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A sectorial element based on Reissner plate theory

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Abstract. In this study, a new functional based on the Reissner theory, for thick plates on a Winkler foundation is obtained. This functional has geometric and dynamic boundary conditions. In deriving the new functional, the Gâteaux differential is used. This functional which is in polar coordinates is also transformable into the classical potential energy equation. Bending and torsional moments, transverse shear forces, rotations and displacements are the basic unknowns of the functional. Two different sectorial elements are developed with 3×8 degrees of freedom (SEC24) and 4×8 degrees of freedom (SEC32). The accuracy of the SEC24 and SEC32 elements together are verified by applying the method to some problems taken from literature.

Key words: sectorial element; Reissner plate; mixed-finite element.

1. Introduction

Plates as structural elements find many areas of application in engineering fields. Therefore plenty of research exist in literature. To provide a more reliable representation of structural behaviour, several refined theories have attempted to include the effects of transverse shear strain, which becomes important as the ratio of plate thickness to characteristic length (h/2a) decreases. Mindlin and Reissner plate theories satisfy these requirements (Reissner 1946, Reissner 1975, Mindlin 1951). The finite element formulation based on these methods requires C^0 continuity. A problem known as “Shear Locking” is encountered when the plate thickness approaches zero, thereby giving incorrect results for thin plates. Shear locking mechanism was studied and explained by numereous authors (Zienkiewicz *et al.* 1977, Pugh *et al*. 1978). We will not attempt to make a detailed review. Interested readers may find additional information in literature (Zienkiewicz 1977, Gallagher 1975, Bathe 1982, Reddy 1993). To the best of our knowledge, we can cite the following two basic approaches in literature to remedy the situation.

In the first approach, a quadrilateral element for thin and thick plates has been developed by Zienkiewicz *et al.* (1971). In this formulation the transverse displacement and two rotations are selected as independent parameters which require only C^0 continuity in the shape function. Hughes *et al.* (1977) have explained the shear locking phenomenon and have developed a very efficient form of bilinear four-node element in which reduced/selective integration are used. Belytschko *et al.*

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(1981) have developed a stabilization matrix. This concept was first proposed by Kavanagh and Kay (1972). In this approach the necessity for a free parameter may be a disadvantage.

The second approach to tackle these problems is the discrete Kirchhoff plate theory (DKT). To capture the behaviour of thin plate theory, the constraint of zero shear strains is imposed at discrete number of points by Katılı (1993a, b). Imposition of the Kirchhoff constraint at a discrete number of points leads to expressions for normal rotations and subsequent operation procedure in the element stiffness matrix.

Recently Eratlı and Aköz (1997) have developed REC32 and TR48 elements using Gâteaux approaches. This method was tested in various problems. In these problems, shear locking phenomenon has not been encountered (In test problem, h/r was taken as 0.001). To understand the shear locking phenomenon, it is helpful to inspect the classical strain energy. The strain energy can be separated into bending energy and shear energy. When thickness of plate decreases, shear terms become the dominant part of the energy. On discretization of energy term and subsequent minimization, a system of equations the form;

$$[\boldsymbol{K}_b+\beta\boldsymbol{K}_s]\ \boldsymbol{u}=\boldsymbol{f} \tag{1}$$

is obtained. In thin plate solution, β increases indefinitly. The matrix $[\boldsymbol{K}_s]$ corresponds to the shear energy. Such a solution will be over-constrained and an unrealistic answer with $\boldsymbol{u}\rightarrow 0$ will be obtained. The inspection of the energy functional (Eratlı and Aköz 1997) shows that when plate thickness decreases, the shear energy also decreases relative to bending energy. Therefore, the shear locking is eliminated in this formulation.

In this study, first; the plate equations are derived in polar coordinates for thick plates. The polar coordinates are very suitable for many problems, such as circular plates and sectorial plates. The closed form element equation is obtained, which eliminates the time consuming numerical integration during FEM analysis. Having the field equations, Hellinger-Reissner, Hu-Washizu or Gâteaux differential approaches can yield a functional that is essential for finite element formulation. There are classic literature for these methods. Using the Hellinger-Reissner or Hu-Washizu principles the stationary functional is constructed, by adding equilibrium or kinematic equations and suitable boundary conditions to the $\Pi(\boldsymbol{u})$ potential energy or $\Pi^*(\boldsymbol{\sigma})$ complementary energy by Lagrange multiplier method (Washizu 1975, Dym and Shames 1973, Reddy 1984, Oden and Reddy 1976). In Gâteaux differential approaches, first the field equations must be potential, which means these field equations must be produced extremizing the functional with respect to independent parameters. In this study, Gâteaux differential approach is employed. This approach is adopted also for some other studies (Aköz 1985, Aköz *et al.* 1991, Aköz and Uzcan (Eratlı) 1992, Omurtag and Aköz 1994, Eratlı and Aköz; Aköz and Kadıoğlu 1996). Both Hellinger-Reissner and Gâteaux approaches can produce the same functional, it is believed that Gâteaux approach has the following advantages over Hellinger-Reissner or Hu-Washizu approaches:

- The field equation must be consistent Goldenweizer (1961) and Morris (1973). Gâteaux differential method provides consistency of field equations (Aköz and Özütok 2000).
- During the potential test, boundary conditions can be constructed.
- All the field equations are enforced to the functional by systematic way.

2. The field equations of thick plate in polar coordinates

The equilibrium equations of a plate element $h\ r\ dr\ d\theta$ (Fig. 1) in polar coordinates based on

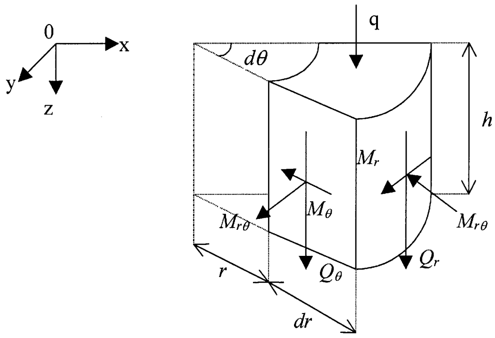


Fig. 1 The positive directions of internal forces

Reissner theory are taken as follows (Panc 1975),

$$\frac{\partial M_r}{\partial r}+\frac{1}{r}\frac{\partial M_{\theta r}}{\partial \theta}+\frac{M_r-M_\theta}{r}-Q_r=0$$

$$\frac{\partial M_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial M_\theta}{\partial \theta}+\frac{2M_{r\theta}}{r}-Q_\theta=0$$

$$\frac{\partial Q_r}{\partial r}+\frac{1}{r}\frac{\partial Q_\theta}{\partial \theta}+\frac{Q_r}{r}+q-kw=0 \tag{2}$$

The positive directions of internal forces are shown in Fig. 1.

$\bar{\Omega}_r(r,\theta,z)$ and $\bar{\Omega}_\theta(r,\theta,z)$ are the components of the rotation of cross-section plane of plate along thickness. Average rotations $\Omega_r(r,\theta,0)$ and $\Omega_\theta(r,\theta,0)$ can be defined by employing energy argument as follows,

$$M_r\Omega_r=\int_{-h/2}^{h/2}\sigma_r\ u\ dz$$

$$M_\theta\Omega_\theta=\int_{-h/2}^{h/2}\sigma_\theta\ v\ dz \tag{3}$$

Assuming linear stress distribution (Panc 1975) we have,

$$\sigma_r=\frac{12M_r}{h^3}z$$

$$\sigma_\theta=\frac{12M_\theta}{h^3}z$$

$$\tau_{\theta r}=\frac{12M_{r\theta}}{h^3}z \tag{4}$$

And inserting Eq. (4) into Eq. (3), we can obtain the following equation for average rotations, similar to Panc (1975).

$$\Omega_r=\frac{12}{h^3}\int_{-h/2}^{+h/2}u\ z\ dz$$

$$\Omega_\theta=\frac{12}{h^3}\int_{-h/2}^{+h/2}v\ z\ dz \tag{5}$$

To obtain stress distribution through thickness, the equilibrium equation in polar coordinates will be used. Inserting linear stress distribution into equilibrium equations, integrating them and using the following boundary conditions:

$$\text{For} \quad \begin{cases} z=\mp h/2 & \tau_{rz}=\tau_{z\theta}=0 \\ z=\ h/2 & \sigma_z=-q \\ z=-h/2 & \sigma_z=0 \end{cases} \tag{6}$$

we will get;

$$\tau_{zr}=\frac{3}{2}\frac{Q_r}{h}\left[1-\left(\frac{2z}{h}\right)^2\right]$$
$$\tau_{z\theta}=\frac{3}{2}\frac{Q_\theta}{h}\left[1-\left(\frac{2z}{h}\right)^2\right]$$
$$\sigma_z=-\frac{q}{4}\left[2-3\left(\frac{2z}{h}\right)+\left(\frac{2z}{h}\right)^2\right] \tag{7}$$

To define the average displacement for the middle plane of plate, we employ energy arguments again similar to Panc approach (1975) as follows:

$$Q_r w(r,\theta,0)=\int_{-h/2}^{h/2}\tau_{rz}\ w^*(r,\theta,z)\ dz$$
$$Q_\theta w(r,\theta,0)=\int_{-h/2}^{h/2}\tau_{r\theta}\ w^*(r,\theta,z)\ dz \tag{8}$$

Inserting the stress distribution for τ_{rz} and $\tau_{r\theta}$ into Eq. (8), we will obtain the following equation for average displacement $w(r,\theta)$;

$$w(r,\theta)=\frac{3}{2h}\int_{-h/2}^{h/2}w^*(r,\theta,z)\left[1-\left(\frac{2z}{h}\right)^2\right]dz \tag{9}$$

To construct the kinematic and constutive equations together, the Panc approach will be accepted (Panc 1975). Assuming energy arguments we have;

$$\int_{-h/2}^{h/2}\gamma_{rz}\tau_{rz}dz=\frac{1}{G}\int_{-h/2}^{h/2}\tau_{rz}^2dz \tag{10}$$

Using Eq. (7) for stress distribution and kinematic relations and substituting;

$$\gamma_{rz}=\frac{\partial u}{\partial z}+\frac{\partial w^*}{\partial r} \tag{11}$$

into Eq. (10) and taking into account the definitions (5) and (9), we end up with,

$$\Omega_r+\frac{\partial w}{\partial r}-\frac{6}{5Gh}Q_r=0 \tag{12}$$

Similarly we obtain,

$$\Omega_\theta+\frac{1}{r}\frac{\partial w}{\partial\theta}-\frac{6}{5Gh}Q_\theta=0 \tag{13}$$

To obtain the next three joint kinematic and constitutive equations we differentiate Eq. (5) and suitably combine such that right hand sides must produce ε_r, ε_θ, $\gamma_{r\theta}$ as follows:

$$\frac{\partial\Omega_r}{\partial r}=\frac{12}{h^3}\int_{-h/2}^{h/2}\varepsilon_r z dz$$

$$\frac{1}{r}\frac{\partial\Omega_\theta}{\partial\theta}+\frac{\Omega_r}{r}=\frac{12}{h^3}\int_{-h/2}^{h/2}\varepsilon_\theta z dz$$

$$\frac{\partial\Omega_\theta}{\partial r}+\frac{1}{r}\frac{\partial\Omega_r}{\partial\theta}-\frac{\Omega_\theta}{r}=\frac{12}{h^3}\int_{-h/2}^{h/2}\gamma_{r\theta} z dz \tag{14}$$

If we use stress-strain equations;

$$\varepsilon_r=\frac{1}{E}[\sigma_r-\nu(\sigma_\theta+\sigma_z)]$$

$$\varepsilon_\theta=\frac{1}{E}[\sigma_\theta-\nu(\sigma_r+\sigma_z)]$$

$$\gamma_{r\theta}=\frac{\tau_{r\theta}}{G} \tag{15}$$

and introducing the stress distribution Eq. (4) into Eq. (14) and Eq. (15) we obtain;

$$\frac{\partial\Omega_r}{\partial r}-\frac{12}{Eh^3}[M_r-\mu M_\theta]=0 \tag{16a}$$

$$\frac{1}{r}\frac{\partial\Omega_\theta}{\partial\theta}+\frac{\Omega_r}{r}-\frac{12}{Eh^3}[M_\theta-\mu M_r]=0 \tag{16b}$$

$$\frac{\partial\Omega_\theta}{\partial r}+\frac{1}{r}\frac{\partial\Omega_r}{\partial\theta}-\frac{\Omega_\theta}{r}-\frac{12}{Eh^3}M_{r\theta}=0 \tag{16c}$$

These equations can be solved for M_r and M_θ as follows:

$$M_r=D\left(\frac{\partial\Omega_r}{\partial r}+\frac{\mu}{r}\frac{\partial\Omega_\theta}{\partial\theta}-\mu\frac{\Omega_r}{r}\right)$$

$$M_\theta=D\left(\mu\frac{\partial\Omega_r}{\partial r}+\frac{1}{r}\frac{\partial\Omega_\theta}{\partial\theta}+\frac{\Omega_r}{r}\right)$$

$$M_{r\theta}=D(1-\mu^2)\left(\frac{\partial\Omega_\theta}{\partial r}+\frac{1}{r}\frac{\partial\Omega_r}{\partial\theta}-\frac{\Omega_\theta}{r}\right) \tag{17}$$

These are general relations. For axisymmetric case, similar relations are obtained by Wang and Lee (1996) and Reddy and Wang (1997).

In obtaining Eq. (16a) and Eq. (16b), the effect of σ_z on the bending moments were ignored because of the terms relating this effect is small comparing with remaining terms. Otherwise these field equations would not pass the Gâteaux potential test as we will see later. The necessity of the neglection of these terms can be detected only by Gâteaux differential approach and it is an evidence of the power of the method as stated in the introduction. If we had used Hellinger-Reissner, Hu-Washizu or weak formulation theories, we could have not recognized incompatibility of this term.

In symbolic form, dynamic boundary conditions can be written as,

$$\boldsymbol{M} - \hat{\boldsymbol{M}} = \boldsymbol{0}$$
$$\boldsymbol{Q} - \hat{\boldsymbol{Q}} = \boldsymbol{0} \tag{18}$$

and geometric boundary conditions are,

$$-\boldsymbol{w} - \hat{\boldsymbol{w}} = \boldsymbol{0}$$
$$-\boldsymbol{\Omega} - \hat{\boldsymbol{\Omega}} = \boldsymbol{0} \tag{19}$$

The explicit form of the boundary conditions will be obtained after some variational manipulations. In Eq. (18) and Eq. (19) quantities with hat are known values on the boundary. $\boldsymbol{M}$, $\boldsymbol{Q}$, $\boldsymbol{\Omega}$, $\boldsymbol{w}$ are the moment, force, rotation and deflection vectors, respectively. Field equations for Reissner plate in polar coordinates can be written in operator form as,

$$\boldsymbol{Q} = \boldsymbol{L}\boldsymbol{y} - \boldsymbol{f} \tag{20}$$

The matrix form of the operator is given in Appendix II.

3. Functional for thick plates in polar coordinates

If the operator $\boldsymbol{Q}$ in Eq. (20) is potential, the equality

$$\langle d\boldsymbol{Q}(\boldsymbol{y},\bar{\boldsymbol{y}}),\boldsymbol{y}^*\rangle = \langle d\boldsymbol{Q}(\boldsymbol{y},\boldsymbol{y}^*),\bar{\boldsymbol{y}}\rangle \tag{21}$$

must be satisfied (Oden and Reddy 1976). $d\boldsymbol{Q}(\boldsymbol{y},\bar{\boldsymbol{y}})$ and $d\boldsymbol{Q}(\boldsymbol{y},\boldsymbol{y}^*)$ are Gâteaux derivatives of the operator in directions of $\bar{\boldsymbol{y}}$ and $\boldsymbol{y}^*$ which are constant elements in the domain. Gâteaux derivative of the operator is defined as;

$$d\boldsymbol{Q}(\boldsymbol{u},\bar{\boldsymbol{u}}) = \left.\frac{\partial Q(\boldsymbol{u} + \tau\bar{\boldsymbol{u}})}{\partial\tau}\right|_{\tau=0} \tag{22}$$

where τ is a scalar. Using this definition, after some simple manipulations it can be shown that the Eq. (21) holds and the operator $\boldsymbol{Q}$ is a potential operator. To satisfy this equality the explicit forms of the boundary conditions must be as follows;

$$[M,\Omega] = \left[\Omega_r,\left(M_r + \frac{1}{r}M_{r\theta}\right)\right] + \left[\Omega_\theta,\left(M_{r\theta} + \frac{1}{r}M_\theta\right)\right]$$
$$[Q,w] = [(Q_r + Q_\theta),w] \tag{23}$$

Since the operator is potential then the functional corresponding to the field equations is obtained as;

$$\boldsymbol{I}(\boldsymbol{y}) = \int_0^1 [\boldsymbol{Q}(s\boldsymbol{y}),\, \boldsymbol{y}]ds \tag{24}$$

where s is a scalar quality. Functional $\boldsymbol{I}(\boldsymbol{y})$ can be obtained after some manipulations as;

$$I = [Q_r,(\Omega_r + w_{,r})] + \left[Q_\theta,\left(\Omega_\theta + \frac{1}{r}w_{,\theta}\right)\right] + [M_r,\Omega_{r_{,r}}] + \left[\frac{1}{r}M_\theta,(\Omega_{\theta_{,\theta}} + \Omega_r)\right]$$
$$+\left[M_{r\theta},\left(\frac{1}{r}\Omega_{r_{,\theta}} + \Omega_{\theta_{,r}} - \frac{1}{r}\Omega_\theta\right)\right] - \frac{6}{Eh^3}\{[M_r,\, M_r] + [M_\theta,\, M_\theta] - 2v[M_r,\, M_\theta] + 2(1+v)[M_{r\theta},\, M_{r\theta}]\}$$

$$-\frac{3}{5Gh}\{[Q_r,Q_r]+[Q_\theta,Q_\theta]\}+\frac{1}{2}[kw,w]-[q,w]-[(\Omega-\hat{\Omega}),M]_\varepsilon-[(w-\hat{w}),Q]_\varepsilon-[\hat{M},\Omega]_\sigma-[\hat{Q},w]_\sigma \tag{25}$$

The braces with the σ index and ε index are valid on the boundary where the dynamic boundary conditions and the geometric boundary conditions are prescribed, respectively. Where [,] is the inner product and defined as follows;

$$[f,g]=\iint f(r,\theta)g(r,\theta)rdrd\theta \tag{26}$$

If the variational derivative of the functional in Eq. (25) is taken, all the field equations and boundary conditions can be reproduced.

The same functional can be obtained by Hellinger-Reissner principle. In this functional the first five terms come directly by adding kinematic Eq. (16) using Lagrange multipliers to the strain energy where Q_r, Q_θ , M_r, M_θ, $M_{r\theta}$ play the Lagrange multiplier role. The sum of other terms in the functional represent the strain energy of the plate in polar coordinates.

4. Finite element formulation for sectorial geometry

Let w be the displacement in z-direction, Ω_r and Ω_θ being the rotations of the cross-sections normal to in rz and θz planes. Q_r, Q_θ are shear forces in polar coordinates, and M_r, M_θ, $M_{r\theta}$ are the bending and torsional moments in polar coordinates. They are the nodal unknowns of the generated finite element and expressed by shape function ψ_i in the element. For example, $w=\sum w_i\ \psi_i$, where w_i are the nodal values and $i=1,\ldots,n$ (n=number of nodes of the element). In the solution of thick plates, two different elements are developed. One of them is SEC24 which has an element with 24 degrees of freedom, and with 8 degrees of freedom per node. The other one is SEC32 which has 32 degrees of freedom element with 8 degrees of freedom per node. The shape functions are given for SEC24 element (Fig. 2) as,

$$\begin{aligned}\psi_1&=(1-s)\\ \psi_2&=s(1-\eta)\\ \psi_3&=s\eta\end{aligned} \tag{27}$$

where,

$$s=\frac{r}{r_2},\quad \eta=\frac{\theta-\theta_1}{\Delta\theta} \tag{28}$$

For SEC32 element (Fig. 3),

$$\begin{aligned}\psi_1&=(1-s)(1-\eta)\\ \psi_2&=s(1-\eta)\\ \psi_3&=s\eta\\ \psi_4&=(1-s)\eta\end{aligned} \tag{29}$$

where,

$$s=\frac{r-r_1}{\Delta r},\quad \eta=\frac{\theta-\theta_1}{\Delta\theta} \tag{30}$$

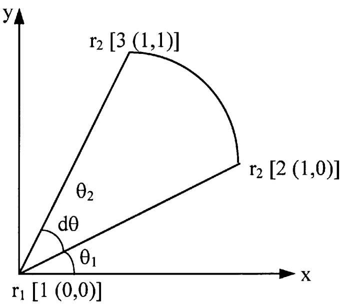


Fig 2 SEC24 element

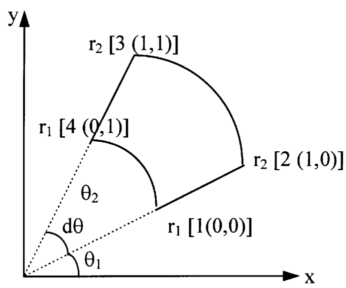


Fig 3 SEC32 element

By variational principles, from Eq. (24), the element matrices $[k]_{24}$, $[k]_{32}$ for SEC24 and SEC32 are obtained as follows:

$$[k]_{24}=\begin{array}{c}\begin{array}{cccccccc} M_r & M_\theta & M_{r\theta} & Q_r & Q_\theta & \Omega_r & \Omega_\theta & w \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}\\ \left[\begin{array}{cccccccc} \gamma_1[k_1]_{24} & \gamma_2[k_1]_{24} & 0 & 0 & 0 & [k_2]_{24} & 0 & 0 \\ & \gamma_1[k_1]_{24} & 0 & 0 & 0 & [k_3]_{24} & [k_4]_{24} & 0 \\ & & \gamma_3[k_1]_{24} & 0 & 0 & [k_4]_{24} & [k_2-k_3]_{24} & 0 \\ & & & \gamma_4[k_1]_{24} & 0 & [k_1]_{24} & 0 & [k_2]_{24} \\ & & & & \gamma_4[k_1]_{24} & 0 & [k_1]_{24} & [k_4]_{24} \\ & \text{Symmetrical} & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & \gamma_5[k_1]_{24} \end{array}\right]\end{array} \qquad (31)$$

$$[k]_{32}=\begin{array}{c}\begin{array}{cccccccc} M_r & M_\theta & M_{r\theta} & Q_r & Q_\theta & \Omega_r & \Omega_\theta & w \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}\\ \left[\begin{array}{cccccccc} \gamma_1[k_1]_{32} & \gamma_2[k_1]_{32} & 0 & 0 & 0 & [k_2]_{32} & 0 & 0 \\ & \gamma_1[k_1]_{32} & 0 & 0 & 0 & [k_3]_{32} & [k_4]_{32} & 0 \\ & & \gamma_3[k_1]_{32} & 0 & 0 & [k_4]_{32} & [k_2-k_3]_{32} & 0 \\ & & & \gamma_4[k_1]_{32} & 0 & [k_1]_{32} & 0 & [k_2]_{32} \\ & & & & \gamma_4[k_1]_{32} & 0 & [k_1]_{32} & [k_4]_{32} \\ & \text{Symmetrical} & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & \gamma_5[k_1]_{32} \end{array}\right]\end{array} \qquad (32)$$

where,

$$\gamma_1=-12/Eh^3,\ \gamma_2=-12\ v/Eh^3,\ \gamma_3=-24(1+v)/Eh^3,\ \gamma_4=-12(1+v)/5Eh,\ \gamma_5=k \qquad (33)$$

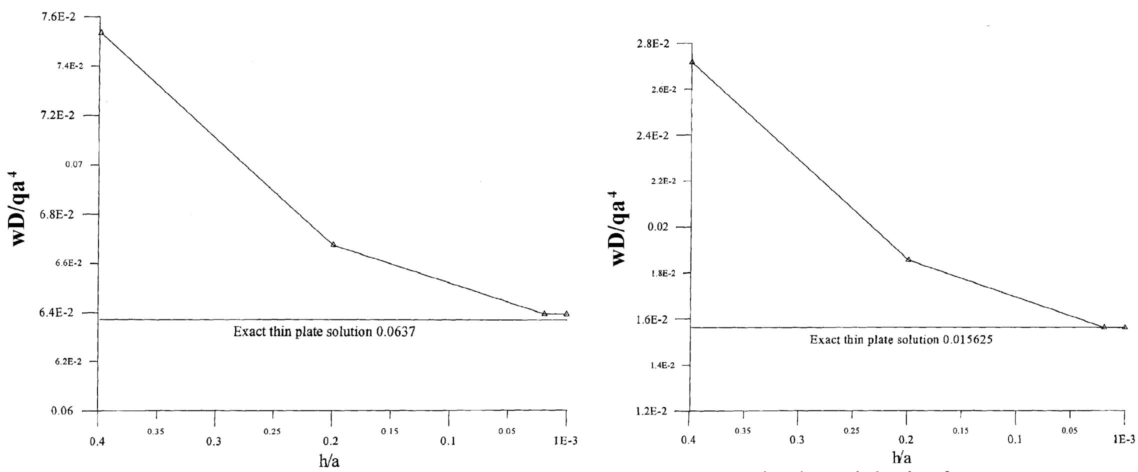
The explicit form of the submatrices $[k_1]_{24}$, $[k_2]_{24}$, $[k_3]_{24}$, $[k_4]_{24}$, $[k_1]_{32}$, $[k_2]_{32}$, $[k_3]_{32}$, $[k_4]_{32}$ are given in Appendix II.

5. Numerical examples

The applicability and accuracy of the proposed sectorial mixed finite element formulation of thick plates are shown on the following plate problems;

5.1. Example 1: The simply supported - clamped circular plates and shear locking test

The formulation obtained for thick plates is applied to thin plate (h/a=10^3) and then The simply supported and clamped circular thick plates subjected uniform loading are analyzed with different meshes. As the plate thickness approaches zero, shear locking might have been encountered, but efficiency of the formulation obtained in this study prevents shear locking phenomenon and Fig. 4 shows the manner in which the sectorial elements behave for a given mesh subdivision (*NEL=288*) in full plate as h/a decreases. The thicknesses of plate are taken as h=0.005, h=0.1, h=1, h=2, respectively. The convergence of displacement w, moment M_r at the center are shown for simply supported and clamped plates in Table 1, 2, and Table 3, 4, respectively. Exact results in these Tables are taken from Katılı (1993). Moment parameter should take the same value, irrespective of radius to thickness ratio. Also, the convergence of moment M_r is sketched in Fig. 5a. The results converge to exact value from above and below depending on even or odd number of element used in the calculation. The radial Q_r shear force converge to exact value (Q_r=$qa/2$=2.5) very rapidly as shown in Fig. 5b. In comparison with the other studies (Katılı 1993, Papadopoulos and Taylor 1990, Batoz and Lardeur 1989, Batoz and Katılı 1992) for the same data, the results obtained in this study show that the convergence of results is good. The results of bending moment M_r, shear force Q_r and displacement w along radial direction are given in Figs. 6-11. Typical circular plate mesh is shown in Fig. 12.



a) Simply supported circular plate b) Clamped circular plate

Fig 4 Behaviour of solution for decreasing values h/a for a given NEL=288

Table 1 Central displacements for the simply supported plate under uniform loading (ν=0.3)

NEL	a/h=50	a/h=5	a/h=2.5
3	0.7104	0.7345	0.8080
7	0.6987	0.7226	0.7948
11	0.6971	0.7208	0.7929
15	0.6964	0.7202	0.7924
19	0.6963	0.7200	0.7921
23	0.6963	0.7198	0.7920
41	0.6961	0.7197	0.7917
43	0.6959	0.7197	0.7917
Exact (Katılı 1993)	0.6959	0.7268	0.8205
Multiplier		$E\,h^3/qa^4$	

Table 2 Central moment M_r for the simply supported plate under uniform loading (ν=0.3)

NEL	a/h=50	a/h=5	a/h=2.5
3	0.2248	0.2219	0.2180
7	0.2127	0.2113	0.2094
11	0.2094	0.2087	0.2077
15	0.2082	0.2077	0.2071
19	0.2076	0.2072	0.2068
23	0.2072	0.2069	0.2067
41	0.2066	0.2065	0.2064
43	0.2065	0.2065	0.2064
Exact (Katılı 1993)	0.2064	0.2064	0.2064
Multiplier		qa^2	

Table 3 Central displacements for the clamped plate under uniform loading (ν=0.3)

NEL	a/h=50	a/h=5	a/h=2.5
3	0.1596	0.1946	0.2931
7	0.1711	0.2022	0.2962
11	0.1712	0.2022	0.2959
15	0.1711	0.2020	0.2956
19	0.1711	0.2020	0.2956
23	0.1711	0.2020	0.2955
41	0.1709	0.2018	0.2955
54	0.1709	0.2018	0.2955
Exact (Katılı 1993)	0.1709	0.2018	0.2955
Multiplier		$E\,h^3/qa^4$	

Table 4 Central moment M_r for the clamped plate under uniform loading (ν=0.3)

NEL	a/h=50	a/h=5	a/h=2.5
3	0.1924	0.1306	0.0892
7	0.1444	0.1070	0.0912
11	0.1166	0.0959	0.0899
15	0.1038	0.0911	0.0892
19	0.0970	0.0886	0.0889
23	0.0929	0.0872	0.0891
41	0.0859	0.0846	0.0884
54	0.0784	0.0819	0.0880
Exact (Katılı 1993)	0.0812	0.0812	0.0812
Multiplier		qa^2	

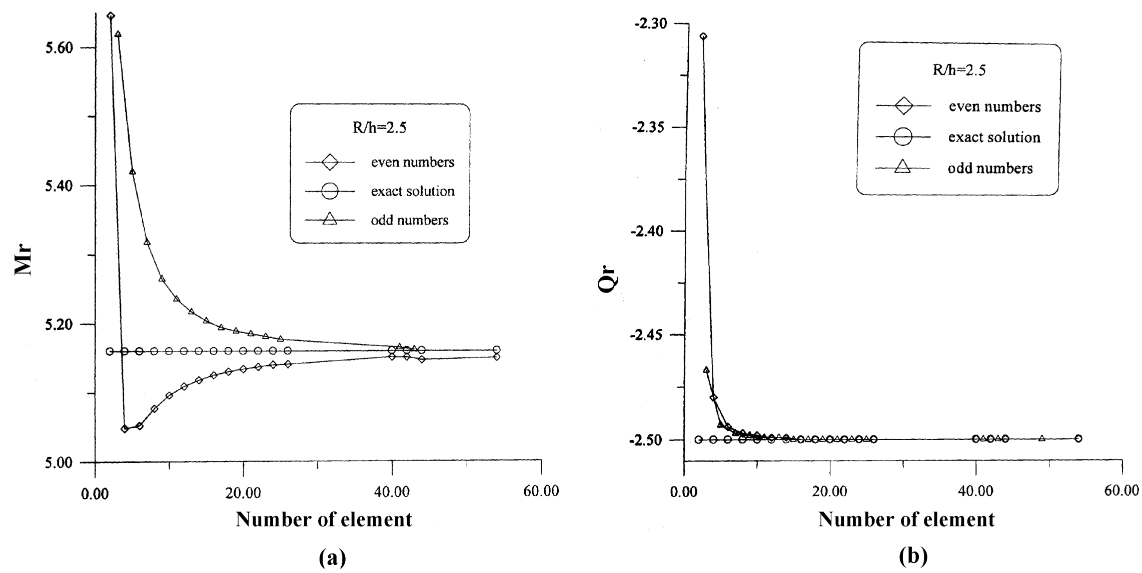


Fig. 5 Convergence of M_r, Q_r in simply supported circular plate for different meshes

5.2. Example 2: The annular plate

Fig. 13 shows an annular plate under uniform loading with simply supported outer edge $r=a$, free inner edge $r=b$. The solution of annular plate is obtained by using SEC32 and compared with theoretical solution. The results are given in Table 5. Theoretical results are obtained using general solution of annular plate given in Panc (1975) for above defined boundary conditions and the constants of integration calculated with general solution are given in Appendix III.

5.3. Example 3: The sectorial plate

The above obtained sectorial elements can be also applied to sectorial plates in bending. The numerical solutions for simply supported sectorial plate (Fig. 14) carrying a uniformly distributed

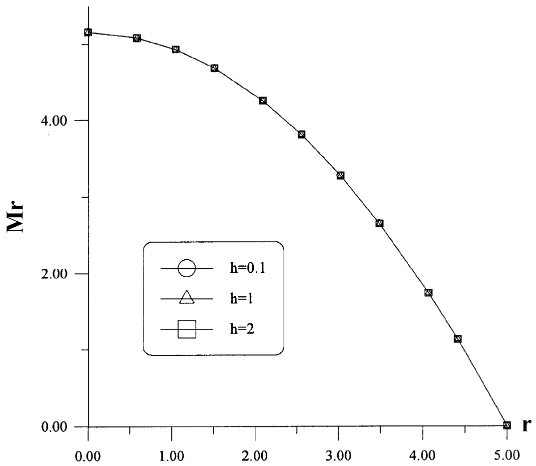


Fig. 6 M_r bending moments along radius for simply supported circular plate

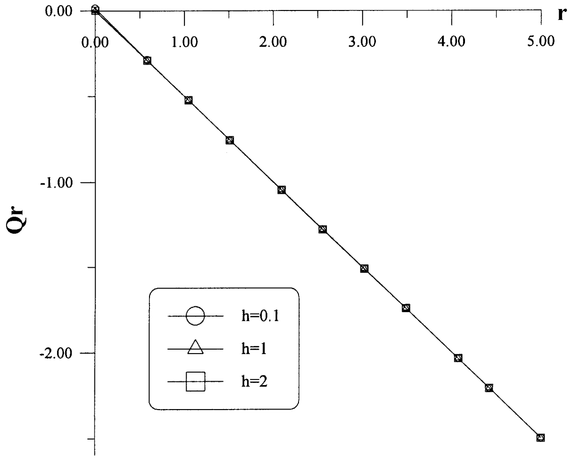


Fig. 7 Q_r shear forces along radius for simply supported circular plate

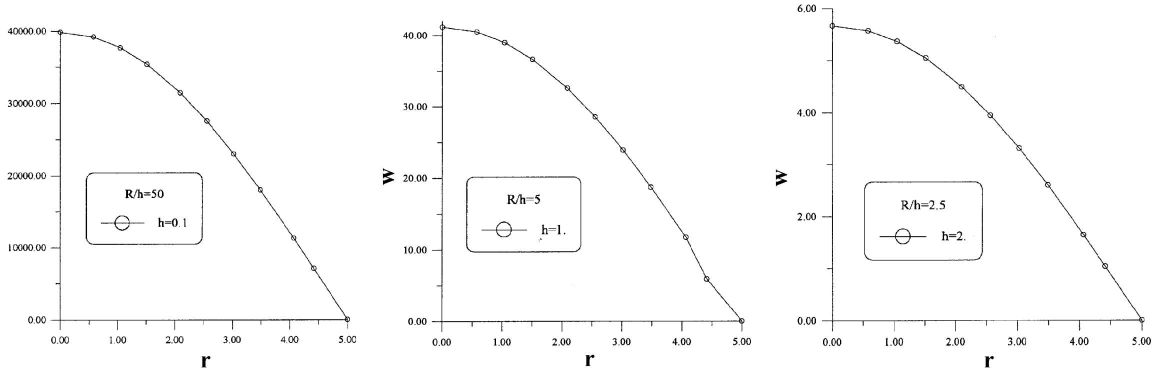


Fig. 8 w displacements along radius for simply supported circular plate

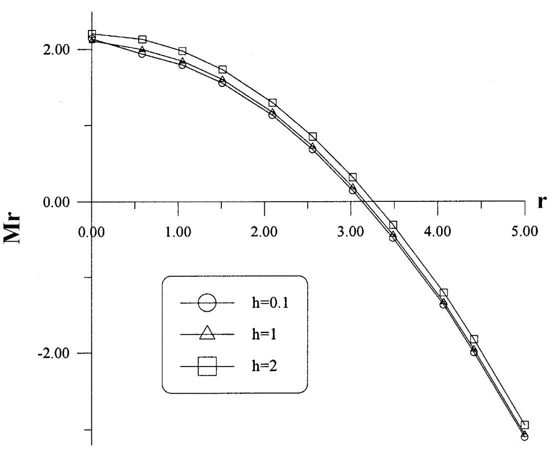


Fig. 9 M_r bending moments along radius for clamped circular plate

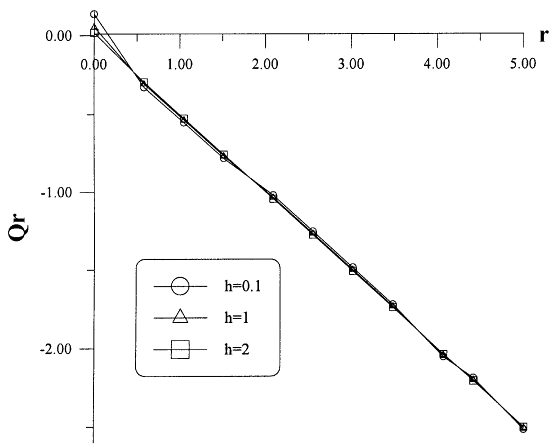


Fig. 10 Q_r shear forces along radius for clamped circular plate

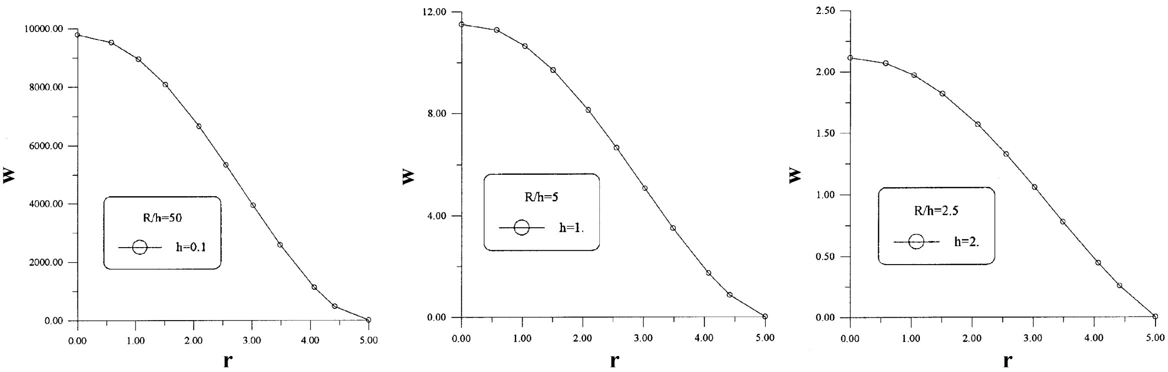


Fig. 11 w displacements along radius for clamped circular plate

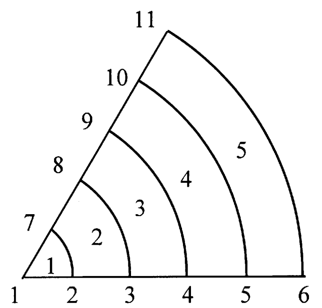


Fig. 12 Typical circular plate mesh with 5 element

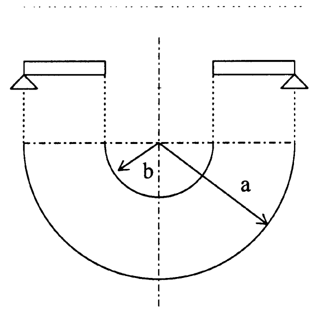


Fig. 13 Annular plate

Table 5 Annular plate under uniform load (E=10.92, ν=0.3, a=5, b=2.5, h=0.2, q=1)

	M_r $r=(a\text{-}b)/2$	M_θ $r=b$	Q_r $r=a$	w $r=b$
SEC32	1.033	6.011	1.875	313.5
Panc (1975)	1.033	5.980	1.875	331.7

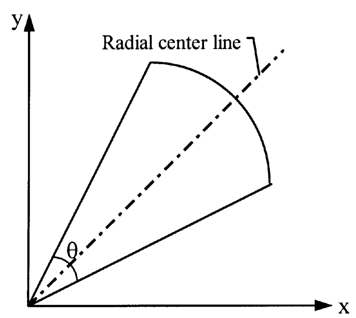


Fig. 14 Simply supported sectorial plate

load are presented in Tables 6 and 7. The excellent agreement can be observed by comparing the numerical results with the theoretical solutions (Timoshenko and Woinowisky-Krieger 1959). The

Table 6 Displacements and moments at points along the radial center line of sectorial plates (ν=0.3)

π/k	r/a=1/4			r/a=1/2			r/a=3/4			r/a=1		
	α	β	β_1	α	β	β_1	α	β	β_1	α	β	β_1
$\pi/3$	0.0021	−0.0026	0.0179	0.0091	0.0155	0.0259	0.0107	0.0247	0.0215	0	0	0.0048
Exact Timoshenko and Woinowisky-Krieger (1959)	0.0021	−0.0025	0.0177	0.0087	0.0149	0.0255	0.0101	0.0243	0.0213	0	0	0.0044
π	0.0666	0.0719	0.0360	0.0904	0.0876	0.0514	0.0622	0.0615	0.0468	0	0	0.0225
Exact Timoshenko and Woinowisky-Krieger (1959)	0.0643	0.0692	0.0357	0.0886	0.0868	0.0515	0.0612	0.0617	0.0468	0	0	0.0221
Multiplier	qa^4/Eh^3	qa^2	qa^2	qa^4/Eh^3	qa^2	qa^2	qa^4/Eh^3	qa^2	qa^2	qa^4/Eh^3	qa^2	qa^2

Table 7 Displacements and moments at points along the radial center line of sectorial plate for different thicknesses (ν=0.3 and $\alpha=\pi/3$)

a/h	$w_{\max}Eh^3/qa^4$	$M_{r\max}/qa^2$
1000	0.01106	0.02424
50	0.01115	0.02444
5	0.01623	0.02713

same problem is solved by finite strip method and similar results are obtained (Cheung and Chan 1981).

5.4. Example 4: The fan shape plate

The simply supported fan shape plate (Fig. 15) with uniform distributed load is solved and the results are given in Tables 8 and 9. Comparison with (Cheung and Chan 1981) shows that an excellent agreement is obtained for moments and reasonable agreement is obtained for displacements.

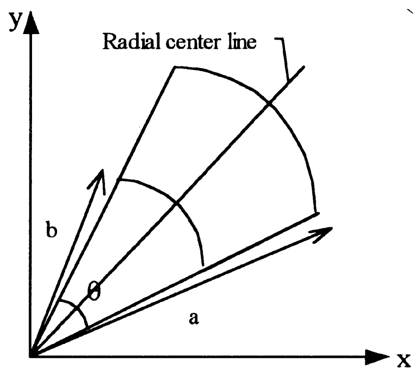


Fig. 15 Simply supported fan shape plate

Table 8 Maximum displacements and moments at points along the radial center line of fan shape plate (ν=0.3 and $\alpha=\pi/3$)

b/a	$w_{max}D/qa^4$ SEC32	$w_{max}D/qa^4$ Cheung and Chan (1981)	M_{rmax}/qa^2 SEC32	M_{rmax}/qa^2 Cheung and Chan (1981)	$M_{\theta max}/qa^2$ SEC32	$M_{\theta max}/qa^2$ Cheung and Chan (1981)
0.00	0.000975	0.000984	0.02469	0.02494	0.02590	0.02506
0.25	0.000950	0.000927	0.02264	0.02485	0.02414	0.02356
0.50	0.000531	0.000500	0.02121	0.02124	0.01231	0.01220
0.75	0.000053	0.000049	0.00752	0.00789	0.00242	0.002438

Table 9 Displacements and moments at points along the radial center line of fan shape plate for different thicknesses (ν=0.3, $\alpha=\pi/3$ and b/a=0.25)

a/h	$w_{max}Eh^3/qa^4$	M_{rmax}/qa^2
1000	0.01038	0.02505
50	0.01050	0.02518
5	0.01568	0.02857

5.5. Example 5: Nonsymmetrical load

In this study, the circular plate under a linear load varying with θ according to $q=q_0\, r/a\cos\theta$ (Fig. 16) is also solved and the results are given in Table 10. The reasonable agreement is obtained as compared with the theoretical solution (Timoshenko and Woinowisky-Krieger 1959).

5.6. Example 6: Simply supported circular plate on Winkler foundation

A few studies exist for circular thick plate on Winkler foundation, using boundary elements (El-Zafrany *et al.* 1995, Al-Hosani *et al.* 1999, Rashed *et al.* 1998). The studies by finite element method were not available to compare the results. To check the numerical results, the coefficient k is taken zero. In this limit case, numerical values converge on the correct value as expected. The results for simply supported circular plate with uniform distributed load on Winkler foundation are

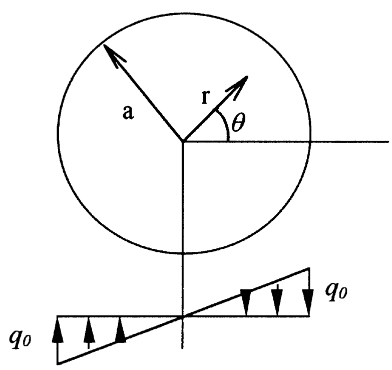


Fig. 16 Nonsymmetrical load

Table 10 Displacements and M_r moments for simply supported circular plate under nonsymmetrical load (ν=0.3)

	$\theta=0^0$				$\theta=45^0$			
	α Sectorial El.	α Exact	β Sectorial El.	β Exact	α Sectorial El.	α Exact	β Sectorial El.	β Exact
r/a=1/4	0.02830	0.02865	0.03308	0.02588	0.02000	0.02026	0.02339	0.01830
r/a=1/2	0.04202	0.04185	0.04516	0.04141	0.02970	0.02959	0.03193	0.02928
r/a=3/4	0.03112	0.03079	0.03789	0.03623	0.02201	0.02177	0.02677	0.02562
r/a=1	0	0	0	0	0	0	0	0
Multiplier	wEh^3/q_0a^4	wEh^3/q_0a^4	M_r/q_0a^2	M_r/q_0a^2	wEh^3/q_0a^4	wEh^3/q_0a^4	M_r/q_0a^2	M_r/q_0a^2

Table 11 Displacements and moments at middle point of simply supported circular plate for different k values

k [kN/m^3]	wEh^3/qa^4	M_r/qa^2
0	0.69590	0.20650
100000	0.04956	0.00738
200000	0.02358	0.00191
300000	0.01499	0.00092

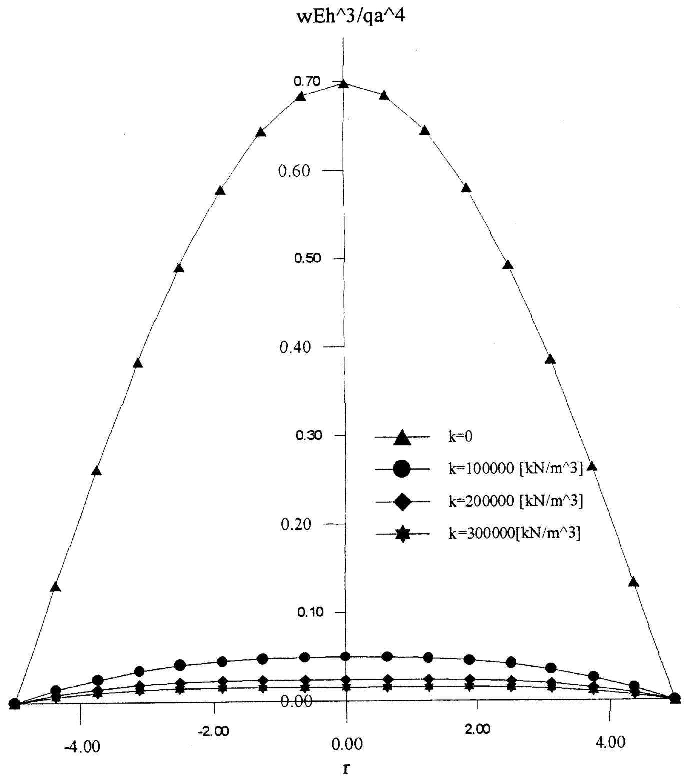


Fig. 17 Displacements along radius of simply supported circular plates for different k values

given in Table 11. Fig. 17 shows displacements along radius for different k values.

6. Conclusions

The development of a finite element for the analysis of thick plate structures on Winkler foundation in polar coordinates has been presented in this paper. To obtain an effective formulation for thick plates, the field equations are written in polar coordinates for the Reissner plate. In these field equations, rotations Ω_r, Ω_θ are introduced similar to Panc approach (Panc 1975). Using field equations, a new functional has been obtained for thin or moderately thick plates on Winkler foundation based on Gâteaux differential approach. This functional has boundary terms which play important roles in numerical solutions for some singular loads. These effects are to be the subject of another paper. A sectorial finite element is obtained which has four nodes and 32 degrees of freedom and it is called SEC32. This element is singular at the center of the circle. To eliminate the singularity another element is developed which has three nodes and 24 degrees of freedom called SEC24. The properties of this formulation briefly are:

- Gâteaux differential method has been used. The functional is obtained by enforcing all field equations in straightforward manner.
- The closed form of element equation is obtained which eliminate the time-consuming numerical inversion of the element matrix.
- This formulation avoids the shear locking, converges to the Kirchhoff solution as the plate thickness goes to zero.
- For the accuracy of solution, one of the requirements is: The well-proportioned elements must be used (Huebner 1975). Five-node elements may be helpful to obtain well-proportioned element.
- This formulation provides accurate and stable solutions.
- This formulation is very suitable for the problem which has symmetry in lateral direction. For this problem a few elements are sufficient to obtain satisfactory results for engineering purposes.
- This formulation is also applicable to any structure if its domain can be represented by sectorial element.
- This formulation is also suitable for the dynamic problems which are under study.

References

Aköz, A.Y. (1985), "A new functional for bars and its applications", IV. *National Applied Mechanics Meeting* (in Turkish).

Aköz, A.Y., Omurtag, M.H. and Doğruoğlu, A.N. (1991), "The mixed finite element formulation for three dimensional bars", *Int. J. Solids Structures*, **28**, 225-234.

Aköz, A.Y. and Uzcan (Eratlı), N. (1992), "The new functional for Reissner plates and its application", *Computers & Structures*, **44**, 1139-1144.

Aköz, A.Y. and Kadıoğlu, F. (1996), "The mixed finite element solution of circular beam on elastic foundation", *Computers & Structures*, **60**, 643-651.

Aköz, A.Y. and Özütok, A. (2000), "A functional for shells of arbitrary geometry and a mixed finite element method for parabolic and circular cylindrical shells" (Accepted for publication in *Int. J. Numer. Methods Eng.*).

Al-Hosani, K., Fadhil, S. and El-Zafrany, A. (1999), "Fundamental solution and boundary element analysis of thick plates on Winkler foundation", *Computers & Structures*, **70**, 325-336.

Bathe, K.J. (1982), *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, Englewood Cliffs, NJ.

Batoz, J.L and Lardeur, A (1989), "Discrete shear triangular nine d.o.f. element for the analysis of thick to very thin plates", *Int. J. Numer. Methods Eng.,* **28**, 533-560.

Batoz, J.L and Katılı, I. (1992), "On a simple triangular Reissner-Mindlin plate element based on incompatible modes and discrete constraints", *Int. J. Numer. Methods Eng.,* **35**,1603-1632.

Belytschko, T., Tsay, C.S. and Liu, W.K. (1981), "A stabilization matrix for the bilinear Mindlin plate element", *Computer Methods in Applied Mechanics and Engineering,* **29**, 313-327.

Cheung, M.S. and Chan, M.Y.T. (1981), "Static and dynamic analysis of thin and thick sectorial plates by the finite strip method", *Computers & Structures*, **14**, 79-88.

Dym, C.L. and Shames, I.H. (1973), *Solid Mechanics A Variational Approach*, McGraw-Hill.

El-Zafrany, A., Fadhil, S. and Al-Hosani, K. (1995), "A new fundamental solution for boundary element analysis of thin plates on Winkler foundation", *Int. J. Numer. Methods Eng.,* **38**, 887-903.

Eratlı, N. and Aköz, A.Y. (1997), "The mixed finite element formulation for the thick plates on elastic foundations", *Computers & Structures*, **65**, 515-529.

Eratlı, N. and Aköz, A.Y., "Mixed finite element formulation for folded plates" (Submitted for publication in *Computers & Structures*).

Gallagher, R.H. (1975), *Finite Element Analysis: Fundamentals*, Prentice-Hall.

Goldenveizer, A.L. (1961), *Theory of Elastic Thin Shells*, Pergomon Press, NewYork.

Huebner, K.H. (1975), *The Finite Element Method for Engineers,* John Wiley & Sons.

Hughes, T.J.R., Taylor, R.L. and Kanok-Nukulchai, W. (1977), "A simple and efficient finite element for plate bending", *Int. J. Numer. Methods Eng.,* **11**, 1529-1543.

Katılı, I. (1993a), "A new discrete Kirchhoff-Mindlin element based on Mindlin-Reissner plate theory and assumed shear strain fields-Part I: An extended DKT element for thick-plate bending analysis", *Int. J. Numer. Methods Eng.,* **36**,1859-1883.

Katılı, I. (1993b), "A new discrete Kirchhoff-Mindlin element based on Mindlin-Reissner plate theory and assumed shear strain fields-Part II: An extended DKQ element for thick-plate bending analysis", *Int. J. Numer. Methods Eng.,* **36**, 1885-1908.

Kavanagh, K.T. and Kay, S.W. (1972), "A note on selective and reduced integration techniques in the finite element method", *Int. J. Numer. Methods Eng.,* **4**, 148-150.

Mindlin, R.D. (1951), "The Influence of rotatory inertia and shear on the flexural motions of elastic plates", *J. Appl. Mech.*, *ASME*, **18**, 31-38.

Morris, A.J. (1973), "Deficiency in current finite elements for thin shell applications", *Int. J. Solids Structures*, **9**, 331-346.

Oden, J.D. and Reddy, J.N. (1976), V*ariational Method in Theoretical Mechanics*, Springer.

Omurtag, M.H. and Aköz, A.Y. (1994), "Hyperbolic paraboloid shell analysis via mixed finite element formulation", *Int. J. Numer. Methods. Eng.*, **37**, 3037-3056.

Panc, V. (1975), *Theories of Elastic Plates*, Noordhoff International Publishing.

Papadopoulos, P. and Taylor, R.L. (1990), "A triangular element based on Reissner-Mindlin plate theory", *Int. J. Numer. Methods Eng.,* **30**, 1029-1049.

Pugh, E.D.L., Hinton, E. and Zienkiewicz, A (1978), "Study of quadrilateral plate bending elements with reduced integration", *Int. J. Numer. Methods Eng.,* **12**, 1059-1079.

Rashed, Y.F., Aliabadi, M.H. and Brebbia, C.A. (1998), "The boundary element method for thick plates on a Winkler foundation", *Int. J. Numer. Methods Eng.,* **41**, 1435-1462.

Reissner, E. (1946), "The effects of transverse shear deformation bending of elastic plates", *J. Appl. Mech.*, ASME, **12**, 69-77.

Reissner, E. (1975), "On transverse bending of plates, including the effect of transverse shear deformation", *Int. J. Solids Structures*, **11**, 569-573.

Reddy, J.N. (1984), *Energy and Variational Methods in Applied Mechanics*, John Wiley & Sons.

Reddy, J.N. (1993), *Finite Element Method*, McGraw-Hill.

Reddy, J.N. and Wang, C.M. (1997), "Relationnships between classical and shear deformation theories of axisymmetric circular plates", *AIAA Journal,* **35**(12), 1862-1868.

Timoshenko, S. and Woinowisky-Krieger, S. (1959), *Theory of Plates and Shells*, McGraw-Hill.

Wang, C.M. and Lee, K.H. (1996), "Deflection and stress-resultants of axisymmetric Mindlin plates in terms of corresponding Kirchhoff solutions", *Int. J. Mech. Sci.*, **38**(11),1179-1185.

Washizu, K. (1975), V*ariational Methods in Elasticity and Plasticity,* Pergamon Press.

Zienkiewicz, O.C., Taylor, R.L. and Too, J. (1971), "Reduced integration technique in general analysis of plates and shells", *Int. J. Numer. Methods Eng.*, **3**, 275-290.

Zienkiewicz, O.C. (1977), *The Finite Element Method*, McGraw-Hill, London.

Zienkiewicz, O.C., Bauer, J., Morgan, K. and Oñate, E. (1977), "A simple and efficient element for axisymmetric shells", *Int. J. Numer. Methods Eng.*, **11**, 1545-1558.

Appendix I

Notation

M_r, M_θ, $M_{r\theta}$	: bending moments
Q_r, Q_θ	: shear forces
q	: distributed load
k	: modulus of subgrade reaction
w	: displacement of plate
Ω_r, Ω_θ	: the components of the rotation of a normal to the middle plane of the plate, respectively
a	: radius of plate
h	: thickness of plate
E, ν, G	: modules of elasticity, Poisson's ratio and shear modules of elasticity respectively
$I(\boldsymbol{y})$	: functional
$<,>$,$[\,,\,]$	: inner product
$[\,,\,]_\varepsilon$	: geometric boundary condition
$[\,,\,]_\sigma$	: dynamic boundary condition
ψ_i	: shape functions (i=1, ..., 3 for SEC24 or i=1, ..., 4 for SEC32)
s, η	: nondimensional coordinates of a master element
$[k]_{24}$, $[k]_{32}$	: SEC24 and SEC32 finite element matrices
$\boldsymbol{L}$	: coefficient matrix
$\boldsymbol{f}$	: load vector
$\boldsymbol{y}$	: unknown vectors

Appendix II

The explicit form of the $\boldsymbol{Q}$ operator,

$$\begin{bmatrix} P_{11} & 0 & 0 & 0 & 0 & 0 & P_{17} & P_{18} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{24} & P_{25} & P_{26} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{35} & P_{36} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & P_{42} & 0 & P_{44} & P_{45} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{52} & P_{53} & P_{54} & P_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{62} & P_{63} & 0 & 0 & P_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{71} & 1 & 0 & 0 & 0 & 0 & P_{77} & 0 & 0 & 0 & 0 & 0 \\ P_{81} & 0 & 1 & 0 & 0 & 0 & 0 & P_{88} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ \Omega_r \\ \Omega_\theta \\ M_r \\ M_\theta \\ M_{r\theta} \\ Q_r \\ Q_\theta \\ w \\ \Omega \\ M \\ Q \end{bmatrix} = \begin{bmatrix} q \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hat{Q} \\ \hat{M} \\ -\hat{\Omega} \\ -\hat{w} \end{bmatrix} \tag{A.1}$$

where,

$$P_{11}=k,\ P_{17}=-\frac{\partial}{\partial r}-\frac{1}{r},\ P_{18}=-\frac{1}{r}\frac{\partial}{\partial\theta},\ P_{24}=-\frac{\partial}{\partial r}-\frac{1}{r},\ P_{25}=\frac{1}{r},\ P_{26}=-\frac{1}{r}\frac{\partial}{\partial\theta}$$

$$P_{35}=-\frac{1}{r}\frac{\partial}{\partial\theta},\ P_{36}=-\frac{\partial}{\partial r}-\frac{2}{r},\ P_{42}=\frac{\partial}{\partial r},\ P_{44}=-\frac{12}{Eh^3},\ P_{45}=\frac{12v}{Eh^3}$$

$$P_{52}=\frac{1}{r},\ P_{53}=\frac{1}{r}\frac{\partial}{\partial\theta},\ P_{54}=\frac{12v}{Eh^3},\ P_{55}=-\frac{12}{Eh^3}$$

$$P_{62}=\frac{1}{r}\frac{\partial}{\partial\theta},\ P_{63}=\frac{\partial}{\partial r}-\frac{1}{r},\ P_{66}=-\frac{12}{Eh^3}$$

$$P_{71}=\frac{\partial}{\partial r},\ P_{77}=-\frac{6}{5Gh},\ P_{81}=\frac{1}{r}\frac{\partial}{\partial\theta},\ P_{88}=-\frac{6}{5Gh} \tag{A.2}$$

Submatrices for SEC24 element,

$$[k_1]_{24}=\int_0^{r_2}\int_{\theta_1}^{\theta_2}\psi_i\psi_j r\ dr\ d\theta=\begin{bmatrix} A_1B_1/12 & A_1B_1/24 & A_1B_1/24 \\ A_1B_1/24 & A_1B_1/12 & A_1B_1/24 \\ A_1B_1/24 & A_1B_1/24 & A_1B_1/12 \end{bmatrix} \tag{A.3}$$

$$[k_2]_{24}=\int_0^{r_2}\int_{\theta_1}^{\theta_2}\psi_i\psi_{j,r} r\ dr\ d\theta=\begin{bmatrix} -B_2/6 & B_2/12 & B_2/12 \\ -B_2/6 & B_2/9 & B_2/18 \\ -B_2/6 & B_2/18 & B_2/9 \end{bmatrix} \tag{A.4}$$

$$[k_3]_{24}=\int_0^{r_2}\int_{\theta_1}^{\theta_2}\psi_i\psi_j\ dr\ d\theta=\begin{bmatrix} B_2/3 & B_2/12 & B_2/12 \\ B_2/12 & B_2/9 & B_2/18 \\ B_2/12 & B_2/18 & B_2/9 \end{bmatrix} \tag{A.5}$$

$$[k_4]_{24}=\int_0^{r_2}\int_{\theta_1}^{\theta_2}\psi_i\psi_{j,\theta}\ dr\ d\theta=\begin{bmatrix} 0 & -r_2/6 & r_2/6 \\ 0 & -r_2/6 & r_2/6 \\ 0 & -r_2/6 & r_2/6 \end{bmatrix} \tag{A.6}$$

where,

$$A_1=r_2^2,\ B_1=\theta_2-\theta_1,\ B_2=r_2B_1 \tag{A.7}$$

Submatrices for SEC32 element,

$$[k_1]_{32}=\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}\psi_i\psi_j\ r\ dr\ d\theta=\begin{bmatrix} A_3A_4B_9/36 & A_3A_6B_9/36 & A_3A_6B_9/72 & A_3A_4B_9/72 \\ A_3A_6B_9/36 & A_3A_5B_9/36 & A_3A_5B_9/72 & A_3A_6B_9/72 \\ A_3A_6B_9/72 & A_3A_5B_9/72 & A_3A_5B_9/36 & A_3A_6B_9/36 \\ A_3A_4B_9/72 & A_3A_6B_9/72 & A_3A_6B_9/36 & A_3A_4B_9/36 \end{bmatrix} \tag{A.8}$$

$$[k_2]_{32}=\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}\psi_i\psi_{j,r}\ r\ dr\ d\theta=\begin{bmatrix} A_1B_9/18 & -A_1B_9/18 & -A_1B_9/36 & A_1B_9/36 \\ A_2B_9/18 & -A_2B_9/18 & -A_2B_9/36 & A_2B_9/36 \\ A_2B_9/36 & -A_2B_9/36 & -A_2B_9/18 & A_2B_9/18 \\ A_1B_9/36 & -A_1B_9/36 & -A_1B_9/18 & A_1B_9/18 \end{bmatrix} \tag{A.9}$$

$$[k_3]_{32}=\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} \psi_i \psi_j \, dr \, d\theta=\begin{bmatrix} A_8/9 & A_8/18 & A_8/36 & A_8/18 \\ A_8/18 & A_8/9 & A_8/18 & A_8/36 \\ A_8/36 & A_8/18 & A_8/9 & A_8/18 \\ A_8/18 & A_8/36 & A_8/18 & A_8/9 \end{bmatrix} \tag{A.10}$$

$$[k_4]_{32}=\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} \psi_i \psi_{j,\theta} \, dr \, d\theta=\begin{bmatrix} A_3/6 & A_3/12 & -A_3/12 & -A_3/6 \\ A_3/12 & A_3/6 & -A_3/6 & -A_3/12 \\ A_3/12 & A_3/6 & -A_3/6 & -A_3/12 \\ A_3/6 & A_3/12 & -A_3/12 & -A_3/6 \end{bmatrix} \tag{A.11}$$

where,

$$A_1=2r_1+r_2,\ A_2=2r_2+r_1,\ A_3=r_1-r_2,\ A_4=3r_1+r_2,\ A_5=3r_2+r_1,\ A_6=r_1+r_2,\ A_8=A_3 B_9,\ B_9=\theta_1-\theta_2 \tag{A.12}$$

Appendix III

The solution of annular plate is:

$$w=A_0+A_2\frac{r^2}{a^2}+A_3\ln\frac{r}{a}+A_4\left(\frac{r^2}{a^2}-\varepsilon\right)\ln\frac{r}{a}+\frac{qa^4}{64D}\left(\frac{r^4}{a^4}-4\varepsilon\frac{r^2}{a^2}\right) \tag{A.13}$$

$$M_r=-2(1+v)\frac{D}{a^2}A_2+(1-v)\frac{D}{r^2}A_3-\frac{D}{a^2}A_4\left[(3+v)+2(1+v)\ln\frac{r}{a}\right]-(3+v)\frac{qr^2}{16} \tag{A.14}$$

$$M_\vartheta=-2(1+v)\frac{D}{a^2}A_2-(1-v)\frac{D}{r^2}A_3-\frac{D}{a^2}A_4\left[(1+3v)+2(1+v)\ln\frac{r}{a}\right]-(1+3v)\frac{qr^2}{16} \tag{A.15}$$

$$Q_r=-\frac{4D}{a^2 r}A_4-\frac{qr}{2} \tag{A.16}$$

where;

$$D=\frac{Eh^3}{12(1-v^2)}$$

$$\varepsilon=\frac{4}{5(1-v)}\left(\frac{h}{a}\right)^2 \tag{A.17}$$

The constants A_0, A_2, A_3 and A_4 must be determined from the boundary conditions prescribed at the edges $r=a$ and $r=b$. The boundary conditions,

$$\begin{array}{llll} r=b & Q_r=0, & r=a & M_r=0, \\ r=a & w=0, & r=b & M_r=0 \end{array} \tag{A.18}$$

$$A_4=-\frac{qa^2b^2}{8D} \tag{A.19}$$

$$A_3=-\frac{qa^2b^2}{16D}\left[\frac{(3+v)}{(1-v)}+\frac{4b^2}{(a^2-b^2)}\frac{(1+v)}{(1-v)}\ln\frac{b}{a}\right] \tag{A.20}$$

$$A_2 = -\frac{qa^2}{32D}\left[(a^2-b^2)\frac{(3+v)}{(1-v)} + \frac{4b^4}{(a^2-b^2)}\ln\frac{b}{a}\right] \tag{A.21}$$

$$A_0 = \frac{qa^2}{64D}\left[2(a^2-b^2)\frac{(3+v)}{(1-v)} + \frac{4b^4}{(a^2-b^2)}\ln\frac{b}{a} - a^2(1-4\varepsilon)\right] \tag{A.22}$$