Smart Structures and Systems, *Vol. 13, No. 1 (2014) 099-109*
DOI: http://dx.doi.org/10.12989/sss.2014.13.1.099

Quantity vs. Quality in the Model Order Reduction (MOR) of a Linear System

Sara Casciati[1a] and Lucia Faravelli[*2]

[1]*Department DICA, University of Catania, Piazza Federico di Svevia, 96100 Siracusa, Italy*
[2]*Department DICAR, University of Pavia, Via Ferrata 1, 27100 Pavia, Italy*

(Received March 13, 2013, Revised April 20, 2013, Accepted May 4, 2013)

Abstract. The goal of any Model Order Reduction (MOR) technique is to build a model of order lower than the one of the real model, so that the computational effort is reduced, and the ability to estimate the input-output mapping of the original system is preserved in an important region of the input space. Actually, since only a subset of the input space is of interest, the matching is required only in this subset of the input space. In this contribution, the consequences on the achieved accuracy of adopting different reduction technique patterns is discussed mainly with reference to a linear case study.

Keywords: dynamic analysis; model order reduction (MOR); numerical model; truncation

1. Introduction

Numerical tools for structural dynamics are widely available both on the software market ((MSC 2013) among others) and as open access codes ((SDTools 2012) among others). They also include special features as sub-structuring and "model order reduction" (MOR) which become quite useful when iterative procedures are required, as in the design of a control system or, along an optimization process, in conducting a sequence of numerical analyses allowing one to estimate the sensitivity of the selected utility function to the different design variables (Benjeddou 2009; Casciati 2008, 2010).

Sub-structuring is a process where models are divided into components and component models are reduced before a coupled system prediction is performed. This process is known as Component Mode Synthesis in the literature. The reference (Craig 1987) details the historical perspective, which relies on the assumption of linear system response; the references (MacNeal 1971) and (Hurty *et al.* 1971) can be regarded as the pioneering papers in the area.

Model order reduction (MOR) is a term used in several different situations. The reference (Preumont and Seto 2008) is mainly using it to address the reduction of a continuous system to a model with a finite number of degrees of freedom. Actually, finite element models of structures need to have many degrees of freedom to represent the geometrical detail of complex structures. For models of structural dynamics, one is however interested in:

*Corresponding author, Professor, E-mail: lucia@dipmec.unipv.it
[a]Assistant Professor, E-mail: saracasciati@msn.com

http://www.techno-press.org/?journal=sss&subpage=8 ISSN: 1738-1584 (Print), 1738-1991 (Online)

- a restricted frequency range;
- a small number of inputs and outputs;
- a limited parameter space.

These restrictions on the expected predictions allow the creation of low order models that accurately represent the dynamics of the full order model in all the considered loading/parameter conditions.

Model Order Reduction (MOR) technique (Schilders *et al.* 2008) offers a low computational effort coupled with the ability to estimate, in an important region of the input space, the input-output mapping of the original system.

Within the MOR techniques, the current research efforts (Carlberg *et al.* 2011, Kaczmarek 2010) are mainly addressed to capture lower order models for nonlinear systems. In particular the proper generalized decomposition is attracting the attention of several research groups (see (Chinestaa *et al.* 2011), among others). A further aspect which should be mentioned is how to incorporate the effects induced by uncertainties and lack of knowledge.

Nevertheless, also the correct application of MOR when dealing with linear systems still requires attention. In this paper, a linear, time invariant structural problem is studied and the manner in which the reduction cascade can affect the accuracy of the final reduced model is discussed. The investigation was motivated by the fact that many benchmark problems ((Ohtori *et al.* 2004), from where the case study is taken, but also (Ni *et al.* 2012)) were recently suggested in the literature covering structural control and structural health monitoring. All of them assume a real structure as reference and then introduce a model of reduced order to compute the structural response. Often, however, the way to generate the reduced order model is not publicized and this leaves a missed link between the structure one presumes to study and the responses one simulates.

2. Governing relations

The numerical models adopted in structural engineering consist of partial differential equations whose spatial derivatives (the elliptic components) are easily accounted by algebraic equations after finite element discretization. For a dynamic system, then the problem is rearranged in terms of ordinary differential equations where the second derivative with respect to time appears. The number of equations represents the order of the model, say N. Introducing the so-called state space representation, only linear derivative of time are considered, with the number of equations being doubled. One writes

$$\dot{z}\,(t) = A\,z(t) + B\,u(t) \tag{1}$$

where z is the state variable vector of size $2N$, the superimposed dot denotes time derivative, u is the vector of the external excitations, of size p, and A and B are time invariant matrices of sizes $2N$ by $2N$ and $2N$ by p, respectively. The state variables are not supposed to have any physical meaning. But they are linked to any set of observables variables $y(t)$ (denoted as "observed variables") by a second set of equations, this time of the algebraic type

$$y(\,t) = C\,z(t) + D\,u(t) \tag{2}$$

Provided the vector y is ordered to give N (relative to the base) displacements followed by N (relative) velocities, a further vector of length N can be computed as the vector of the (absolute) accelerations, provided no external action enters the equilibrium equation (i.e., p=1 and $u(t)$

represents the base excitation only)

$$y_a(t) = -\left[\left(\frac{K}{M}\right)\ \left(\frac{c}{M}\right)\right] y(t) \tag{3}$$

where M, c and K are the matrices of mass, damping and stiffness, respectively. D is assumed to be 0 in that follows.

Model reduction procedures are discrete versions of Ritz/Galerkin analyses: they seek solutions in the subspace generated by a reduction matrix T. Assuming $\psi = T\psi_R$, the second order finite element model for structural dynamics in the frequency domain (with $Z(s) = Ms^2 + cs + K$), is projected as follows

$$[T^T MTs^2 + T^T cTs + T^T KT]_{N_R \times \backslash N_R} \quad \psi_R(s) = T^T b\ \upsilon(s) \tag{4}$$

Modal analysis, model reduction, component mode synthesis, and related methods, all deal with an appropriate selection of singular projection bases ($T_{Nx\ NR}$ with $N_R << N$). An accurate model is defined by the fact that the input/output relation is preserved for a given frequency and parameter range (say α)

$$C\,[Z(s, \alpha)]^{-1}\, b = [CT]\,[T^T Z(s, \alpha)T]^{-1}\,[T^T b] \tag{5}$$

Three main policies for reducing the model order in such a linear context can be foreseen:

A) Assume that some subsets of z variables can be replaced by fictitious variables, with all the z in the subset equal to its replacement; the approximation propagates to Eq. (2) where the number of different observed variables decreases (the nodes in the geometry discretization are regarded as masters and slaves);

B) Assume that the left end side of some equations in Eq. (1) is negligible; then those equations can be solved as algebraic equation to obtain an expression for the associated state variable; this expression is used in Eq. (1) to reduce the order and in Eq. (2) to express all the original observed variables in terms of a reduced number of state variables (often referred to as static condensation);

C) Re-write Eq. (1) in a different basis system and apply to the obtained balanced system a truncation using Hankel singular values; the basis transformation also applies to Eq. (2) and after truncation just a bit of information is lost.

The last path requires further details. From the model of Eqs. (1) and (2), the Gramian matrices of controllability and observability, W_c and W_o, satisfy the following pair of Lyapunov equations

$$A\,W_c + W_c\,A^T + BB^T = 0 \tag{4}$$

$$A^T\,W_o + AW_o + C^T C = 0 \tag{5}$$

respectively. The steps to perform a reduction by balanced transformation are described as follows.

1) Find the Gramian matrices W_c and W_o as solutions of the Lyapunov equations ;

2) Perform the Cholesky factorizations of the Gramian matrices

$$W_c = L_c L_c^T\ ;\ W_o = L_o L_o^T \tag{6}$$

3) Consider the Singular Values Decomposition (SVD) of the Cholesky factors

$$L_o^T L_c = U \Lambda V^{\boldsymbol{T}} \tag{7}$$

where U is the matrix of the eigenvectors of the matrix in the r.h.s., say Q, by its transpose, V is the matrix of the eigenvectors of $Q^T Q$ and Λ is the diagonal matrix of the singular values.

4) Define the balanced transformation

$$T = L_c V \Lambda^{-1/2}; \quad T^{-1} = \Lambda^{-1/2} U^T L_o^T \tag{8}$$

5) Build the state space matrix balanced representation by introducing the matrices

$$A_b = T^{-1} A T = \Lambda^{-1/2} U^T L_o^T \ A \ L_c V \Lambda^{-1/2} \tag{9}$$

$$B_b = T^{-1} B = \Lambda^{-1/2} U^T L_o^T \ B \tag{10}$$

$$C_b = CT = C L_c V \Lambda^{-1/2} \tag{11}$$

The Hankel singular values are then introduced to operate a model reduction.

3. Discussing the effects of the different policies on a case study

The different policies of Model Order Reduction (MOR) introduced in the previous section were applied to the structural system in Fig. 1, used for the benchmark defined in (Ohtori *et al.* 2004). In particular Table 1 gives the different model adopted. Indeed, the starting model consists of 138 nodes for a total of 396 degrees of freedom (full model, say F), but they are reduced to 291 (model A) by assuming that the horizontal displacements of all the nodes in the same floor are identical. The dynamic excitation is restricted to the horizontal ground acceleration which follows a recorded time history (the El Centro record is selected among the 4 offered by the benchmark in (Ohtori *et al.* 2004)).

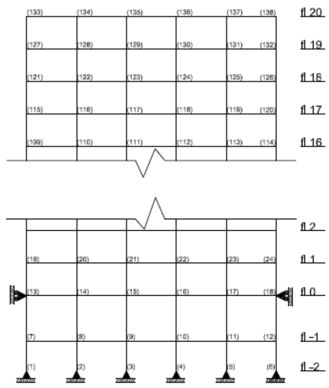
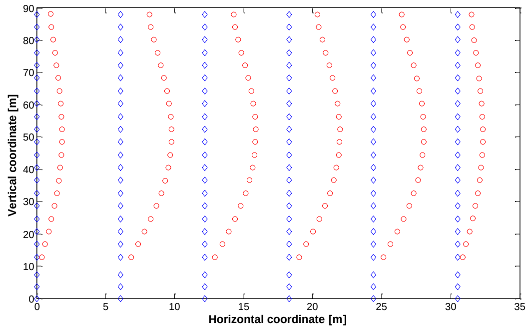


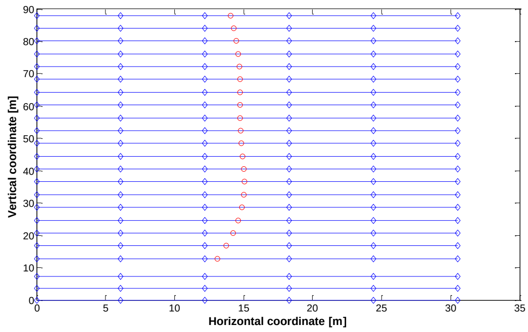
Fig. 1 The frame analyzed in (Ohtori *et al.* 2004) and studied in this paper

Table 1 Modelling the plane frame in Figure 1 (dof = degree of freedom)

Model	Description	Number of states	Solution time
F (full)	3 independent dof for every node	**792**	**4h 30’**
FC66	F balanced and truncated	66	2’ 27”
FC44	F balanced and truncated	44	2’ 19”
FC20	F balanced and truncated	20	1’ 52”
A (replacing some *z* variables by a subset)	2 independent dof for every node + 1horizontal dof for each floor	**582**	**3h 00’**
AC20	A balanced and truncated	20	1’ 30”



(a)



(b)

Fig. 2 The diamonds represent the initial positions of the nodes, the circles the deformed positions at t = 6.86 s (to be regarded as any generic time step) . In (a) all the deformed positions are provided; in (b) only the master nodes is shown, since all the floor displacements are forced to be equal to that of the master node

In Fig. 2 a summary of the solutions discussed in this paper is graphically given. Again, model F means that the degrees of freedom for the nodes of each storey are all independent one of the other, while model A means that the horizontal motion is forced to be the same for all nodes in every single storey. Since the stiffness of the external column is lower than the one of the internal columns, and the associated node mass is halved, in model A the external columns show quite lower horizontal displacements than those of the internal nodes at the same floor. The horizontal displacements of nodes 135 and 138 (as in Figure 1), as computed by model F, are compared in Fig. 3. This different behaviour of the two nodes is fully lost in model A.

Indeed model A provides, for each floor, a horizontal displacement response which fits well the average of the six displacements of that specific floor, as shown in Fig. 4, where a zoom also allows one to appreciate some minor discrepancies. Fig. 5 is organized as Fig. 4, but the time histories are those of the velocity associated with the horizontal degree of freedom, and the discrepancies are now on the peak values.

Fig. 6, eventually, provides a comparison of the accelerations. In this figure the horizontal acceleration of node 135, obtained from model A, is compared with the average of the accelerations computed for the six nodes of floor 20. It is seen that also the accelerations show a good agreement. But, in Figure 6, also the actual horizontal accelerations of nodes 135 and 138, as resulting from model F are drawn, to provide evidence of the high frequency components in their plots. In other words, here the average is a poor information of the behavior of the single node in the same floor.

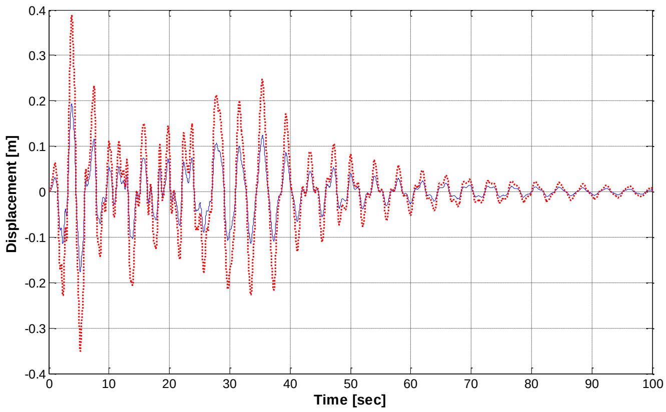
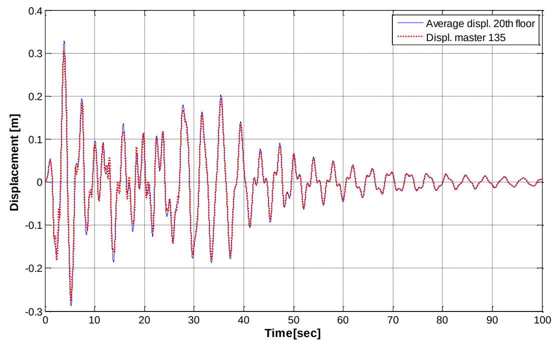
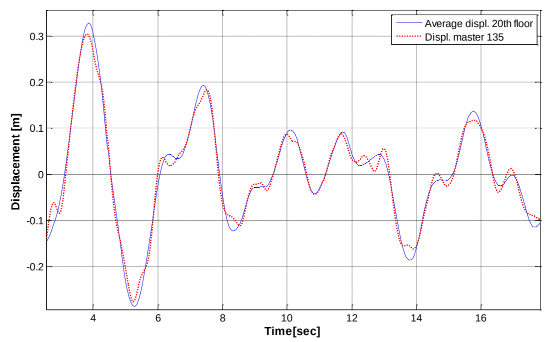


Fig. 3 Comparison between the actual horizontal displacements of nodes 135 (dotted line) and 138 (solid line)

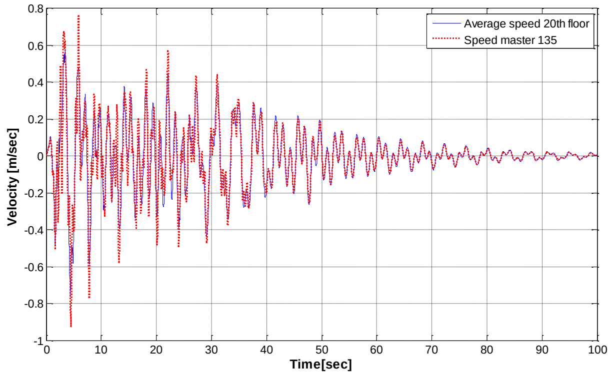


(a)

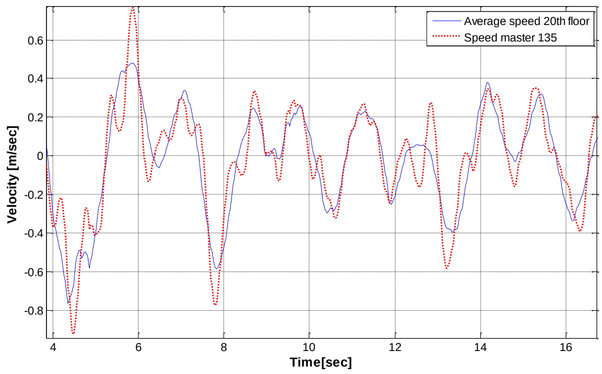


(b)

Fig. 4 Top-floor response displacement time histories: (a) the dotted line results from model A, while the solid line is the average of the floor displacements as computed in model F; (b) a zoom from (a)



(a)



(b)

Fig. 5 Top-floor horizontal-response velocity time-histories: (a) the dotted line results from model A, while the solid line is the average of the floor velocities as computed in model F; (b) a zoom from (a)

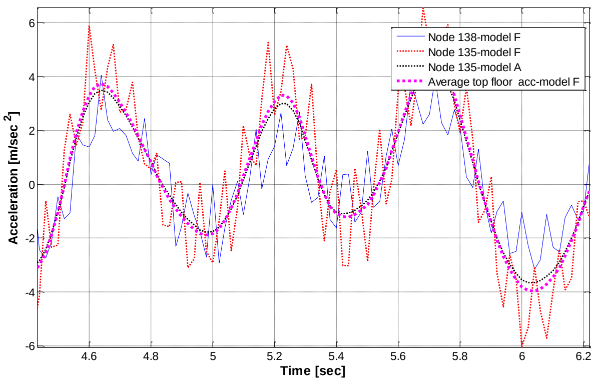
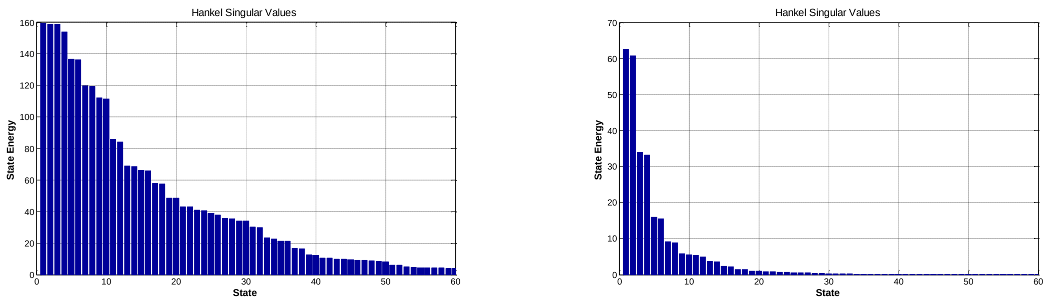


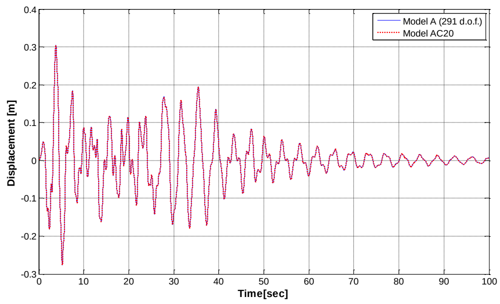
Fig. 6 Top-floor horizontal-response acceleration time-histories (zoom)

It is worth noting that in (Casciati *et al.* 2012), two further models were investigated, achieved by static condensation of the rotation degrees of freedom (model B1; 153 degrees f freedom) and of both the rotation and the vertical degrees of freedom (model B2; 42 degrees of freedom). In the latter case the so called shear type model is achieved. In (Casciati *et al.* 2012) balanced and truncated models (B1C and B2C) were introduced for these two further models. In both cases a reduced order model made of 20 states is achieved, each different from the other. In this specific paper such balanced and truncated models are achieved directly from models F and A, only. They will be marked as FC*n* or AC*n*, with *n* denoting the number of reduced states.

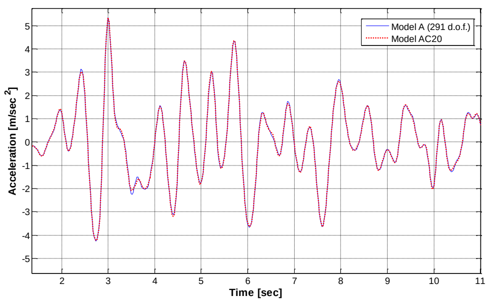


(a) (b)

Fig. 7 Plot of the Hankel singular values for model F (a) and model A (b)



(a)

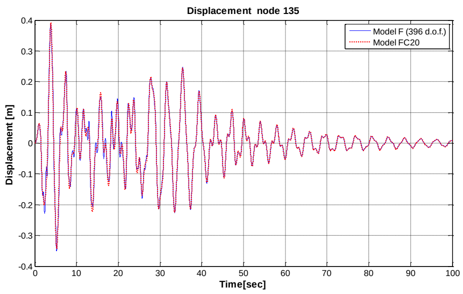


(b)

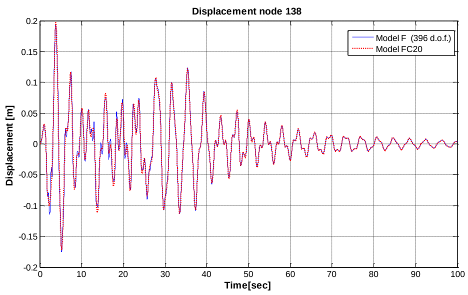
Fig. 8 Comparison: (a) between the horizontal displacements of node 135 achieved by models A and AC20 and (b) between the horizontal acceleration of node 135 achieved by models A and AC20

The authors of (Ohtori *et al.* 2004) introduced balancement and truncation on model A. With the plot of the Hankel singular values in Fig. 7(b)), the truncation was introduced at 20 states (model AC20). A comparison of the response achieved by model A and model AC20 is given in Fig. 8. It is seen that both displacements and accelerations are in perfect agreement.

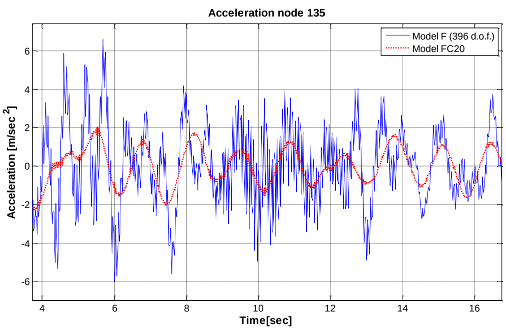
A trivial repetition of the procedure starting from model F and, forgetting to check the plot of the Hankel singular values in Fig. 7(b)), truncation at 20 states, would provide the results in Figure 9 (model FC20), where it is seen a good agreement in displacement but not in accelearation. The exam of Fig. 7(a)) suggests that one cannot truncate before 60: in Fig. 10 (model FC66), one adopted 66 states to achieve a good agreement also in terms of acceleration. In order to provide a quantitative measure of the approximation, Table 2 gives the sum of the squares of the differences between the response, in terms of acceleration, from truncated models and the real response.



(a)



(b)



(c)

Fig. 9 Comparison of the results achieved by model F and FC20: (a) between the horizontal displacements of node 135, (b) between the horizontal displacement of node 138 and (c) between the acceleration of node 135

Table 2 Sum of the squares (divided by the number of points) of the differences between the acceleration response from truncated models and the true one, for the single nodes of the top floor

	Sum of squares of the differences from the true response			
	A-AC20	F-FC20	F-FC44	F-FC66
Node 138		0.2392	0.0831	0.0758
Node 137		0.6684	0.087	0.0643
Node 136		0.7546	0.2016	0.1746
Node 135 (master)	0.0000518	0.7546	0.2016	0.1746
Node 134		0.6684	0.087	0.0643
Node 133		0.2392	0.0831	0.0758

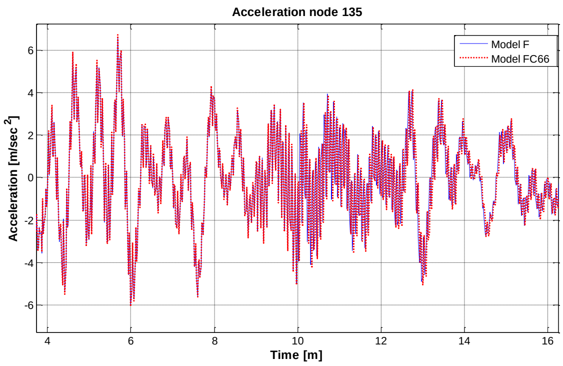


Fig. 10 Comparison: a) between the horizontal acceleration of node 135 achieved by models F and FC66

4. Conclusions

The benchmark structural frame adopted in reference (Ohtori *et al.* 2004) was studied by a simplified model obtained after a first reduction of the mechanical degrees of freedom and then after the standard balancement-truncation procedure. Alternatively one can directly study the reduced order model achieved by applying the balancement-truncation procedure to the initial full numerical model.

The investigation on this paper is intended to emphasize the bit of information lost on the kinematics of the structural system, associated with the true independent relative motion of nodes in the same floor. A direct model order reduction, on the other side, would have required a larger amount of states if the accuracy is pursued not only at displacement and velocity level, but also for the acceleration. It is worth noticing however that this increased number of states comes with CPU solution times of the order of a few minutes, still consistent with the envisaged use in structural monitoring and structural control.

Future work is expected to cover 3D structural models to facilitate the investigation of monumental cultural heritage systems (Casciati-Borja 2004, Casciati-Osman 2005, Casciati-Al Saleh 2010).

Acknowledgements

The research activity summarized in this contribution was developed within the framework of the Marie Curie European project SMARTEN (http://www.smarten-itn.eu/index.html). The first author acknowledges the support of the 2010 Athenaeum Research Fund.

Moreover, the authors acknowledge the cooperation of Marco Baldin who developed the numerical analyses as partial fulfillment of his duties toward the Master in Civil Engineering at the University of Pavia.

References

Benjeddou, A. (2009), "New insights in piezoelectric free-vibrations using simplified modeling and analyses" , *Smart Struct. Syst.*, **5**(6), 591-612.

Carlberg, K., Bou Mosleh, C. and Farhat, C. (2011), "Efficient non-linear model reduction via a least-squares Petrov-Galerkin projection and compressive tensor approximations", *Int. J. Numer. Method.*, **86**(2),155-181.

Casciati, F., Casciati, S., Faravelli, L. and Franchinotti, M. (2012), "Model order reduction vs. structural control", *Proceedings of the ACMA2012*, Fez (Marocco).

Casciati, S. (2008), "Stiffness identification and damage localization via differential evolution algorithms", *Struct.Control Health Monit.*, **15**(3), 436-449.

Casciati, S. (2010), "Response surface models to detect and localize distributed cracks in a complex continuum", *J. Eng. Mech. - ASCE*, **136**(9), 1131-1142.

Casciati, S. and Al-Saleh, R. (2010), "Dynamic behavior of a masonry civic belfry under operational conditions", *Acta Mech.*, **215**(1-4), 211-224.

Casciati, S. and Borja, R.L. (2004), "Dynamic FE analysis of south memnon colossus including 3D soil-foundation-structure interaction", *Comput. Struct.*, **82**(20-21), 1719-1736.

Casciati, S. and Osman, A. (2005), "Damage assessment and retrofit study for the luxor memnon colossi", *Struct.Control Health Monit.*, **12**(2), 139-156.

Chinestaa, F., Ammarb, A., Leyguea, A. and Keuningsc, R. (2011), "An overview of the proper generalized decomposition with applications in computational rheology", *J. Non-Newton. Fluid.*, **166**(11), 578-592.

Craig, R.J. (1987), "A review of time-domain and frequency domain component mode synthesis methods", *Int. J. Anal. Exp. Modal Anal.*, **2**(2), 59-72.

Hurty, W.C., Collins, J.D. and Hart, G.C. (1971), "Dynamic analysis of large structures by modal synthesis techniques", *Comput. Struct.*, **1**(4), 535-563.

Kaczmarek, J. (2010), "Collection of dynamical systems with dimensional reduction as a multiscale method of modelling for mechanics of materials", *Interact. Multiscale Mech.*, **3**(1), 1-22.

MacNeal, R.H. (1971), "A hybrid method of component mode synthesis", *Comput. Struct.*, **1**(4), 581-601.

Ni, Y.Q., Xia, Y., Lin, W., Chen, W.H. and Ko, J.M. (2012), "SHM benchmark for high-rise structures: a reduced-order finite element model and field measurement data", *Smart Struct. Syst.*, **10**(4), 411-426.

MSC (2013), http://www.mscsoftware.com.

Ohtori, Y., Christenson, R.E. and Spencer, B.F. (2004), "Benchmark control problems for seismically excited nonlinear buildings", *J. Eng. Mech. - ASCE.*, **130**, 366-376.

Preumont, A. and Seto, K. (2008), *Active control of structures*, John Wiley & Sons, Chichester.

Schilders, W.H.A., van der Vorst, H.A. and Rommes, J. (2008), *Model order reduction: theory, research aspects and applications*, Springer, Berlin.

SDTools (2012), *Structural dynamics toolbox*, User's Guide on December 5, 2012.

FC