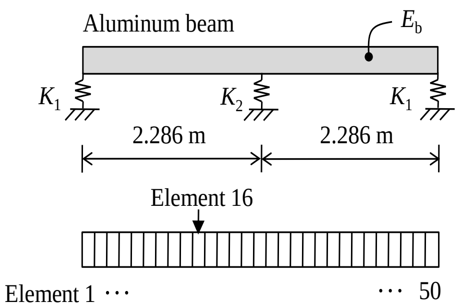


Fig. 3 Schematic of the example beam and the FEM

Table 1 FEM properties for the example beam

Property	Value
Cross section area, A	1.05×10^{-3}m^2
Second moment of area about vertical axis z, I_z	9.57×10^{-7}m^4
Second moment of area about transverse axis y, I_y	7.23×10^{-7} m^4
Mass density, ρ	2710 kg/m^3
Young's modulus, E_b	70 GPa
Poisson's ratio, v	0.33
Stiffness for support springs, K_1	6.0×10^{-5} N/m
Stiffness for support spring, K_2	1.2×10^{-4} N/m

To estimate the modeling error due to the TDD method following the flowchart in Fig. 2, the minimum sample size, ne, needs to be determined first. If a significance confidence level 5% is chosen, and the desired confidence intervals for μ_{ef} and σ_{ef} are set to be $\left[\hat{\mu}_{ef} - \hat{S}_{ef}/5, \hat{\mu}_{ef} + \hat{S}_{ef}/5\right]$ and $\left[0.864\%\hat{S}_{ef}^{\,2}, 1.164\% S_{ef}^{\,2}\right]$ respectively, one can obtain $n_{\mu f}$ = 97 and $n_{\sigma f}$ = 349 through Eqs. (8) and (9). Following the steps for determining nφ, with selected α= 0.05, γ = 0.9, ES = 0.15, and p_e = 6, then n_φ= 123 is obtained. Thus, ne = max (n_f, n_φ) ≥ 350. In this study, n_e = 400 is used. As mentioned previously, E_b, K_1, and K_2 are unknown. Then 400 FEMs are generated with the same configurations as the preliminary FEM by varying the three stiffness properties of the structures. In this study, we use [0.5$\mathbf{x}_0$, 2$\mathbf{x}_0$] as the ranges for E_b, K_1, and K_2, where x_0 are the best guessed values for E_b, K_1, and K_2. Particularly, we let $\mathbf{x}_0$ = [1.1$E_{b,\text{ true}}$, 1.05$K_{1,\text{true}}$, 1.05$K_{2,\text{true}}$], where $E_{\text{b,true}}$, $K_{1,\text{true}}$, $K_{2,\text{true}}$ are the values used for the damaged beam are listed on Table 1.

For each of the 400 FEMs, the modal analysis is conducted to obtain f_i and φ_i, and a pulse excitation is applied to the FEM to obtain the vibration responses, from which $\hat{f}_i$ and $\hat{\boldsymbol{\varphi}}_i$ are extracted through TDD. Then, the modeling errors, e_{fi} and $\mathbf{e}_{\varphi i}$, are assessed using Eqs. (5) and (6). As described previously, the modeling error due to the TDD method is a function of the corresponding influence factors. Given the influence factors, the statistics of modeling error can be estimated. In this illustration, the recorded response length is 4 s and the sampling rate is 1000 Hz, and we use ±1 Hz for the filter width in the TDD method. By changing only one influence factor, we can observe how the modeling errors vary with that influence factor. Fig. 4 shows how the mean and standard deviation of modeling error e_{fi} for the first four frequencies varies when one of the influencing factors, the recorded response length in the vibration test, increases. As the lower modes have the longer periods, given the same response length, the standard deviation of the modeling errors are higher for the lower modes. In other words, longer the response length, more reliable the extracted modal frequencies is. However, after exceeding a certain response length, the error does not change significantly. Other influencing factors can be studied similarly.

The regression models for the mode shape error $\mathbf{e}_{\varphi i}$ of the first three modes are written following the formulation given in Eq. (7) as

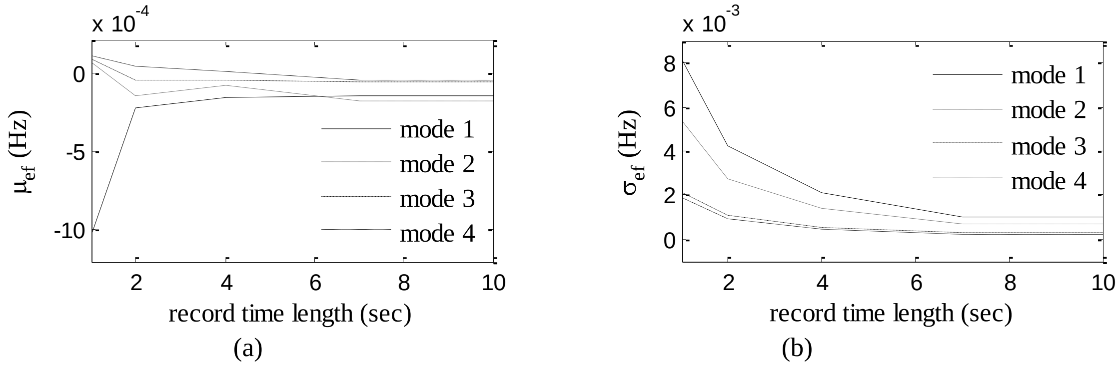


Fig. 4(a) Mean and (b) standard deviation of the estimated modeling error in modal frequencies

$$\mathbf{e}_{\varphi_1} = \alpha_{1,1}\boldsymbol{\varphi}_2 + \alpha_{2,1}\boldsymbol{\varphi}_3 + \alpha_{3,1}\boldsymbol{\varphi}_4 + \beta_{1,1}\left(f_2 - f_1\right)^{\theta_{1,1}}\boldsymbol{\varphi}_2 + \left(\sigma_{0,1}\mathbf{I} + \sigma_{1,1}\left|\boldsymbol{\varphi}_2\right|\right)\varepsilon + \sigma_1\boldsymbol{\varepsilon} \tag{12}$$

$$\mathbf{e}_{\varphi_2} = \alpha_{1,2}\boldsymbol{\varphi}_1 + \alpha_{2,2}\boldsymbol{\varphi}_3 + \alpha_{3,2}\boldsymbol{\varphi}_4 + \beta_{1,2}\left(f_2 - f_1\right)^{\theta_{1,2}}\boldsymbol{\varphi}_1 + \left(\sigma_{0,2}\mathbf{I} + \sigma_{1,2}\left|\boldsymbol{\varphi}_1\right|\right)\varepsilon + \sigma_2\boldsymbol{\varepsilon} \tag{13}$$

$$\mathbf{e}_{\varphi_3} = \alpha_{1,3}\boldsymbol{\varphi}_1 + \alpha_{2,3}\boldsymbol{\varphi}_3 + \alpha_{3,3}\boldsymbol{\varphi}_4 + \beta_{1,3}\left(f_4 - f_3\right)^{\theta_{1,3}}\boldsymbol{\varphi}_4 + \left(\sigma_{0,3}\mathbf{I} + \sigma_{1,3}\left|\boldsymbol{\varphi}_4\right|\right)\varepsilon + \sigma_3\boldsymbol{\varepsilon} \tag{14}$$

where the statistics of the model parameters $\alpha_{j,i}$, $\beta_{j,i}$, $\theta_{j,i}$, and $\sigma_{j,i}$ are shown in Table 2 and they are estimated using the Bayesian updating rule (Box and Tiao 1992).

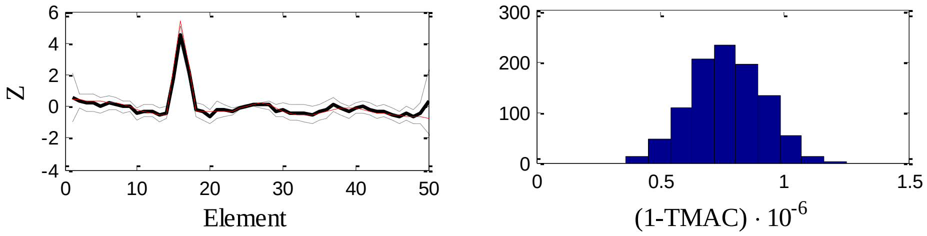
With the consideration of the measurement and modeling errors, the DIM (Stubbs and Kim 1996) is applied using the extracted modal parameters. Since the stiffness properties of the structures, E_b, K_1, and K_2, are unknowns, one could use the extracted modal frequencies to identify a baseline FEM for the study structure following a system identification method proposed by Huang *et al*. (2010). The DIM calculates a damage index, Z, for each element in the FEM based on the mode shapes from the baseline FEM and the mode shapes extracted from the dynamic responses of the study structure. The damage indexes of the damaged elements can be differentiated from the ones of the undamaged elements through statistical approaches, such that the damage location can be identified.

The normalized damage index, Z, is calculated for 50 elements of the beam. For brevity, the procedure for calculating Z is not included here but it can be found in Stubbs and Kim (1996). The left column plots in Fig. 5 give the normalized damage index of each element for the beam under different levels of measurement noise, while the left plot in Fig. 6 shows the damage indexes when only the modeling error due to TDD is considered. The solid lines and the dotted lines are the mean and mean ± 1 standard deviation of Z, respectively.

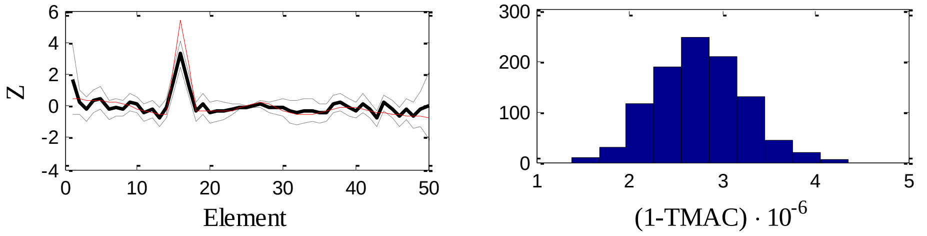
The variability of Z shown in the left column plots of Fig. 5 reflects the influences of the measurement and modeling errors, which are not negligible. This result is consistent with the conclusion drawn from the study conducted by Doebing and Farrar (1997), where they found that the statistical significance of the modal parameters could not be ignored when the modal data are used in the damage detection applications. Moreover, with the increment of the noise level shown in the left column plots of Fig. 5, the mean value of Z of the damaged element decreases, while the standard deviation of Z increases. The larger the standard deviation is, the more uncertainties are brought into the damage detection. Under the high noise level 2%, the Z value of the damaged element is hardly differentiated from other elements, which might make the damage detection difficult. The results show that it is important to account for the statistical significance of modal parameters when modal data are applied for damage detection.

Table 2 Statistics of model parameters

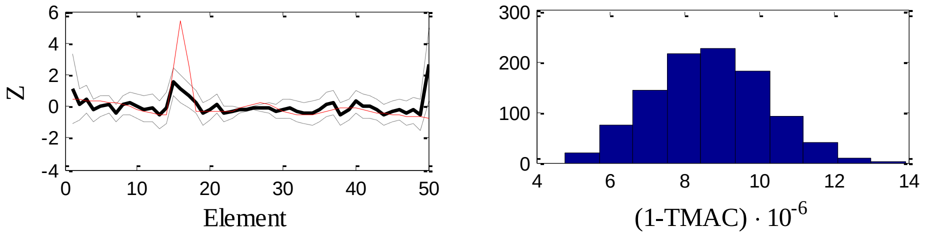
		α_1	α_2	α_3	β_1	θ_1	σ_0	σ_1	σ
Mode 1	Mean	-1.41E-04	-2.57E-05	-1.96E-05	-2.03E+00	1.41E-01	5.38E-05	2.68E-04	8.51E-07
	St. Dev	-9.38E-07	-2.80E-07	-2.70E-07	-2.15E-03	4.82E-04	2.98E-07	2.13E-06	–
Mode 2	Mean	9.10E-05	-4.10E-05	-2.59E-05	-1.70E+00	1.51E-01	1.23E-05	1.08E-03	1.61E-06
	St. Dev.	1.71E-06	-4.26E-07	-3.60E-07	-1.32E-03	3.31E-04	1.19E-07	8.50E-07	–
Mode 3	Mean	1.68E-04	2.20E-04	-1.34E-03	-3.86E+00	-3.55E+01	8.98E-04	6.70E-03	4.10E-05
	St. Dev..	3.21E-05	3.19E-05	-3.56E-05	-2.55E-02	-1.44E+00	8.55E-06	6.11E-05	–



(a) measurement noise level 0.5%



(b) measurement noise level 1.0%



(c) measurement noise level 2.0%

Fig. 5 (left) Z values (thick solid line: mean, dotted line: mean ± 1 standard deviation, thin solid line: no error considered) for the beam elements and (right) $1-\text{TMAC}$ for mode shapes

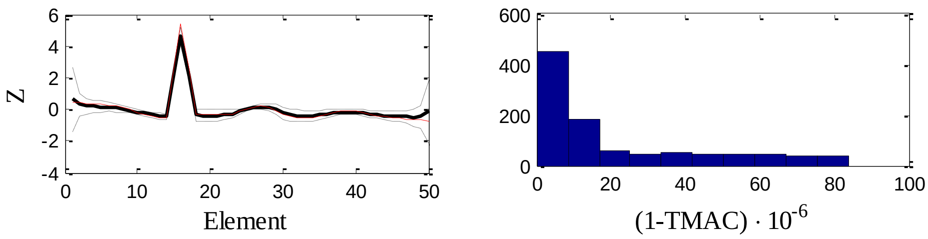


Fig. 6 (left) Z values (thick solid line: mean, dotted line: mean ± 1 standard deviation, thin solid line: no error considered) for the beam elements and (right) $1-\text{TMAC}$ for mode shapes considering the estimated modeling error due to TDD

Another observation is that the influence on Z from the measurement and the modeling errors shown in the left column plots of Fig. 5 is larger than the influence from the estimated modeling error due to the TDD method alone shown in the left plot of Fig. 6. Therefore, more emphasis needs to be put on reducing the measurement noise when the dynamic responses are collected.

To exam the influence of the measurement and modeling errors on the mode shape only, the modal assurance criteria (MAC) (Allemang and Brown 1982) is used for estimating the degree of correlation between two mode shape vectors. When N_T modes are considered, a Total MAC (TMAC) between two sets of mode shapes from two records, records *a* and *b*, can be calculated by

$$\text{TMAC} = \prod_{i}^{N_T} \frac{\left|\boldsymbol{\varphi}_{i,a}^T \boldsymbol{\varphi}_{i,b}\right|^2}{\left(\boldsymbol{\varphi}_{i,a}^T \boldsymbol{\varphi}_{i,a}\right)\left(\boldsymbol{\varphi}_{i,b}^T \boldsymbol{\varphi}_{i,b}\right)} \tag{15}$$

The value of TMAC is between 0 and 1. When TMAC is close to 1, it indicates the mode shapes obtained from the two records *a* and *b* are very similar. In other words, the bigger the value of (1 − TMAC) is, the bigger the difference between the mode shapes influenced by the measurement and modeling errors is. Thus, (1 − TMAC) reflects the variability in mode shapes in an overall sense.

The right column plots in Fig. 5 show the values of (1 − TMAC) for the mode shapes of the beam under different levels of measurement noise. The right plot in Fig. 6 shows the values of (1 − TMAC) considering the modeling error due to TDD alone. All the values of (1 − TMAC) are smaller than 1.5×10^{-4}, indicating that TMAC is close to 1. Therefore, the variability in the mode shapes due to measurement and modeling errors can be considered to be low. However, as discussed previously and shown in the right plots of Figs. 5 and 6, this low variability has a significant impact on the damage detection results. Moreover, as expected, when the measurement noise level becomes higher the values of TMAC become smaller, which means the variability in mode shapes is higher as shown in the right column plots of Fig. 5.

Interestingly, the values of TMAC in the right column plots of Fig. 5 are much larger than the values in the right plot of Fig. 6, indicating that the different sets of mode shapes due to measurement and modeling errors are more similar to each other than the sets of mode shapes considering only the modeling errors. However, the Figs. 5 and 6shows that the influence on the damage detection due to measurement and modeling errors (shown in the right column plots of Fig. 5) is significantly larger than the influence due to modeling error (shown in the right plot of Fig. 6). Therefore, using the TMAC values cannot be used for measuring the influence of the measurement and modeling errors on the damage detection.

As discussed previously, when the measurement noise level becomes high, the damage detection may become impossible. Since the level of the measurement noise is unknown in reality, it is desire to have a quality that can be used to measure the variability of mode shapes so that by examining this quality one can determine whether the extracted modal parameters are useful for the damage detection and how reliable the damage detection results are. One option of such quantity could the standard deviation of the damage index and the standard deviation can be calculated using the data pool of modal shapes generated following the proposed procedure shown in Fig. 1. However, the further study in this regard is needed.

5. Conclusions

This paper presented procedures to account for the uncertainties in the modal parameters extracted from ambient vibration tests. In particular, the focus is on the measurement errors and the modeling error resulting from Time Domain Decomposition (TDD). To estimate the statistics of the extracted modal parameters considering these errors, a bootstrap approach is used to generate artificial records based on limited number of response records obtained from the ambient vibration test. As it is critical to estimate the modeling errors associated with the extraction process particularly for system identification applications, the model prediction expansion is adopted in this study to assess the modeling errors. The proposed procedure to evaluate the modeling error associated with TDD can be extended to other extraction methods.

The proposed procedures are illustrated by a numerical example of a two-span continuous aluminum beam. The influences from measurement and the modeling errors are studied by applying a damage detection method, the Damage Index Method (DIM), on the example beam using the extracted modal parameters. Future work should include applying the proposed procedures using laboratory or field data.

References

Allemang R.J. and Brown, D.L. (1982), "A correlation coefficient for modal vector analysis", *Proceedings of the 1st Int. Modal Analysis Conference*, Orlando, FL, Nov.

Ang, H.S. and Tang, W. (2006), *Probability Concepts in Engineering: Emphasis on Applications to Civil and Environmental Engineering*. 2nd Ed., New York, NY: John Wiley & Sons.

Arici, Y. and Mosalam, K.M. (2005), "Statistical significance of modal parameters of bridge systems identified from strong motion data", *Earthq. Eng. Struct. D.*, **34**(10), 1323-1341.

Banan, M.R., Banan, M.R. and Hjelmstad, K.D. (1994), "Parameter estimation of structures from static response.II. Numerical simulation studies", *J. Struct. Eng. – ASCE*, **120**(11), 3259-3283.

Beck, J.L. and Au, S.K. (2002), "Bayesian updating of structural models and reliability using Markov chain Monte Carlo simulation", *J. Eng. Mech. – ASCE*, **128**(4), 380-391.

Brinker, R., Zhang, L. and Andersen, P. (2000), "Modal identification from ambient responses using frequency domain decomposition", *Proceedings of the 18th International Modal Analysis Conference*, San Antonio, TX.

Box, G.E.P. and Tiao, G.C. (1992), *Bayesian Inference in Statistical Analysis*, Wiley, New York, New York, USA.

Ceracolo, R., Pescatore, M. and De Stefano A. (2009), "Symptom-based reliability and generalized repairing cost in monitored bridges", *Reliab. Eng. Syst. Safe.*, **94**(8), 1331-1339.

Ching, J., Muto, M. and Beck, J.L. (2006), "Structural modal updating and health monitoring with incomplete modal data using gibbs sampler", *Comput. - Aided Civil Infrastruct. E.* , **21**(4), 242-257.

Cohen, J. (1988), *Statistical Power Analysis for the Behavioral Sciences* (2nd Ed.), Lawrence Earlbaum Associates, Hillsdale, New Jersey, USA.

Doebling, S.W. and Farrar, C.R. (1997), "Effects of measurement statistics on the detection of damage in the Alamosa Canyon Bridge", *Proceedings of the 15th International Modal Analysis Conference*, Orlando, FL, Feb.

Doebling, S.W., Farrar, C.R. and Prime, M.B. (1998), "A summary review of vibration-based damage identification methods", *Shock Vib. Digest*, **30**(2), 91-105.

Doebling, S.W. and Farrar, C.R. (2001), "Estimation of statistical distribution for modal parameters identified from average Frequency response function data", *J. Vib. Control*, **7**, 603-624.

Efron, B. (1982), *The Jackknife, the Bootstrap and Other Resampling Plans*, Society for industry and Applied Mathematics, Philadelphia, Pennsylvania, USA.

Efron, B. (1987), "Better bootstrap confidence intervals (with discussion)", *J. American Statistical Association*, **82** (397), 171-185.

Gardoni, P., Der Kiureghian, A. and Mosalam, K.M. (2002), "Probabilistic capacity models and fragility estimates for RC columns based on experimental observations", *J. Eng. Mech. – ASCE*, **128**(10), 1024-1038.

Haukaas, T. and Gardoni, P. (2011), "Model uncertainty in finite-element analysis: Bayesian finite elements", *J. Eng. Mech. – ASCE*, **137** (8), 519-526.

Huang, Q., Gardoni, P. and Hurlebaus, S. (2010), "A probabilistic damage detection approach using vibration-based nondestructive testing", *Struct. Saf.*, **38**, 11-21.

Kim, B., Stubbs, N. and Park, T. (2005), "A new method to extract modal parameters using output-only responses", *J. Sound Vib.*, **282**, 215-230.

Liu, P.L. (1995),"Identification and damage detection of trusses using modal data", *J. Struct. Eng. - ASCE*, **121**(4), 599-608.

Longman, Rw and Juang, J.N. (1987), "A variance based confidence criterion for ERA identified modal parameters", *Proceedings of the AAS/AIAA Astrodynamics Specialist Conference*, Kalispell, MT, August.

Mckay, M.D., Beckman, R.J. and Conover, W.J. (1979), "Comparison of three methods for selecting values of input variables in the analysis of output from a computer code", *Technometrics*, **21**, 239-245.

McKenna, F. and Fenves, G.L. (2002), *The Open Sees Command Language Primer*, Department of Civil and Environmental Engineering, University of California, Berkeley, California, USA.

Oberkampf, W.L., DeLand, S.M., Rutherford, B.M., Diegert, K.V. and Alvin, K.F. (2002), "Error and uncertainty in modeling and simulation", *Reliab. Eng. Syst. Safe,* **75**(3), 333-357.

Papadopoulos, L. and Garcia, E. (1998), "Structural damage identification: a probabilistic approach", *AIAA J.*, **36**(11), 2137-2145.

Peterson, L.D., Bullock, S.J. and Doebling, S.W. (1996), "The statistical sensitivity of experimental modal frequencies and damping ratio to measurement noise", *Modal Anal.*, **11**(1), 63-75.

Pothisiri, T. and Hjelmstad, K.D. (2003), "Structural damage detection and assessment from modal response", *J. Eng. Mech. - ASCE* , **129**(2), 135-145.

Reinert, J.M. and Apostolakis, G.E. (2006), "Including model uncertainty in risk-informed decision making", *Ann. Nucl. Energy*, **33**, 354-369.

Stubbs, N. and Kim, J.-T.(1996), "Damage localization in structures without baseline modal parameters", ,*AIAA J.*, **34**(8), 1644-1649.

Xia, Y., Hao, H., Brownjohn, J.M.W. and Xia, P.Q. (2002), "Damage identification of structures with uncertain frequency and mode shape data", *Earthq. Eng. Struct. D.*, **31**, 1053-1066.

Zio, Z. (2009), "Reliability engineering: Old problems and new challenges", *Reliability Engineering & System Safety* 94(2), 125-141.