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Exact solutions for free vibration of multi-step orthotropic shear plates

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Abstract. The governing differential equations for free vibration of multi-step orthotropic shear plates with variably distributed mass, stiffness and viscous damping are established. It is shown that a shear plate can be divided into two independent shear bars to determine the natural frequencies and mode shapes of the plate. The *jk*-th natural frequency of a shear plate is equal to the square root of the square sum of the *j*-th natural frequency of a shear bar and the *k*-th natural frequency of another shear bar. The *jk*-th mode shape of the shear plate is the product of the *j*-th mode shape of a shear bar and the *k*-th mode shape of another shear bar. The general solutions of the governing equations of the orthotropic shear plates with various boundary conditions are derived by selecting suitable expressions, such as power functions and exponential functions, for the distributions of stiffness and mass along the height of the plates. A numerical example demonstrates that the present methods are easy to implement and efficient. It is also shown through the numerical example that the selected expressions are suitable for describing the distributions of stiffness and mass of typical multi-storey buildings.

Key words: vibration; plates; tall buildings; natural frequencies; mode shapes.

1. Introduction

Experimental results obtained in dynamic testing of buildings (e.g., Wang 1978, Li 1995, Li *et al.* 1996, Jeary 1997) have shown that for certain cases the shear deformation is dominant in the total deformation of multi-storey buildings in their horizontal vibrations. Such buildings are usually called shear-type buildings. Korqingskee (1953) investigated the free vibration of frame buildings that are considered as a multi-step cantilever shear bar and in which each step of the bar has constant parameters (mass and stiffness). Wang (1978) suggested that frame buildings and other shear-type buildings could be treated as a one-step cantilever shear bar with variably distributed mass and stiffness along the height of a bar for the analysis of free vibration. He derived the closed-form solutions for such a problem. But, he assumed that the mass of the shear bar is proportional to its stiffness. Li *et al.* (1997) recently proposed an approach to determine the dynamic characteristics of cantilever shear bars with variably distributed mass and stiffness. In their study, the value of mass of a shear bar is not necessarily proportional to its stiffness. However, if a building has a narrow rectangular plane configuration, $B/L < 1/4$, where B and L are the width and length of the rectangular plane, the stiffness of each floor of the building cannot be treated as infinitely rigid. This building

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may not be simplified as a cantilever shear bar for free vibration analysis. The field measurements conducted by Ishizaki and Hatakeyana (1960), Wang (1978), Li *et al.* (1994) and Jeary (1997) revealed that for multi-storey buildings with narrow rectangular plane configuration (narrow buildings), shear deformation is usually dominant in the total deformation in their horizontal vibrations. They reported that not only did the parallel motions between floors occur, but also the relative motions between parallel frames were observed. Thus, when analysing free vibration of narrow buildings, it is reasonable to regard such structures as a shear plate with variably distributed stiffness and mass. Because the stiffness of such a plate in the x-axis in Fig. 1 is different from that in the y-axis, the shear plate considered in this paper is an orthotropic shear plate.

In fact, there are very few equations of vibrating plates with variable cross-section where exact analytical solutions can be obtained. An analytical approach for the vibration of a simply supported plate with one change in thickness was developed by Chopra (1974). Guo *et al.* (1997) found the closed-form solutions for the free vibration of a stepped, simply supported plate with uniform thickness and abrupt thickness changes. The concept of orthotropic shear plates was developed and used by Beiner and Librescu (1984). They have presented an analysis of weight minimisation for rectangular flat panels with fixed flutter speed. To simplify the problem, a structural model that considers transverse shear deformation only and neglects the bending stiffness of the plate was adopted in their study. This has the effect of reducing the linear partial differential equation for this problem from the fourth to the second order. However, Beiner and Librescu (1984) did not consider free vibration analysis of orthotropic shear plates. Wang (1978) derived the exact analytical solutions for free vibration of cantilever shear plates with uniformly distributed mass and stiffness. However, it is obvious that the distributions of mass and shear stiffness of most narrow buildings are actually not uniform, especially, along the building heights. In general, the variation of mass and stiffness along the longitudinal axis of a narrow building (the x-axis in Fig. 1) can be neglected (Li *et al*. 1994). Thus, it is assumed that the narrow buildings considered in this paper have uniformly distributed mass and shear stiffness along the longitudinal axis, but variably distributed mass and shear stiffness along the heights of the narrow buildings. The distributions of mass and shear stiffness are described by selecting suitable functions, such as power functions and exponential functions. The general solutions of one-step shear plates with variably distributed mass, shear stiffness and viscous damping corresponding to various boundary conditions are derived. The analysis of free vibration of a multi-step shear plate with variably distributed mass and shear stiffness is a complex problem and the exact solution of this problem has not previously been obtained. Use of the general solution of a one-step shear plate with variable cross-section together with a transfer matrix method is presented in this paper in order to resolve this problem. It is shown through a numerical example that the selected expressions are suitable for describing the distributions of shear stiffness and mass of typical multi-storey buildings. It should be pointed out that the method presented in this paper is not only suitable for free vibration analysis of narrow buildings, but also for that of orthotropic shear plates.

In this paper, exact analytical solutions for free vibrations of orthotropic shear plates with variably distributed mass, stiffness and viscous damping are derived. In the absence of the exact solutions, this problem can be solved using approximated methods (e.g., the Rize method) or numerical methods (e.g., the finite element method and the finite strip method). However, the present exact solutions can provide adequate insight into the physics of the problem and can be easily implemented. The availability of the exact solutions will help in examining the accuracy of the approximate or numerical solutions. Therefore, it is always desirable to obtain the exact solutions to such problems.

2. The governing equations

The governing differential equation for vibration of a multi-step orthotropic shear plate (Fig. 1) which considers transverse shear deformation only and neglects its bending stiffness can be established as follows:

It is assumed that a multi-step orthotropic shear plate is subjected to a horizontal dynamic load, $q(x, y, t)$. In order to establish the differential equation of vibration of this plate, an infinitesimal element of the plate is cut from the i-th step plate, as shown in Fig. 2. The size of the element is $dx \times dy$. The dynamic loading acting on the element is $q(x, y, t)dxdy$. The initial force is $(-\overline{m}_{ixy}(\partial^2 w_i/\partial t^2)dxdy)$ and the damping force is $(-C_{ixy}(\partial w_i/\partial t)dxdy)$, where w_i and C_{ixy} are the dynamic displacement of the plate and the viscous damping coefficient in the z-axis at the point (x, y), respectively. Fig. 2 shows the element that is rotated over an angle of 90°. Considering the equilibrium conditions for all the forces acting on the element (Fig. 2), using d'Almbert priciple, leads to

$$\left[\left(Q_{iy}+\frac{\partial Q_{iy}}{\partial y}dy\right)-Q_{iy}\right]dx+\left[\left(Q_{ix}+\frac{\partial Q_{ix}}{\partial x}dx\right)-Q_{ix}\right]dy+q(x,y,t)dxdy$$
$$-\overline{m}_{ixy}\frac{\partial^2 w_i}{\partial t^2}dxdy-C_{ixy}\frac{\partial w_i}{\partial t}dxdy=0 \tag{1a}$$

Thus

$$\frac{\partial Q_{ix}}{\partial x}+\frac{\partial Q_{iy}}{\partial y}=\overline{m}_{ixy}\frac{\partial^2 w_i}{\partial t^2}dxdy+C_{ixy}\frac{\partial w_i}{\partial t}-q(x,y,t) \tag{1b}$$

in which $\overline{m}_{ixy}$ is the mass intensity (mass per unit area) at the point (x, y) in the i-th step plate.

The shear forces can be expressed as

$$Q_{ix}=K_{ix}\frac{\partial w_i}{\partial x},\qquad Q_{iy}=K_{iy}\frac{\partial w_i}{\partial y} \tag{2}$$

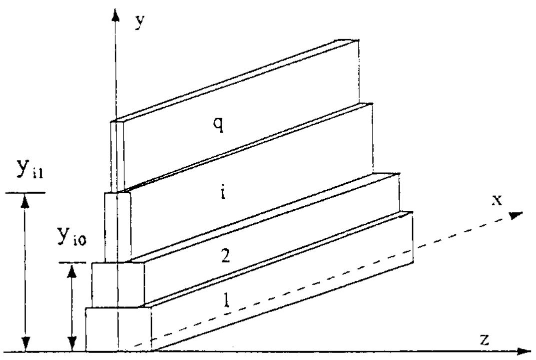


Fig. 1 A multi-step shear plate

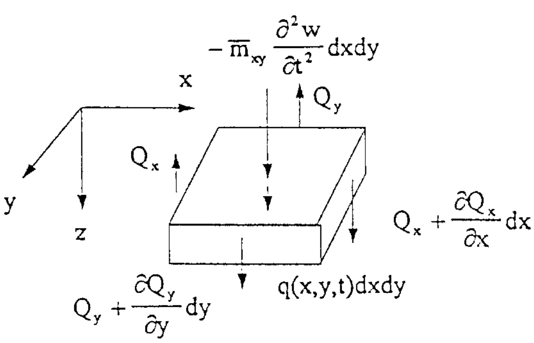


Fig. 2 An element of the shear plate

in which K_{ix} and K_{iy} are the transverse shear stiffness in the x-axis and in the y-axis, respectively.

Substituting Eq. (2) into Eq. (1b) gives

$$\frac{\partial}{\partial x}\left(K_{ix}\frac{\partial w_i}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{iy}\frac{\partial w_i}{\partial y}\right)=\overline{m}_{ixy}\frac{\partial^2 w_i}{\partial t^2}+C_{ixy}\frac{\partial w_i}{\partial t}-q(x,y,t) \tag{3}$$

This is the governing equation for vibration of the i-th step of a multi-step orthotropic shear plate with variably distributed mass and stiffness along the height of the plate. Setting $q(x,y,t)=0$ gives the governing equation for free vibration of the i-th shear plate as follows

$$\frac{\partial}{\partial x}\left(K_{ix}\frac{\partial w_i}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{iy}\frac{\partial w_i}{\partial y}\right)-\overline{m}_{ixy}\frac{\partial^2 w_i}{\partial t^2}-C_{ixy}\frac{\partial w_i}{\partial t}=0 \tag{4}$$

The governing equation for undamped free vibration can be obtained by setting $C_{ixy}=0$ as

$$\frac{\partial}{\partial x}\left(K_{ix}\frac{\partial w_i}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{iy}\frac{\partial w_i}{\partial y}\right)-\overline{m}_{ixy}\frac{\partial^2 w_i}{\partial t^2}=0 \tag{5}$$

It is assumed that

$$w_i(x,y,t)=Z_i(x,y)\sin(\omega t+\gamma_0) \tag{6}$$

where $Z_i(x, y)$ is the vibration mode function, ω is the undamped circular natural frequency, γ_0 is the initial phase.

Substituting Eq. (6) into Eq. (5) leads to

$$\frac{\partial}{\partial x}\left(K_{ix}\frac{\partial Z_i}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{iy}\frac{\partial Z_i}{\partial y}\right)-\overline{m}_{ixy}Z_i=0 \tag{7}$$

For solving Eq. (4), it is assumed that

$$w_i(x,y,t)=Z_i(x,y)\exp(\lambda t) \tag{8}$$

in which λ is a complex value.

Substituting Eq. (8) into Eq. (3) leads

$$\frac{\partial}{\partial x}\left(K_{ix}\frac{\partial Z_i}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{iy}\frac{\partial Z_i}{\partial y}\right)-\lambda(2C_0+\lambda)\overline{m}_{ixy}Z_i=0 \tag{9}$$

If it is assumed that

$$C_{ixy}=2C_0\overline{m}_{ixy} \tag{10}$$

Setting

$$\omega^2=-\lambda(2C_0+\lambda) \tag{11}$$

and substituting Eq. (11) into Eq. (9) obtains the same expression as that of Eq. (7). This suggests that the damped mode shape is the same as the undamped mode shape, and the relationship between the damped frequency and undamped frequency can be found by solving Eq. (11) as follows

$$\lambda=-C_0\pm i\omega\sqrt{1-\left(\frac{C_0}{\omega}\right)^2} \tag{12}$$

The real part of λ is damping coefficient and the imaginary part is the damped circular natural frequency, denoted as ω_d,

$$\omega_d = k_d\omega,\ k_d=\sqrt{1-\left(\frac{C_0}{\omega}\right)}=\sqrt{1-\xi^2} \tag{13}$$

in which ξ is the critical damping ratio.

It can be seen from Eq. (13) that the effect of damping on natural frequency can be neglected for the case of light damping. Even if the damping coefficient is large, the effect of damping can be also not considered in free vibration analysis. After the undamped natural frequencies have been found, the damped natural frequencies can be determined from Eq. (13). This suggests that if the distribution of the damping coefficient of a shear plate is assumed to be proportional to that of the mass ($C_x=C_0\overline{m}_x$), the damped natural frequency is equal to the corresponding undamped natural frequency multiplied by the coefficient, k_d, and the damped mode shape is the same as the corresponding undamped mode shape.

In order to determine the undamped natural frequencies and mode shapes, using the method of separation of variable gives

$$Z_i(x,y)=X_i(x)Y_i(y) \tag{14a}$$

It is assumed that K_y is a function of y, K_x and $\overline{m}_{xy}$ are also functions of y.

$$K_{iy}=K_{i1}\ f_1(y),\quad K_{ix}=K_2\varphi_i(y),\quad \overline{m}_{ixy}=\overline{m}\,\varphi_i(y) \tag{14b}$$

i.e., We assume that K_{ix} is directly proportional to $\overline{m}_{ixy}$, since the values of K_{ix} and $\overline{m}_{ixy}$ are mainly dependent on the size and materials of building floors.

Substituting Eqs. (14a) and (14b) into Eq. (7) obtains

$$\frac{\frac{d}{dy}\left[K_{i1}\ f_i(y)\frac{dY_i(y)}{dy}\right]}{Y_i(y)\varphi_i(y)}=-K_2\frac{\frac{d^2X_i(x)}{dx^2}}{X_i(x)}-\overline{m}\,\omega^2 \tag{15}$$

The left hand side of the above equation is a function of y and the right hand side is a function of x. Thus, both sides should be equal to a constant. It is assumed that the constant is $-\overline{m}\,\theta^2$, then, the following two ordinary differential equations are obtained from Eq. (15)

$$K_2\frac{d^2X(x)}{dx^2}+\overline{m}\Omega^2X=0 \tag{16}$$

$$\frac{d}{dy}\left[K_{i1}f_i(y)\frac{dY_i(y)}{dy}\right]+m\,\varphi_i(y)\,\theta^2\,Y_i(y)=0 \tag{17}$$

where

$$\Omega^2=\omega^2-\theta^2 \qquad \omega=\sqrt{\Omega^2+\theta^2} \tag{18}$$

It is obvious that Eqs. (16) and (17) are two governing equation of vibration mode shape of two shear bars. K_2, $\overline{m}$, Ω are the stiffness, mass intensity and circular natural frequency of a shear bar, respectively. The boundary conditions of this shear bar are the same as those of the shear plate in the x-axis. On the other hand, $K_{i1}\ f_i(y)$, $\overline{m}\varphi_i(y)$, θ, are the stiffness, mass intensity and circular natural frequency of another shear bar, respectively. Its boundary conditions are the same as those of the shear plate in the y-axis. The natural frequency of the plate is equal to the square root of the square sum of the two natural frequencies of the two bars. The mode shape of the plate is the product of the corresponding two mode shapes of the two bars. This suggests that free vibration analysis for a shear plate can be carried out by analysing two independent shear bars, i.e., by solving the two independent ordinary differential equations, Eq. (16) and Eq. (17).

3. Exact solutions of the governing equations

The governing equation for mode shape of the shear bar in the x-axis for each step (see Fig. 1) is the same as Eq. (16). The general solution of Eq. (16) is found as

$$X(x)=D_1\sin\frac{\Omega}{\alpha_2}x+D_2\cos\frac{\Omega}{\alpha_2}x \tag{19}$$

where

$$\alpha_2=\sqrt{\frac{K_2}{\bar{m}}} \tag{20}$$

The k-th circular natural frequency and mode shape are as follows

$$\left.\begin{aligned}\Omega_k&=\frac{(k-1)\pi}{L}\sqrt{\frac{K_2}{\bar{m}}}\\ X_k(x)&=\cos\frac{(k-1)\pi x}{L}\end{aligned}\right\}\text{ for free-free edges of the shear plate in the } x\text{-axis} \tag{21}$$

or

$$\left.\begin{aligned}\Omega_k&=\frac{k\pi}{L}\sqrt{\frac{K_2}{\bar{m}}}\\ X_k(x)&=\sin\frac{k\pi x}{L}\end{aligned}\right\}\text{ for fixed-fixed edges, or freely supported edges} \tag{22}$$

or

$$\left.\begin{aligned}\Omega_k&=\frac{(2k-1)}{2L}\sqrt{\frac{K_2}{\bar{m}}}\\ X_k(x)&=\sin\frac{(2k-1)\pi x}{2L}\end{aligned}\right\}\text{ for fixed-free edges of the shear plate} \tag{23}$$

where L is the length of the shear plate in the x-axis.

The general solution of Eq. (17) for multi-step shear plate can be obtained by use of a transfer matrix method as described below:

The general solution of Eq. (17) can be expressed as

$$Y_i(y)=D_{i1}S_{i1}(y)+D_{i2}S_{i2}(y) \tag{24}$$

where i denotes the i-th step and q is the total number of steps, $S_{i1}(y)$ and $S_{i2}(y)$ are the special solutions of Eq. (17), which are dependent on the expressions of $f_i(y)$ and $\varphi_i(y)$.

As suggested by Tuma and Cheng (1983), Li *et al.* (1994) and Li *et al.* (1998), the functions which can be used to approximate the variation of mass and stiffness are algebraic polynomials, exponential functions, trigonometric series, or their combinations. In this paper, two important cases of $f_i(y)$ and $\varphi_i(y)$ are considered and discussed as follows:

Case 1: The expressions of $f_i(y)$ and $\varphi_i(y)$ are described by the following exponential functions

$$\left.\begin{aligned} & f_i(y)=e^{-\beta_i \frac{y}{H}} \\ & \varphi_i=e^{-b_i \frac{y}{H}} \end{aligned}\right\} \tag{25}$$

The parameters β_i, b_i are constants that can be determined in terms of the real distributions of mass and shear stiffness of the shear plate. H is the height of the shear plate.

The special solutions of this case are found as

$$\left.\begin{aligned} & S_{i1}(y)=\xi_i^{\nu_i} J_{\nu_i}(\gamma_i \xi_i) \\ & S_{i2}(y)=\xi_i^{\nu_i} J_{-\nu_i}(\gamma_i \xi_i) \quad \nu_i=\text{non integer} \\ & \text{or } S_{i1}(y)=\xi_i^{\nu_i} Y_{\nu_i}(\gamma_i \xi_i) \quad \nu_i=\text{ integer} \end{aligned}\right\} \tag{26}$$

in which the parameters ν_i, γ_i and the variable ξ_i are

$$\left.\begin{aligned} & \nu_i=\frac{\beta_i}{\beta_i-b_i} \\ & \gamma_i=\frac{4\overline{m}\theta^2 H^2}{K_1(\beta_i-b_i)^2} \\ & \xi_i=e^{\frac{(\beta_i-b_i)y}{2H}} \end{aligned}\right\} \tag{27}$$

If $\beta_i=b_i$, then

$$\left.\begin{aligned} & S_{i1}(y)=e^{\frac{\beta_i y}{2H}} \cos\gamma_i y \\ & S_{i2}(y)=e^{\frac{\beta_i y}{2H}} \sin\gamma_i y \\ & \gamma_i=\frac{\theta^2}{\alpha_1^2}-\frac{\beta_i^2}{4H^2},\ \frac{\beta_i^2}{4H^2}<\frac{\theta^2}{\alpha_1^2},\ \alpha_1^2=\frac{K_1}{\overline{m}} \end{aligned}\right\} \tag{28}$$

where $J_\nu(\xi)$ is Bessel function of the first kind, of order ν; $Y_\nu(\xi)$ is Bessel function of the second kind, of order ν.

Case 2: The expressions of $f_i(y)$ and $\varphi_i(y)$ are described by the following power functions

$$\left.\begin{aligned} f_i(y) &= (1+\beta_i y)^{p_i} \\ \varphi_i(y) &= (1+\beta_i y)^{c_i} \end{aligned}\right\} \tag{29}$$

where β_i, p_i, c_i are constants which can be determined in terms of the real distributions of mass and stiffness.

The special solutions of this case are found as

$$\left.\begin{aligned} & S_{i1}(y) = \left(\frac{c_i - p_i + 2}{2n_i}\xi_i\right)^{\nu_i} J_{\nu_i}(\gamma_i \xi_i) \\ & S_{i2}(y) = \left(\frac{c_i - p_i + 2}{2n_i}\xi_i\right)^{\nu_i} J_{-\nu_i}(\gamma_i \xi_i) \quad \nu_i = \text{non integer} \\ & \text{or } S_{i2}(y) = \left(\frac{c_i - p_i + 2}{2n_i}\xi_i\right)^{\nu_i} Y_{\nu_i}(\xi_i) \quad \nu_i = \text{ integer} \end{aligned}\right\} \tag{30}$$

in which

$$\left.\begin{aligned} \nu_i &= \frac{1 - p_i}{c_i - p_i + 2} \\ \xi_i &= \frac{2n_i}{c_i - p_i + 2}(1+\beta_i y)^{\frac{c_i + p_i + 2}{2}} \\ n_i^2 &= \frac{\bar{m}\theta^2}{K_{i1}\beta_i^2} \end{aligned}\right\} \tag{31}$$

If $p_i = c_i + 2$, then

$$\left.\begin{aligned} & S_{i1}(y) = (1+\beta_i y)^{\frac{1-p_i}{2}} \cos[\sqrt{D_i}\ln(1+\beta_i y)] \\ & S_{i2}(y) = (1+\beta_i y)^{\frac{1-p_i}{2}} \sin[\sqrt{D_i}\ln(1+\beta_i y)] \\ & D_i = n_i^2 - \frac{(1-p_i)^2}{4} > 0, \; n_i = \frac{\bar{m}\theta^2}{K_{i1}\beta_i^2} \end{aligned}\right\} \tag{32a}$$

If $\beta_i = 0$, i.e., for a uniform shear plate, then

$$\left.\begin{aligned} S_{i1}(y) &= \sin\frac{\theta}{\alpha_1} y \\ S_{i2}(y) &= \cos\frac{\theta}{\alpha_1} y \\ \alpha_1 &= \sqrt{\frac{K_1}{\bar{m}}} \end{aligned}\right\} \tag{32b}$$

A transfer matrix method is introduced herein to establish the equation of mode shape in the y-axis and the frequency equation of a multi-step shear plate. The mode shape function of $Y_i(y)$ displacement and the shear force, $Q_i(y)$, can be expressed as a matrix

$$[Y_i(y)Q_i(y)]^T = \begin{bmatrix} S_{i1}(y) & S_{i2}(y) \\ K_{iy}S'_{i1}(y) & K_{iy}S'_{i2}(y) \end{bmatrix} \begin{bmatrix} D_{i1} \\ D_{i2} \end{bmatrix} \tag{33}$$

The relationship between the parameters, Y_{i1}, Q_{i1} at the end 1 and, Y_{i0}, Q_{i0} at the end 0 of the i-th step (Fig. 3) can be expressed as

$$\begin{bmatrix} Y_{i1} \\ Q_{i1} \end{bmatrix} = [T_i] \begin{bmatrix} Y_{i0} \\ Q_{i0} \end{bmatrix} \tag{34}$$

in which

$$[T_i] = [S(y_{i1})][S(y_{i0})]^{-1}$$

$$[S(y_{i0})] = \begin{bmatrix} S_{i1}(y_{i0}) & S_{i2}(y_{i0}) \\ K_{iy}(y_{i0})S'_{i1}(y_{i0}) & K_{iy}(y_{i0})S'_{i2}(y_{i0}) \end{bmatrix}$$

$$[S(y_{i1})] = \begin{bmatrix} S_{i1}(y_{i1}) & S_{i2}(y_{i1}) \\ K_{iy}(y_{i1})S'_{i1}(y_{i1}) & K_{iy}(y_{i1})S'_{i2}(y_{i1}) \end{bmatrix}$$

$$Y_{i0} = Y_i(y_{i0}) \qquad Y_{i1} = Y_i(y_{i1})$$

$$Q_{i0} = Q_i(y_{i0}) \qquad Q_{i1} = Q_i(y_{i1})$$

$[T_i]$ is called the transfer matrix, because it transfers the parameters at the end 0 to those at the end 1 of a step.

The relationship between the parameters at the end 1 for the top step (Fig. 1) and those at the end 0 for the bottom step can be established by using Eq. (34) repeatedly as:

$$\begin{bmatrix} Y_{q1} \\ Q_{q1} \end{bmatrix} = [T] \begin{bmatrix} Y_{10} \\ Q_{10} \end{bmatrix} \tag{35}$$

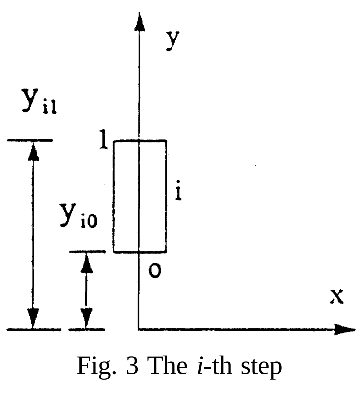


Fig. 3 The i-th step

in which

$$[T]=[T_q][T_{q-1}]\dots[T_1] \tag{36}$$

$[T]$ is a matrix which can be expressed as

$$[T]=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{37}$$

For the case of fixed-free edges of a shear plate in the y-axis (Fig. 4), Eq. (35) becomes

$$\begin{bmatrix} Y_{q1} \\ 0 \end{bmatrix}=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{bmatrix} 0 \\ Q_{10} \end{bmatrix} \tag{38}$$

From Eq. (38),

$$T_{22}Q_{10}=0$$

Because of $Q_{10} \neq 0$, we have

$$T_{22}=0 \tag{39}$$

This is the frequency equation of a multi-step shear plate with fixed-free edges in the y-axis.

For the case of fixed-fixed edges of a shear plate in the y-axis (Fig. 5), Eq. (35) becomes

$$\begin{bmatrix} 0 \\ Q_{q1} \end{bmatrix}=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{bmatrix} 0 \\ Q_{10} \end{bmatrix} \tag{40}$$

The frequency equation for this case is

$$T_{12}=0 \tag{41}$$

For the case of free-free edges of a shear plate in the y-axis (Fig. 6), Eq. (35) becomes

$$\begin{bmatrix} Y_{q1} \\ 0 \end{bmatrix}=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{bmatrix} Y_{10} \\ 0 \end{bmatrix} \tag{42}$$

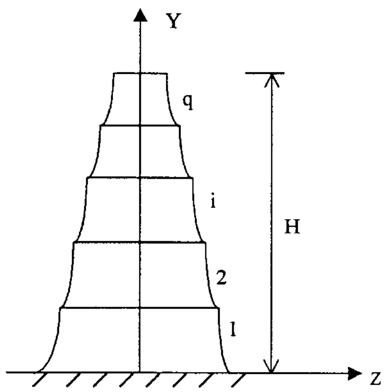


Fig. 4 The case of fixed-free edges

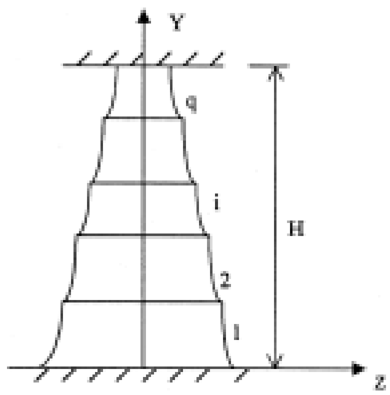


Fig. 5 The case of fixed-fixed edges

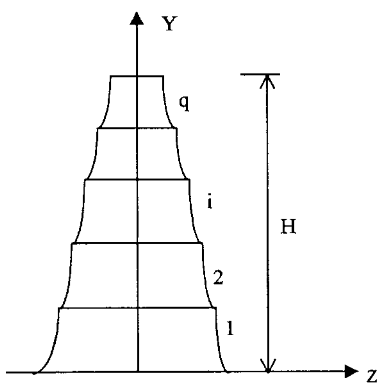


Fig. 6 The case of free-free edges

The frequency equation for this case is

$$T_{21}=0 \tag{43}$$

Solving the frequency equation obtains θ_j (j=1, 2, ...), then substituting θ_j into Eq. (34) and setting $Y_{10}=1$ (if $Y_{10}\neq 0$) or $Q_{10}=1$ (if $Q_{10}\neq 0$) give the j-th mode shape of the shear plate in the y-axis.

The frequency equation and the j-th mode shape of an one-step orthotropic shear plate in the y-axis for the case 1 can be obtained by setting q=1 in the above equations as follows

$$\left.\begin{array}{l}
J_{-(v-1)}(\gamma_j)J_{v-1}\left(\gamma_j e^{\frac{\beta-b}{2}}\right)=J_{v-1}(\gamma_j)J_{-(v-1)}\left(\gamma_j e^{\frac{\beta-b}{2}}\right)\quad v=\text{non integer}\\
Y_j(y)=\xi^{v}[J_v(\gamma_j\xi)-\dfrac{J_{v-1}(\gamma_j)}{J_{-(v-1)}(\gamma_j)}J_{-v}(\gamma_j\xi)]\\
Y_{v-1}(\gamma_j)J_{v-1}\left(\gamma_j e^{\frac{\beta-b}{2}}\right)=J_{v-1}(\gamma_j)Y_{v-1}\left(\gamma_j e^{\frac{\beta-b}{2}}\right)\quad v=\text{ integer}\\
Y_j(y)=\xi^{v}[J_v(\gamma_j\xi)-\dfrac{J_{v-1}(\gamma_j)}{J_{v-1}(\gamma_j)}Y_v(\gamma_j\xi)]
\end{array}\right\}\text{for free-free edges} \tag{44}$$

or

$$\left.\begin{array}{l}
J_{-v}(\gamma_j)J_v\left(\gamma_j e^{\frac{\beta-b}{2}}\right)=J_v(\gamma_j)J_{-v}\left(\gamma_j e^{\frac{\beta-b}{2}}\right)\quad v=\text{non integer}\\
Y_j(y)=\xi^{v}[J_v(\gamma_j\xi)-\dfrac{J_v(\gamma_j)}{J_{-v}(\gamma_j)}J_{-v}(\gamma_j\xi)]\\
Y_v(\gamma_j)J_v\left(\gamma_j e^{\frac{\beta-b}{2}}\right)=J_v(\gamma_j)Y_v\left(\gamma_j e^{\frac{\beta-b}{2}}\right)\quad v=\text{ integer}\\
Y_j(y)=\xi^{v}[J_v(\gamma_j\xi)-\dfrac{J_v(\gamma_j)}{Y_v(\gamma_j)}Y_v(\gamma_j\xi)]
\end{array}\right\}\text{for fixed-fixed edges} \tag{45}$$

or

$$\left.\begin{array}{l}
J_{-v}(\gamma_j)J_{v-1}\left(\gamma_j e^{\frac{\beta-b}{2}}\right)=-J_v(\gamma_j)J_{-(v-1)}\left(\gamma_j e^{\frac{\beta-b}{2}}\right)\quad v=\text{non integer}\\
Y_j(y)=\xi^{v}[J_v(\gamma_j\xi)-\dfrac{J_v(\gamma_j)}{J_{-v}(\gamma_j)}J_{-v}(\gamma_j\xi)]\\
Y_v(\gamma_j)J_{v-1}\left(\gamma_j e^{\frac{\beta-b}{2}}\right)=J_v(\gamma_j)Y_{v-1}\left(\gamma_j e^{\frac{\beta-b}{2}}\right)\quad v=\text{ integer}\\
Y_j(y)=\xi^{v}[J_v(\gamma_j\xi)-\dfrac{J_v(\gamma_j)}{Y_v(\gamma_j)}Y_v(\gamma_j\xi)]
\end{array}\right\}\text{for fixed-fixed edges} \tag{46}$$

where

$$\left.\begin{aligned}&\nu=\frac{\beta}{\beta-b}\\&\gamma_j=\frac{4\overline{m}\,\theta_j^2H^2}{K_1(\beta-b)^2}\\&\xi=e^{\frac{(\beta-b)y}{2H}}\end{aligned}\right\} \tag{47}$$

After γ_j is found, θ_j can be determined from Eq. (47).

If β=b, the frequency equation and the j-th mode shape are

$$\left.\begin{aligned}&\tan\ \gamma_jH=-\frac{2\gamma_jH}{\beta}\\&Y_j(y)=e^{\frac{\beta y}{2H}}\sin\gamma_jy\end{aligned}\right\}\text{ for fixed-free edges} \tag{48}$$

or

$$\left.\begin{aligned}&\gamma_j=\frac{j\pi}{H}\\&Y_j(y)=e^{\frac{\beta y}{2H}}\sin\gamma_jy\end{aligned}\right\}\text{ for fixed-fixed edges} \tag{49}$$

or

$$\left.\begin{aligned}&\gamma_j=\frac{(j-1)\pi}{H}\\&Y_j(y)=e^{\frac{\beta y}{2H}}\left(\cos\gamma_jy-\frac{\beta}{2H\gamma_j}\sin\gamma_jy\right)\end{aligned}\right\}\text{ for free-free edges} \tag{50}$$

After γ_j is found from Eq. (48) or Eq. (49) or Eq. (50), θ_j can be determined by

$$\theta_j=\sqrt{\gamma_j^2+\frac{\beta^2}{4H^2}}\sqrt{\frac{K_1}{\overline{m}}} \tag{51}$$

For a one-step orthotropic shear plate for the case 2, we have

$$\left.\begin{aligned}&J_{-(\nu-1)}(\eta_j)J_{\nu-1}(\eta_j\psi)=J_{\nu-1}(\eta_j)J_{-(\nu-1)}(\eta_j\psi)\quad \nu=\text{non integer}\\&Y_j(y)=\left(\frac{c-p+2}{2n_j}\xi\right)^{\nu}[J_\nu(\xi)+\frac{J_{\nu-1}(\eta_j)}{J_{-(\nu-1)}(\eta_j)}J_{-\nu}(\xi)]\\&Y_{\nu-1}(\eta_j)J_{\nu-1}(\eta_j\psi)=J_{\nu-1}(\eta_j)Y_{\nu-1}(\eta_j\psi)\quad \nu=\text{ integer}\\&Y_j(y)=\left(\frac{c-p+2}{2n_j}\xi\right)^{\nu}[J_\nu(\xi)-\frac{J_{\nu-1}(\eta_j)}{J_{\nu-1}(\eta_j)}Y_\nu(\xi)]\end{aligned}\right\}\text{ for free-free edges} \tag{52}$$

or

$$\left.\begin{array}{l} J_{-\nu}(\eta_j)J_{\nu-1}(\eta_j\psi)=-J_\nu(\eta_j)J_{-(\nu-1)}(\eta_j\psi) \quad \nu=\text{non integer} \\ Y_j(y)=\left(\dfrac{c-p+2}{2n_j}\xi\right)^\nu[J_\nu(\xi)+\dfrac{J_\nu(\eta_j)}{J_{-\nu}(\eta_j)}J_{-\nu}(\xi)] \\ Y_\nu(\eta_j)J_{\nu-1}(\eta_j\psi)=J_\nu(\eta_j)Y_{\nu-1}(\eta_j\psi) \quad \nu=\text{ integer} \\ Y_j(y)=\left(\dfrac{c-p+2}{2n_j}\xi\right)^\nu[J_\nu(\xi)-\dfrac{J_\nu(\eta_j)}{Y_\nu(\eta_j)}Y_\nu(\xi)] \end{array}\right\} \text{for fixed-free edges} \tag{53}$$

or

$$\left.\begin{array}{l} J_{-\nu}(\eta_j)J_{\nu}(\eta_j\psi)=J_\nu(\eta_j)J_{-\nu}(\eta_j\psi) \quad \nu=\text{non integer} \\ Y_j(y)=\left(\dfrac{c-p+2}{2n_j}\xi\right)^\nu[J_\nu(\xi)+\dfrac{J_\nu(\eta_j)}{J_{-\nu}(\eta_j)}J_{-\nu}(\xi)] \\ Y_\nu(\eta_j)J_{\nu}(\eta_j\psi)=J_\nu(\eta_j)Y_{\nu}(\eta_j\psi) \quad \nu=\text{ integer} \\ Y_j(y)=\left(\dfrac{c-p+2}{2n_j}\xi\right)^\nu[J_\nu(\xi)-\dfrac{J_\nu(\eta_j)}{Y_\nu(\eta_j)}Y_\nu(\xi)] \end{array}\right\} \text{for fixed-fixed edges} \tag{54}$$

where

$$\left.\begin{array}{l} \eta_j=\dfrac{2n_j}{c-p+2} \\ \psi=(1+\beta H)^{\frac{c-p+2}{2}} \\ n_j^2=\dfrac{\bar{m}\theta_j^2}{K_1\beta^2} \end{array}\right\} \tag{55}$$

After η_j is found, θ_j can be determined by

$$\theta_j=\beta\eta_j\sqrt{\frac{c-p+2}{2}}\sqrt{\frac{K_1}{\bar{m}}} \tag{56}$$

If p=c+2, then the frequency equation and the j-th mode shape in the y-axis are as follows

$$\left.\begin{array}{l} D_j\ln(1+\beta H)=(j-1)\pi \\ Y_j(y)=(1+\beta y)^{\frac{1-p}{2}}\left\{\cos[D_j\ln(1+\beta y)]-\dfrac{1-p}{2D_j}\sin[D_j\ln(1+\beta y)]\right\} \end{array}\right\} \text{for free-free edges} \tag{57}$$

or

$$\left.\begin{array}{l} \tan[D_j(1+\beta H)]=\dfrac{2D_j}{p-1} \\ Y_j(y)=(1+\beta y)^{\frac{1-p}{2}}\sin[D_j\ln(1+\beta y)] \end{array}\right\} \text{for fixed-free edges} \tag{58}$$

or

$$\left.\begin{aligned} &D_j \ln(1+\beta H)=j\pi \\ &Y_j(y)=(1+\beta y)^{\frac{1-p}{2}} \sin[D_j \ln(1+\beta y)] \end{aligned}\right\} \text{for fixed-fixed edges} \tag{59}$$

where

$$D_j^2=n_j^2-\frac{(1-p)^2}{4}>0 \tag{60}$$

After D_j is solved, θ_j can be determined by

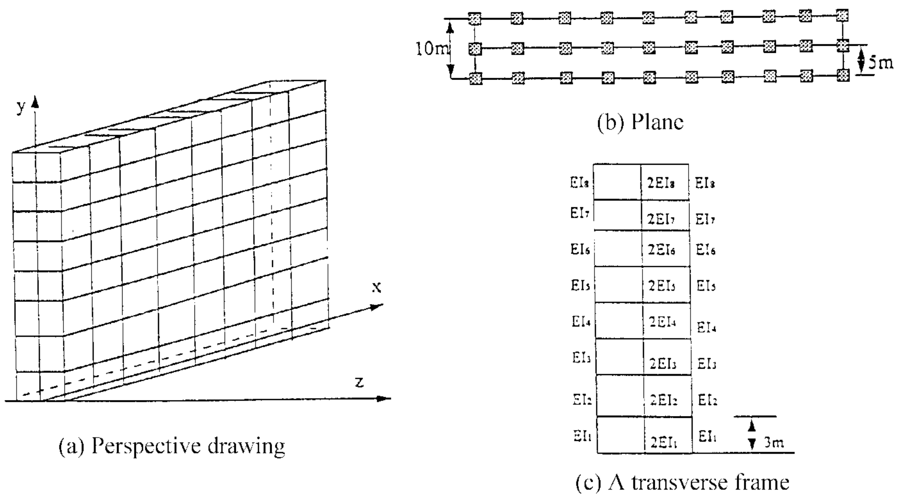
$$\theta_j=\beta\sqrt{D_j^2+\frac{(1-p)^2}{4}}\sqrt{\frac{K_1}{\overline{m}}} \tag{61}$$

4. Numerical example

Fig. 7 shows a sketch of a narrow building that is treated as a cantilever shear plate for free vibration analysis. The masses of columns are lumped to each floor. The masses in different storeys are as follows

m_1=2.16×10^5 kg
m_1=m_3=m_4=2.00×10^5 kg
m_5=m_6=m_7=1.93×10^5 kg
m_8=1.735×10^5 kg

The stiffness of columns in different storeys is
EI_1=8.4×10^4 KN· m^2



(a) Perspective drawing (b) Plane (c) A transverse frame

Fig. 7 A narrow building

$EI_2=EI_3=EI_4=8.0\times10^4$ KN· m^2
$EI_5=EI_6=EI_7=7.5\times10^4$ KN· m^2
$EI_8=6.747\times10^4$ KN· m^2

The stiffness of each floor in the y-axis is treated as infinitely rigid and the stiffness of that in the x-axis is
$GF_1=8.4\times10^8$ KN
$GF_2=GF_3=GF_4=8.0\times10^8$ KN
$GF_5=GF_6=GF_7=7.5\times10^8$ KN
$GF_8=7.0\times10^8$ KN

where GF_i is the shear stiffness of the i-th floor.

The storey height is a constant, h=3.0 m. The distance between the transverse frames is 6 m. The procedure for determining the frequencies and mode shapes of this narrow building is as follows

4.1. Determination of the mass per unit area

Because the mass of a storey is distributed on the area that is equal to the product of the length, 9×6 m, of the building and the storey height, 3 m, thus, the mass per unit area of this building for the first storey is determined by

$$\overline{m}_1=\frac{2.16\times10^5}{3\times5\times9}=1.60\times10^3\text{kg}/\text{m}^2$$

The mass per unit area of this building from the second storey to the fourth storey is

$$\overline{m}_2=1.5257\times10^3\text{kg}/\text{m}^2$$

For the fifth storey to the seventh storey,

$$\overline{m}_3=1.43\times10^3\text{kg}/\text{m}^2$$

For the eighth storey,

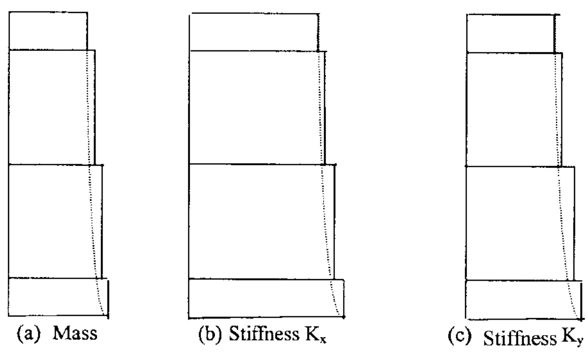


Fig. 8 The distribution of mass and stiffness
Note: The solid lines represent the real distributions and the dash lines represent the assumed exponential distributions

$$\overline{m}_4 = 1.285 \times 10^3 \text{kg/m}^2$$

The four-step distribution of the mass per unit area is shown in Fig. 8a. In order to apply the methods proposed in this paper to analyse free vibration of this narrow building, the four-step distribution of $\overline{m}$ (Fig. 8a) is represented by a continuous curve and it can be seen that the mass distribution can be described reasonably well by the exponential function as follows

$$\overline{m}_{xy} = 1.60 \times 10^3 e^{-0.2192\frac{y}{H}}$$

4.2. Evaluation of the shear stiffness K_x

K_x is the value of GF divided by the storey height.
For the first storey,

$$K_{1x} = \frac{8.4 \times 10^6}{3} = 2.80 \times 10^6 \text{Nm}^{-1}$$

For the second storey to the fourth storey,

$$K_{2x} = 2.67 \times 10^6 \text{N} \cdot \text{m}^{-1}$$

For the fifth storey to the seventh storey,

$$K_{3x} = 2.5 \times 10^6 \text{Nm}^{-1}$$

For the eighth storey,

$$K_{4x} = 2.249 \times 10^6 \text{Nm}^{-1}$$

The four-step distribution of K_x is shown in Fig. 8b. The steped distribution of K_x (Fig. 8b) is also represented by a continuous curve. The distribution of K_x is described reasonably well by the following exponential function (Fig. 8b).

$$K_x = 2.8 \times 10^6 e^{-0.2192\frac{y}{H}} \text{kN/m}$$

4.3. Evaluation of shear stiffness K_y

The total number of transverse frames is ten, thus, the first storey stiffness is

$$D_{1y} = \left[\frac{12EI_1}{h^3} \times 2 + \frac{12(2EI_1)}{h^3}\right] \times 10 = 1.49 \times 10^6 \text{N/m}$$

The stiffness for the second storey to the fourth storey is

$$D_{2y} = 1.42 \times 10^6 \text{N/m}$$

The stiffness for the fifth storey to the seventh storey is

$$D_{3y} = 1.33 \times 10^6 \text{N/m}$$

The eighth storey stiffness is

$$D_{4y}=1.197\times 10^{6}\text{N/m}$$

K_{iy} (i=1, 2, 3, 4) are the values of $D_{iy}h$ divided by the distance between the transverse frames.

$$K_{1y}=\frac{1.49\times 3}{6}\times 10^{6}=7.45\times 10^{5}\text{N/m}$$

$$K_{2y}=7.1\times 10^{5}\text{N/m}$$

$$K_{3y}=6.67\times 10^{5}\text{N/m}$$

$$K_{4y}=5.98\times 10^{5}\text{N/m}$$

The four-step distribution of K_y is shown in Fig. 8c. The distribution of K_y is also described reasonably well by the following exponential function (Fig. 8c)

$$K_y=7.45\times 10^{5}e^{-0.2192\frac{y}{H}}$$

4.4. Determination of natural frequencies

The frequency equation for this shear plate with four-step uniform cross-section is

$T_{22}=0$

where T_{22} is the element of $[T]$ that is given by

$$[T]=[T_1][T_2][T_3][T_4]$$

and $[T_i]$ (i=1, 2, 3, 4) is

$$[T_i]=\begin{bmatrix} \sin\frac{\theta}{\alpha_1}y_{i1} & \cos\frac{\theta}{\alpha_1}y_{i1} \\ K_{iy}\frac{\theta}{\alpha_1}\cos\frac{\theta}{\alpha_1}y_{i1} & -K_{iy}\frac{\theta}{\alpha_1}\sin\frac{\theta}{\alpha_1}y_{i1} \end{bmatrix}\begin{bmatrix} \sin\frac{\theta}{\alpha_1}y_{i0} & \cos\frac{\theta}{\alpha_1}y_{i0} \\ K_{iy}\frac{\theta}{\alpha_1}\cos\frac{\theta}{\alpha_1}y_{i0} & -K_{iy}\frac{\theta}{\alpha_1}\sin\frac{\theta}{\alpha_1}y_{i0} \end{bmatrix}^{-1}$$

Solving the frequency equation obtains

$\theta_1=5.64$, $\theta_2=9.58$, $\theta_3=12.51$, $\theta_4=14.57$

Because the two opposite edges of the cantilever shear plate in the x-axis are free edges, thus, Ω_k is found from Eq. (23) as

$$\Omega_k=\frac{173.4477(k-1)\pi}{54},\quad (k=1,\ 2,\ 3,\ 4)$$

i.e., $\Omega_1=0$, $\Omega_2=10.0908$, $\Omega_3=20.1816$, $\Omega_4=30.2724$

ω_{jk} is given by

$$\omega_{jk}=\sqrt{\theta_j^2+\Omega_k^2}$$

The values of ω_{jk} calculated based on the four-step uniformly distributed mass and stiffness are

Table 1 The circular natural frequencies of the narrow building

ω_{11}	ω_{21}	ω_{12}	ω_{31}	ω_{22}	ω_{41}	ω_{32}	ω_{42}
5.64 (5.65)	9.58 (9.60)	11.5 (11.6)	12.5 (12.4)	13.9 (12.9)	14.6 (14.6)	16.1 (16.0)	17.7 (17.8)
ω_{13}	ω_{23}	ω_{33}	ω_{43}	ω_{14}	ω_{24}	ω_{34}	ω_{44}
21.0	22.3	23.7	24.9	30.8	31.8	32.8	33.6

Note: The data in parentheses are the values calculated based on the assumed exponential functions for the distributions of mass and stiffness.

listed in Table 1. It is necessary to point out that ω_{jk} is corresponding to the j-th mode shape in the y-axis and the k-th mode shape in the x-axis.

As shown in Fig. 8, the narrow building has variably distributed mass and stiffness along the building height. The stiffness and mass of the building are described reasonably well by the exponential functions that are the special case, β=b, of the Case 1 discussed previously. Thus, this building can be treated as a one-step cantilever shear plate for free vibration analysis. The values of ω_{jk} computed based on a one-step cantilever shear plate with variably distributed mass and stiffness are also presented in Table 1 for comparison purposes. It can be seen that results calculated in terms of the two methods are almost identical.

4.5. Determination of mode shape

For the first step, we have,

$$\begin{bmatrix} Y_{11} \\ Q_{11} \end{bmatrix} = [T_1] \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For the second, third and fourth step, we have

$$\begin{bmatrix} Y_{21} \\ Q_{21} \end{bmatrix} = [T_2][T_1] \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

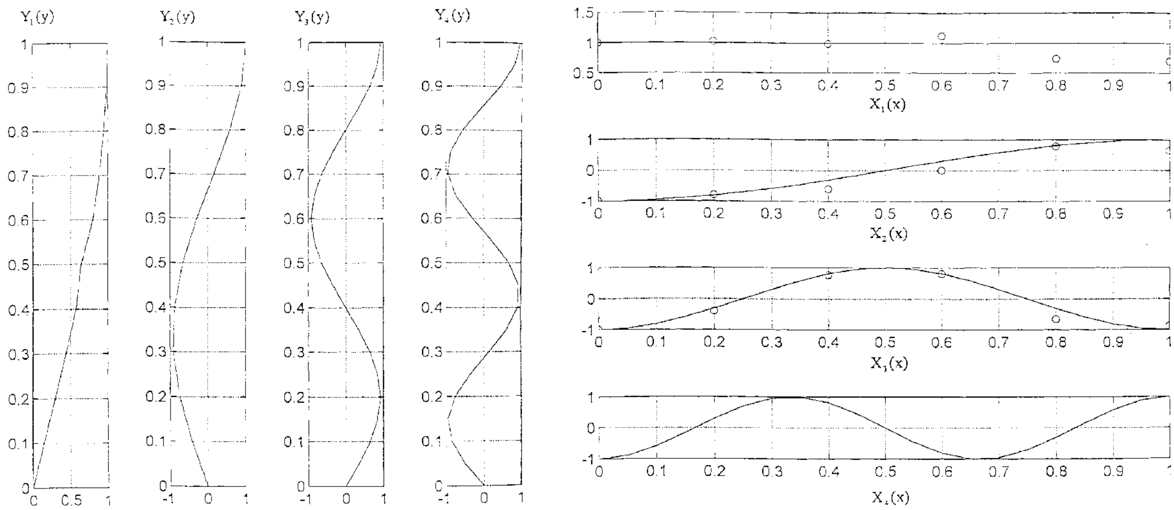
$$\begin{bmatrix} Y_{31} \\ Q_{31} \end{bmatrix} = [T_3][T_2][T_1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [T_3] \begin{bmatrix} Y_{21} \\ Q_{21} \end{bmatrix}$$

$$\begin{bmatrix} Y_{41} \\ Q_{41} \end{bmatrix} = [T_4][T_3][T_2][T_1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [T_4] \begin{bmatrix} Y_{31} \\ Q_{31} \end{bmatrix}$$

Substituting θ_j and the expressions of $[T_1]$, Eq. (62), into the above equations obtains the j-th (j=1, 2, 3, 4) mode shapes of displacement which are shown in Fig. 9.

The k-th mode shape function in the x-axis can be written as

$$X_k(x) = \cos\frac{(k-1)\pi x}{54}$$



(a) (b)

Fig. 9 (a) The first four mode shapes in the y-direction, (b) The first four mode shapes in the x-direction
Note: The solid lines represent the calculated mode shapes and the circle symbols represent the measured results by Wang (1978)

Fig. 9 shows the mode shapes in the y-axis, $Y_j(y)$, and in the x-axis, $X_k(x)$, (j=1, 2, 3, 4; k=1, 2, 3, 4), respectively. Wang (1978) had measured structural dynamic characteristics of many multi-storey and tall buildings. He found that narrow buildings that can be treated as a cantilever shear plate have the same or almost the same mode shapes in the x-axis. The averages of the experimental data obtained by Wang (1978) are also plotted in Fig. 9 for comparison purposes. It is clear that the calculated results are in good agreement with the measured field data.

The mode shape functions, $Z(x, y)$, of this building can be found as

$$Z_{jk}(x, y)=Y_j(y)X_k(x)$$

Fig. 10 shows the obtained mode shapes, $Z_{11}(x, y)$, $Z_{22}(x, y)$, $Z_{33}(x, y)$ and $Z_{44}(x, y)$.

If $K_2 \to \infty$, then ω_{j1} (j >1) is found to be less than ω_{1k} (k >1), i.e., only Ω_1 will occur. In this case, $X_2(x)$ may not appear in the vibration of the building, thus, this narrow building can be simplified as a shear bar in the analysis of free vibration.

5. Conclusions

In fact, there are very few equations of vibrating plates with variable cross-section where exact solutions can be obtained. In this paper, an approach to determine the natural frequencies and mode shapes of orthotropic shear plates with variably distributed mass and stiffness corresponding to several boundary conditions is proposed. It has been shown that a one-step shear plate and a multi-step shear plate can be divided into two independent one-step shear bars and multi-step shear bars with the same boundary conditions as those of the shear plates in analysing their free vibrations.

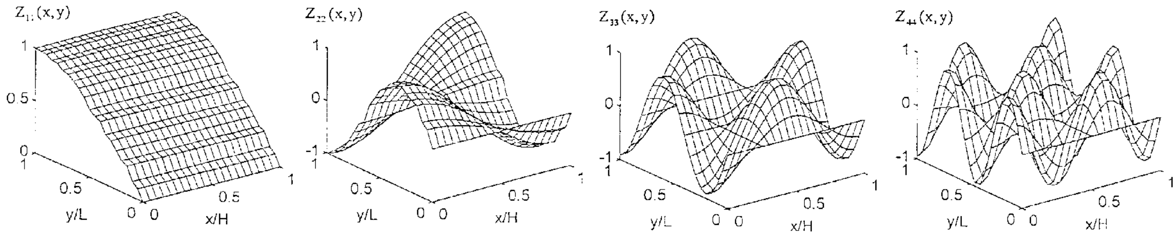


Fig. 10 The mode shapes of the narrow building

The *jk*-th natural frequency of a shear plate is equal to the square root of the square sum of the *j*-th natural frequency of a shear bar and the *k*-th natural frequency of another shear bar. The *jk*-th mode shape of the shear plate is the product of the *j*-th mode shape of a shear bar and the *k*-th mode shape of another shear bar. The exact solutions that are expressed in terms of Bessel and trigonometric functions are derived by selecting suitable expressions, such as power functions and exponential functions, for the distributions of stiffness and mass along the height of the plates. These closed form expressions presented herein can be also used as benchmarks for checking the results obtained from numerical or approximate methods. The numerical example demonstrates that the present methods are easy to implement and efficient. It is also shown through the numerical example that the selected expressions are suitable for describing the distributions of stiffness and mass of typical multi-storey buildings.

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