*Structural Engineering and Mechanics, Vol. 8, No. 6 (1999) 547-560*
DOI: http://dx.doi.org/10.12989/sem.1999.8.6.547


# An asymptotic analysis on non-linear free vibration of squarely-reticulated circular plates

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**Abstract.** In this paper an asymptotic iteration method is adopted to analyze non-linear free vibration of reticulated circular plates composed of beam members placed in two orthogonal directions. For the resulting linear ordinary differential equations in the process of iteration, the power series with rapid convergence has been applied to obtain an analytical solution for non-linear characteristic relation between the amplitude and frequency of the structure. Numerical examples are given, and the phenomena indicating hardening of such structures have been presented for the (immovable or movable) simply-supported and clamped circular plates.

**Key words:** reticulated plates; non-linear; free vibration; asymptotic iteration method.

## 1. Introduction

Recently applications of large-span space reticulated structures have been demonstrated in numerous kinds of building structures, such as sport stadia, exhibition halls, aircraft hangars, supermarkets, leisure centers and swimming pools, etc.

The understandings of vibrating behaviours of the reticulated structures are of great importance to the design for such structures. Many concerns have focused on these problems. Ellington and McCallion (1959) give an analysis on free vibration of grillages by the finite difference method. Applying the exact representation of the stiffness, Anderson and Williams (1982, 1986) investigate the vibration of periodic lattice structures. Cheung *et al.* (1988) adopt a double U-transformation approach to solve the free vibration problem of rectangular networks. Yamada and Takeuchi (1993) propose a method to estimate the free vibration frequency of latticed cylindrical panels on the basis of the mode similarity to the static buckling mode.

However, so far there have been no much work on the non-linear vibration analyses of single-layer reticulated structures with large-number beam members due to difficulties in mathematical treatment or complexity in numerical computations. It is found that this kind of structures takes on very strong nonlinearity, and it is thus necessary to pay much attention to them, especially including exact analyses on non-linear natural frequencies.

Based on the analyses on the internal forces and deformations of single-layer reticulated shallow shell structures with or without imperfections whose beam members are placed in two orthogonal directions, Nie (1991, 1994a) and Liu *et al.* (1991) choose a continuum (plate) shell model to perform non-linear analyses of the structures. The model has been examined by

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numerical computations and experiment studies, and comparisons of results show that the use of the proposed model is enough exact and effective (Nie and Cheung 1995, Nie and Liu 1995). The model has also been applied to analysis of multi-mode vibration of the structures with a rectangular boundary (Nie 1994b).

This paper aims to present an analytical solution for non-linear amplitude-frequency relation by solving the axisymmetrical fundamental governing equations (coupled dynamic equilibrium equation and compatibility equation) in terms of nondimensional transverse displacement (deflection) and radial membrane force with the help of an asymptotic method. For the resulting linear ordinary differential equations in the process of iteration, the power series with rapid convergence has been adopted to obtain an asymptotic solution for non-linear characteristic relation between the amplitude and frequency of the structure. Numerical examples for four cases are given, and the phenomena indicating hardening of such structures have been demonstrated.

## 2. The mathematical formulation for the problem

Let us consider a square reticulated circular plate with radius $a$, as shown in Fig. 1, each beam member has the same material properties and sizes with length $L$, area of cross-section $A$, transverse and lateral bending stiffness $EI$, $EI_0$, and twisting stiffness $GJ$. By analyzing the internal forces and deformations of a typical element of such discrete structure, applying the principle of energy, the mathematical expressions for equilibrium equation in the normal direction and compatibility equation in terms of transverse (joint) displacement (deflection) $w$ and a force function $\phi$ can be written as follows (Nie and Cheung 1995).

$$\frac{EI}{L} w_{,1111} + \frac{2GJ}{L} w_{,1122} + \frac{EI}{L} w_{,2222} + m_0 \ddot{w} = w_{,11}\phi_{,22} - 2w_{,12}\phi_{,12} + w_{,22}\phi_{,11} \qquad (1)$$

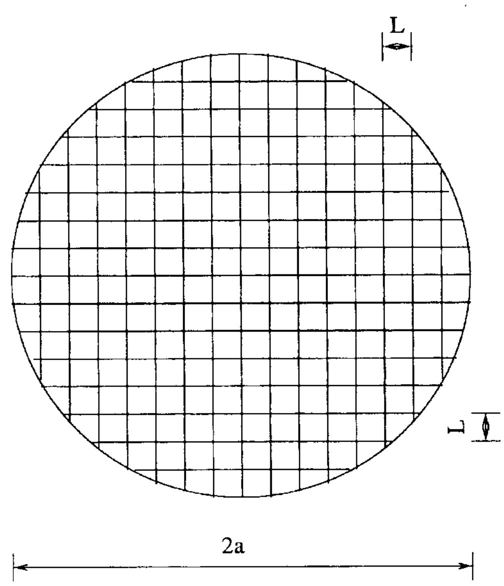


Fig. 1 The basic geometry of the reticulated circular plate

$$\frac{L}{EA}\,\phi_{,1111} + \frac{L^3}{6EI_0}\phi_{,1122} + \frac{L}{EA}\,\phi_{,2222} = (w_{,12})^2 - w_{,11}\,w_{,22} + \frac{L^2}{8EI_0}\,(GJ - EI)\,[(\nabla^2 w)\,w_{,12}]_{,12} \quad (2)$$

in which the trem $\ddot{w}$ due to the inertia force in the normal direction is included, and $m_0$ is mass density. The above two equations are anisotropic Kármán-type equations. Introducing the corresponding nondimensional variables and quantities, the above fundamental governing equations can be transformed into the following nondimensional forms for an axisymmetrical case (Nie and Cheung 1995).

$$\mathcal{L}(W) + \frac{\partial^2 W}{\partial \tau^2} = \frac{1}{\rho}\,\frac{d}{d\rho}\left(T\frac{dW}{d\rho}\right) \quad (3)$$

$$\hbar\,(\rho T) = -\frac{1}{2}\left(\frac{dW}{d\rho}\right)^2 + M_5\left[\left(\frac{d^2W}{d\rho^2}\right)^2 - \left(\frac{dW}{\rho d\rho}\right)^2\right] + \frac{M_5}{2}\,\rho\frac{d}{d\rho}\left[\left(\frac{d^2W}{d\rho^2}\right)^2 - \left(\frac{dW}{\rho d\rho}\right)^2\right] \quad (4)$$

in which

$$\rho = \frac{r}{a},\quad \tau = \sqrt{\frac{3EI + GJ}{4Lm_0a^4}}\,t$$

$$W = \frac{4w}{\sqrt{(3EI + GJ)\left(\frac{3}{EA} + \frac{L^2}{12EI_0}\right)}}$$

$$M_5 = \frac{(GJ - EI)L^2}{32EI_0a^2}$$

$$T = \frac{4La^2\rho}{3EI + GJ}N_r,\quad N_r = \frac{1}{r}\,\frac{d\phi}{dr}$$

and the expressions for differential operators $\mathcal{L}$, $\hbar$ are

$$\mathcal{L}(\cdots) = \frac{1}{\rho}\,\frac{d}{d\rho}\,\rho\frac{d}{d\rho}\,\frac{1}{\rho}\,\frac{d}{d\rho}\,\rho\frac{d}{d\rho}(\cdots)$$

$$\hbar\,(\cdots) = \rho\frac{d}{d\rho}\,\frac{1}{\rho}\,\frac{d}{d\rho}(\cdots)$$

For the case of hinged structure, boundary conditions can be expressed by (Nie and Cheung 1995).

$$W = 0,\ \frac{d^2W}{d\rho^2} + m\frac{dW}{d\rho} = 0,\ \frac{dT}{d\rho} - nT = 0,\ \ \rho = 1 \quad (5)$$

$$\frac{dW}{d\rho} = 0,\ \ T = 0,\ \ \rho = 0 \quad (6)$$

where $m = \dfrac{EI - GJ}{3EI + GJ}$, $n = \dfrac{\sqrt{2}}{4 + \sqrt{2} - \dfrac{8}{\pi}}$. It is obvious that when $n \to \infty$ in Eq. (5), then Eqs. (5) and (6) correspond to the case of movable simply-supported edge. Further, the two cases can be transformed into immovable and movable clamped edges respectively when $m \to \infty$ in Eq. (5). Hence, in the following analysis only the case of hinged edge is considered, then the corresponding results for the other three cases can also be derived.

To solve the coupled Eqs. (3) and (4), it is assumed that

$$W(\rho,\ \tau) = V(\rho)\cos\omega\tau \tag{7}$$
$$T(\rho,\ \tau) = S(\rho)\cos^2\omega\tau \tag{8}$$

in which $V(\rho)$ and $S(\rho)$ are undetermined functions. Owing to the equilibrium Eq. (3), the corresponding equation in the form of variation can be written as follows

$$\int_0^{\frac{2\pi}{\omega}} \int_0^1 \left[ \mathcal{L}(W) - \omega^2 V\cos\omega\tau - \frac{1}{\rho}\frac{d}{d\rho}\left(T\frac{dW}{d\rho}\right)\right] \delta W\, \rho d\rho d\tau = 0 \tag{9}$$

in which $\delta W = \cos\omega\tau\delta V$. Integrating with repect to $\tau$ in the above equation, the following equation concerning $V(\rho)$ can be deduced considering that $\delta V$ can take arbitrary value

$$\mathcal{L}[V(\rho)] - \omega^2 V(\rho) = \frac{3}{4\rho}\frac{d}{d\rho}\left[S(\rho)\frac{dV}{d\rho}\right] \tag{10}$$

On the other hand, inserting Eqs. (7) and (8) into Eq. (4) yields

$$\hbar(\rho S) = -\frac{1}{2}\left[\frac{dV(\rho)}{d\rho}\right]^2 + M_5\left\{\left[\frac{d^2V(\rho)}{d\rho^2}\right]^2 - \left[\frac{dV(\rho)}{\rho d\rho}\right]^2\right\} + \frac{M_5}{2}\rho\frac{d}{d\rho}\left\{\left[\frac{d^2V(\rho)}{d\rho^2}\right]^2 - \left[\frac{dV(\rho)}{\rho d\rho}\right]^2\right\} \tag{11}$$

The corresponding boundary conditions are rewritten by

$$V = 0,\ \frac{d^2V}{d\rho^2} + m\frac{dV}{d\rho} = 0,\ \frac{dS}{d\rho} - nS = 0\quad \rho = 1 \tag{12}$$

$$\frac{dV}{d\rho} = 0,\ S = 0,\quad \rho = 0 \tag{13}$$

The following analysis will focus on the solution for $V$ and $S$.

## 3. The asymtotic solution

### 3.1. The first iteration

From Eq. (10), a linear boundary-value problem only concerning $V(\rho)$ is expressed by

$$\mathcal{L}[V^{(1)}] - \omega_0^2 V^{(1)} = 0 \tag{14}$$

$$V^{(1)} = 0,\ \frac{d^2V^{(1)}}{d\rho^2} + m\frac{dV^{(1)}}{d\rho} = 0,\quad \rho = 1 \tag{15}$$

$$\frac{dV^{(1)}}{d\rho} = 0,\quad \rho = 0 \tag{16}$$

and denote

$$V^{(1)}(\rho)\,|_{\rho=0} = W_c \tag{17}$$

in which $V^{(1)}$ corresponds to the solution for the first iteration, $W_c$ is the amplitude of the structure and $\omega_0$ is the corresponding linear frequency. Meanwhile, $S^{(1)}$ directly satisfy Eq. (11) with condition Eqs. (12) and (13) related to the unknown variable $V^{(1)}$, i.e.,

$$\hbar\,[\rho S^{(1)}] = -\frac{1}{2}\left[\frac{dV^{(1)}}{d\rho}\right]^2 + M_5\left\{\left[\frac{d^2V^{(1)}}{d\rho^2}\right]^2 - \left[\frac{dV^{(1)}}{\rho d\rho}\right]^2\right\} + \frac{M_5}{2}\rho\frac{d}{d\rho}\left\{\left[\frac{d^2V^{(1)}}{d\rho^2}\right]^2 - \left[\frac{dV^{(1)}}{\rho d\rho}\right]^2\right\} \quad (18)$$

$$\frac{dS^{(1)}}{d\rho} - nS^{(1)} = 0, \quad \rho = 1 \quad (19)$$

$$S^{(1)} = 0, \quad \rho = 0 \quad (20)$$

The general solution for Eq. (14) is expressed by using two power series as follows

$$V^{(1)}(\rho) = A_1\sum_{k=0}^{\infty} b_k\rho^{4k} + A_2\sum_{k=0}^{\infty} c_k\rho^{4k+2} \quad (21)$$

where $A_1$, $A_2$ are unknown constants, the coefficients $b_k$, $c_k$ ($k$=1, 2, ...) depend on $\omega_0$ and (see also Appendix)

$$\begin{gathered} b_0 = c_0 = 1 \\ b_k = \omega_0^{2k} g_k,\ c_k = \omega_0^{2k} f_k,\ b_k = (2k+1)^2 c_k \quad (k = 1,\ 2,\ \cdots) \\ f_k = \frac{1}{16^k[(2k+1)!]^2},\ g_k = \frac{1}{16^k[2k)!]^2} \quad (k = 1,\ 2,\ \cdots) \end{gathered} \quad (22)$$

Using Eqs. (15) and (16), the following equation is obtained

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(4k-4j-2)(4k+4j+1+m)b_k c_j = 0$$

namely

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(2k+1)^2(4k-4j-2)(4k+4j+1+m)c_k c_j = 0 \quad (23)$$

It follows that the linear frequency $\omega_0$ can be obtained by solving the following non-linear equation

$$\sum_{k=0}^{\infty} F_k\,\omega_0^{2k} = 0 \quad (24)$$

in which

$$\begin{aligned} F_0 &= 1 \\ F_1 &= -6f_1H_5 \\ F_2 &= (9f_1^2 - 70f_2)H_9 \\ F_3 &= (2f_1f_2 - 238f_3)H_{13} \\ F_4 &= (-102f_1f_3 + 25f_2^2 - 558f_4)H_{17} \\ F_5 &= (26f_2f_3 - 342f_1f_4 - 1078f_5)H_{21} \\ F_6 &= (49f_3^2 - 766f_1f_5 - 118f_2f_4 - 1846f_6)H_{25} \\ &\cdots \end{aligned}$$

and

$$H_k = \frac{k+m}{1+m} \tag{25}$$

Accordingly, $V^{(1)}$ can be rewritten by

$$V^{(1)}(\rho) = W_c\left(\sum_{k=0}^{\infty} b_k\,\rho^{4k} - a_0\sum_{k=0}^{\infty} c_k\,\rho^{4k+2}\right) \tag{26}$$

in which Eq. (17) has been applied, and

$$a_0 = \frac{\sum_{k=0}^{\infty} b_k}{\sum_{k=0}^{\infty} c_k} = \frac{\sum_{k=0}^{\infty}(2k+1)^2 c_k}{\sum_{k=0}^{\infty} c_k} \tag{27}$$

Substituting Eq. (26) into Eq. (18), $S^{(1)}$ can be solved by applying the corresponding condition Eqs. (19), (20), the result is

$$\begin{aligned} S^{(1)} = \Bigg\{ & c_s^0\,\rho - \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{8kj}{[4(k+j)-2][4(k+j)]}\,b_k b_j\,\rho^{4(k+j)-1} \\ & - a_0^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(4k+2)(4j+2)}{2[4(k+j)+2][4(k+j)+4]}\,c_k\,c_j\,\rho^{4(k+j)+3} \\ & + a_0\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{4k\,(4j+2)}{[4(k+j)][4(k+j)+2]}\,b_k\,c_j\,\rho^{4(k+j)+1} \\ & + M_5\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{4k\,(4j)[(4k-1)(4j-1)-1]}{8(k+j-1)}\,b_k b_j\,\rho^{4(k+j)-3} \\ & + a_0^2 M_5\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{(4k+2)(4j+2)[(4k+1)(4j+1)-1]}{8(k+j)}\,c_k c_j\,\rho^{4(k+j)+1} \\ & + 4a_0^2 M_5\sum_{j=1}^{\infty}(2j+1)\,c_j\,\rho^{4j+1} \\ & - 2a_0 M_5\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{4k(4j+2)[(4k-1)(4j+1)-1]}{8(k+j)-4}\,b_k c_j\,\rho^{4(k+j)-1}\Bigg\} W_c^2 \end{aligned} \tag{28}$$

in which

$$\begin{aligned} c_s^0 = \frac{1}{n-1}\Bigg\{ & -\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{8kj\,[4(k+j)-1-n]}{[4(k+j)-2][4(k+j)]}\,b_k\,b_j \\ & - a_0^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(4k+2)(4j+2)[4(k+j)+3-n]}{2[4(k+j)+2][4(k+j)+4]}\,c_k\,c_j \\ & + a_0\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{4k(4j+2)[4(k+j)+1-n]}{[4(k+j)][4(k+j)+2]}\,b_k\,c_j \end{aligned}$$

$$+M_5\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{2kj[(4k-1)(4j-1)-1][4(k+j-1)+1-n]}{(k+j-1)}b_k b_j$$

$$+a_0^2M_5\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{(4k+2)(4j+2)[(4k+1)(4j+1)-1][4(k+j)+1-n]}{8(k+j)}c_k c_j$$

$$+4a_0^2 M_5\sum_{j=1}^{\infty}(2j+1)(4j+1-n)c_j$$

$$\left.-2a_0M_5\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{4k(4j+2)[(4k-1)(4j+1)-1][4(k+j)-1-n]}{8(k+j)-4}b_k c_j\right\} \tag{29}$$

### 3.2. The second iteration

Adopting the solutions resulting from the first iteration as reference variables, the corresponding differential equation for $V^{(2)}$ which is the solution for the second iteration is formulated as follows

$$\mathcal{L}[V^{(2)}(\rho)]-\omega^2V^{(2)}(\rho)=\frac{3}{4\rho}\frac{d}{d\rho}\left[S^{(1)}(\rho)\frac{dV^{(1)}}{d\rho}\right] \tag{30}$$

in which $\omega$ is the non-linear frequency. The boundary conditions are

$$V^{(2)}=0,\ \frac{d^2V^{(2)}}{d\rho^2}+m\frac{dV^{(2)}}{d\rho}=0,\ \rho=1 \tag{31}$$

$$\frac{dV^{(2)}}{d\rho}=0,\ \rho=0 \tag{32}$$

and

$$V^{(2)}(\rho)\mid_{\rho=0}=W_c \tag{33}$$

The solution of the above equation has the following form

$$V^{(2)}=A_3\sum_{k=0}^{\infty}b_k'\rho^{4k}+A_4\sum_{k=0}^{\infty}c_k'\rho^{4k+2}+V^{(2)*}$$

in which $A_3$, $A_4$ are also two unknown constants, the coefficients $b_k'$, $c_k'$ have the following expressions

$$b_0'=c_0'=1,\ b_k'=\omega^{2k}g_k,\ c_k'=\omega^{2k}f_k\quad(k=1,\ 2,\cdots)$$

and $V^{(2)*}$ is the particular solution for the Eq. (30), its expression is (see also Appendix)

$$V^{(2)*}=-\frac{1}{\omega^2}V^{(2)**}$$

$$V^{(2)**}=\frac{3}{4}[-4c_s^0a_0+(16c_s^0b_1+2a_0^3+64M_5a_0^2b_1)\rho^2]W_c^3$$

$$+\frac{3}{4}\left\{c_s^0[\sum_{k=1}^{\infty}16(k+1)^2b_{k+1}C(k)-a_0\sum_{k=1}^{\infty}(4k+2)^2c_kB(k)]\right.$$

$$
\begin{aligned}
& -\sum_{i,j,k=1}^{\infty} \frac{64ijk\,[2(i+j+k)-1]}{[4(j+k)-2][4(j+k)]}\, b_i b_j b_k B(i+j+k-1) \\
& +a_0 \sum_{i,j,k=1}^{\infty} \frac{2k(i+j+k)}{j+k}\left[\frac{4j(4i+2)}{4(j+k)-2} c_i b_j + \frac{8i(4j+2)}{4(j+k)+2} b_i c_j\right] b_k C(i+j+k-1) \\
& -a_0^2 \sum_{i,j,k=1}^{\infty} \frac{(2j+1)[4(i+j+k)+2]}{4(j+k)+2}\left[\frac{i(4k+2)}{j+k+1} b_i + \frac{2k(4i+2)}{j+k} c_i\right] c_j c_k B(i+j+k) \\
& +a_0^3 \sum_{i,j,k=1}^{\infty} \frac{2(4i+2)(4j+2)(4k+2)(i+j+k+1)}{[4(j+k)+2][4(j+k)+4]} c_i c_j c_k C(i+j+k) \\
& +a_0 \sum_{i,j=1}^{\infty} 128ij(i+j)\left[\frac{1}{8[4(i+j)-2](i+j)} + \frac{1}{4j(4j+2)}\right] b_i b_j C(i+j-1) \\
& -a_0^2 \sum_{i,j=1}^{\infty} 8i(4j+2)[4(i+j)+2]\left[\frac{1}{(4j+2)(4j+4)} + \frac{1}{4i(4i+2)} + \frac{1}{4(i+j)[4(i+j)+2]}\right] b_i c_j B(i+j) \\
& +a_0^3 \sum_{i,j=1}^{\infty} (4i+2)(4j+2)[4(i+j)+4]\left[\frac{1}{[4(i+j)+2][4(i+j)+4]}\right. \\
& \left. + \frac{2}{(4j+2)(4j+4)}\right] c_i c_j C(i+j) - a_0^2 \sum_{i=1}^{\infty} 8i(4i+2)\left[\frac{1}{8} + \frac{2}{4i(4i+2)}\right] b_i B(i) \\
& \left. +a_0^3 \sum_{i=1}^{\infty} 2(4i+2)(4i+4)\left[\frac{1}{8} + \frac{2}{(4i+2)(4i+4)}\right] c_i C(i) \right\} W_c^3 \\
& +\frac{3}{4} M_5 \left\{ \sum_{i,j,k=1}^{\infty} \frac{32ijk[(4j-1)(4k-1)-1](i+j+k-1)}{j+k-1} b_i b_j b_k C(i+j+k-2) \right. \\
& -a_0 \sum_{i,j,k=1}^{\infty} 16(4i+2)jk[4(i+j+k)-2]\left[\frac{(4j-1)(4k-1)-1}{8(j+k-1)}\right. \\
& \left. + \frac{(4i+1)(4k-1)-1}{4(i+k)-2}\right] c_i b_j b_k B(i+j+k-1) + a_0^2 \sum_{i,j,k=1}^{\infty} 16i(4j+2)(4k+2)(i+j+k)\,. \\
& \left[\frac{(4j+1)(4k+1)-1}{8(j+k)} + \frac{(4j+1)(4i-1)-1}{4(j+i)-2}\right] b_i c_j c_k C(i+j+k-1) \\
& -a_0^3 \sum_{i,j,k=1}^{\infty} (4i+2)(4j+2)(4k+2)[(4j+1)(4k+1)-1] \\
& \frac{2(i+j+k)+1}{4(j+k)} c_i c_j c_k B(i+j+k) \\
& -a_0 \sum_{i,j=1}^{\infty} 32ij[4(i+j)-2]\left[\frac{(4i-1)(4j-1)-1}{8(i+j-1)} + 1\right] b_i b_j B(i+j-1) \\
& +2a_0^2 \sum_{i,j=1}^{\infty} 16(4i+2)j(i+j)\left[2 + \frac{(4i+1)(4j-1)-1}{4(i+j)-2}\right] c_i b_j C(i+j-1)
\end{aligned}
$$

$$-a_0^3\sum_{i,j=1}^{\infty}(4i+2)(4j+2)[4(i+j)+2]\left[\frac{(4i+1)(4j+1)-1}{4(i+j)}+2\right]c_i c_j B(i+j)$$

$$\left.+2a_0^2\sum_{i=1}^{\infty}32(i+1)^2 b_{i+1}C(i)-2a_0^3\sum_{i=1}^{\infty}2(4i+2)^2 c_i B(i)\right\} W_c^3 \tag{34}$$

where

$$B(k)=\frac{1}{b'_k}\sum_{l=0}^{k}b'_l\rho^{4l},\ C(k)=\frac{1}{c'_k}\sum_{l=0}^{k}c'_l\rho^{4l+2}$$

and simplified summation notations

$$\sum_{i,j=1}^{\infty}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty},\ \sum_{i,j,k=1}^{\infty}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}$$

Utilizing the condition Eqs. (31)~(33), a nonlinear equation for $\omega$ is resulted in

$$\sum_{k=0}^{\infty}(\alpha_{1k}+\alpha_{2k}W_c^2)\omega^{2k}=0 \tag{35}$$

in which

$$\alpha_{10}=0$$
$$\alpha_{11}=2g_1$$
$$\alpha_{12}=-12H_5f_1g_1$$
$$\alpha_{13}=18H_9f_1^2g_1-140H_9f_2g_1$$
$$\alpha_{20}=-12H_5f_1d_1+2d_3g_1-2e_2g_1-6H_5f_1d_1-6H_5d_2g_1+e_1g_1+2d_2g_1$$
$$\alpha_{21}=18H_9f_1^2d_1-140H_9f_2d_1-12H_5d_3f_1g_1-6H_5e_2f_1g_1+e_1f_1g_1$$
$$a_{22}=18H_9d_3f_1^2g_1-140H_9d_3f_2g_1$$
$$a_{23}=0$$

……

$$d_1=\frac{3}{4}(-36c_s^0a_0c_1-10a_0^2b_1-384M_5a_0b_1^2-144M_5a_0^3c_1)$$
$$d_2=\frac{3}{4}(64c_s^0b_2+\frac{40}{3}a_0b_1^2+16a_0^3c_1+512M_5b_1^3+1664M_5a_0^2b_1c_1+256a_0^2b_2)$$
$$d_3=-3c_s^0a_0$$
$$d_4=\frac{3}{4}(16c_s^0b_1+2a_0^3+64M_5a_0^2b_1)$$
$$e_1=4H_3d_1+6H_5d_2+2d_4$$
$$e_2=d_1+d_2+d_3+d_4$$

Solving the above non-linear equation, the non-linear frequency $\omega$ can be determined for a given value of $W_c$.

## 4. Numerical examples and analyses

Here two reticulated circular plates are considered, i.e., the radius of reticulated circular plate $a$= 1m, each beam member has the same circular cross-section with 5.8 mm diameter, its length $L$= 0.1 m and $L$=0.2 m respectively. The Poisson's ratio $\nu$=0.3.

For all cases of (immovable or movable) simply-supported and clamped edge conditions, from Eqs. (24) and (27) the exact values of the nondimensional linear frequency $\omega_0$ and constant $a_0$ can be solved and computed. It is observed that the two quantities do not depend on the geometrical sizes of the reticulated plate, and are decided by the material constant $m = \dfrac{EI - GJ}{3EI + GJ}$ only for the former case. The values of coefficients $b_k$, $c_k$ (here $k$ is taken up to 6) are listed in Table 1. It can be noted that the coefficients rapidly decrease with the increase of value of $k$, this leads to fast convergence for all undetermined variables and quantities expressed in series.

When $L$=0.2 m, the change of nondimensional non-linear frequencies with the amplitude is displayed in Fig. 2. It is obvious that the linear frequency corresponding to $W_c$=0 for the clamped plate is larger than that for the simply-supported plate, and the non-linear frequency increases with the amplitude, i.e., there is always hardening non-linear behaviour for the plates. This is similar to nonlinearity of solid plate (Chia 1980). Further, immovable edge condition can result in a faster increase in non-linear frequency for the simply-supported or clamped structure. The characteristic relation between nondimensional ratio of non-linear frequency to linear frequency, $\omega/\omega_0$, and the amplitude $W_c$ for $L$=0.1 m is illustrated in Fig. 3. Similarly, for immovable simply-supported or clamped plate, the corresponding hardening non-linear behaviour is stronger, and especially for the hinged reticulated circular plate, the nonlinearity is strongest.

## 5. Conclusions

In this paper an analytical solution of non-linear natural frequency of reticulated circular plate is given with the aid of an asymptotic iteration method. The application of power series with rapid convergence results in good accuracy for the solution of this problem.

Table 1 Coefficients $b_k$ and $c_k$ ($k$=1-6)

| | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
|---|---|---|---|---|
| Hinged | 0.3239 | $2.9137\times10^{-3}$ | $4.1940\times10^{-6}$ | $1.7325\times10^{-9}$ |
| Clamped | 1.6336 | $7.4124\times10^{-2}$ | $5.3816\times10^{-4}$ | $1.1213\times10^{-6}$ |

| | $b_5$ | $b_6$ | $c_1$ | $c_2$ |
|---|---|---|---|---|
| Hinged | $2.7710\times10^{-13}$ | $2.0602\times10^{-17}$ | $3.5986\times10^{-2}$ | $1.1655\times10^{-4}$ |
| Clamped | $9.0455\times10^{-10}$ | $3.3922\times10^{-13}$ | $1.8151\times10^{-1}$ | $2.9650\times10^{-3}$ |

| | $c_3$ | $c_4$ | $c_5$ | $c_6$ |
|---|---|---|---|---|
| Hinged | $8.5592\times10^{-8}$ | $2.1389\times10^{-11}$ | $2.2901\times10^{-15}$ | $1.2191\times10^{-19}$ |
| Clamped | $1.0983\times10^{-5}$ | $1.3843\times10^{-8}$ | $7.4756\times10^{-12}$ | $2.0072\times10^{-15}$ |

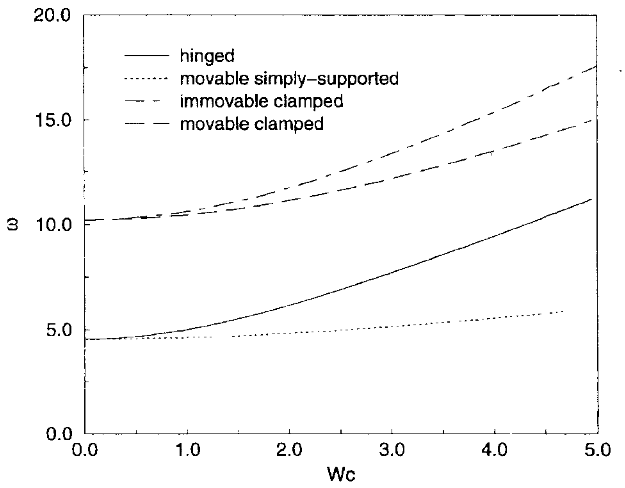


Fig. 2 The change of non-linear frequency with the amplitude of the structure

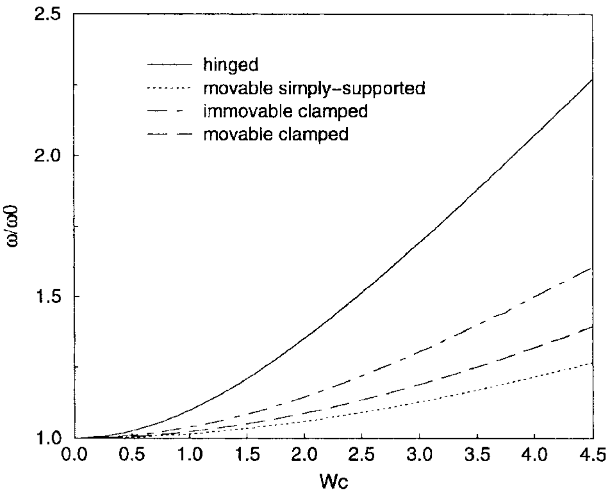


Fig. 3 The characteristic relation between the ratio of non-linear frequency to linear frequency and the amplitude

The computation results show that non-linear vibrations of reticulated circular plates have hardening nonlinearity behaviours. The nonlinearity for the hinged reticulated circular plate is strongest for the four cases of edge conditions.

It can be also concluded that the proposed and used method in this paper can be extended to non-linear free vibration analysis of reticulated shallow spherical shells.

## Acknowledgements

The work is carried out when the author works at the institute for statics and dynamics of the Ruhr-University Bochum, Germany as research fellow under Alexander von Humboldt (AvH) Foundation. The author gratefully acknowledges the kind help from Prof. W.B. Krätzig and the support by the AvH Foundation. The author is also very grateful to the anonymous reviewers for their helpful suggestions which lead to the revisions of the original manuscript.

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## Appendix

The Particular Solution for the Equation $\mathcal{L}[V(\rho)]-\alpha V(\rho)=\rho^m$

(a) Two Independent Solutions for the Equation $\mathcal{L}[V(\rho)]-\alpha V(\rho)=0$

Let infinite power series $\sum_{k=0}^{\infty}\beta_k\rho^k$ be the form of solution for the following equation

$$\mathcal{L}[V(\rho)]-\alpha V(\rho)=0$$

in which

$$\mathcal{L}(\cdots)=\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}(\cdots)$$

Substituting it into the equation, a recursion formula can be obtained as follows

$$\beta_k = \frac{\alpha}{k^2(k-2)^2}\beta_{k-4}$$

Note that $\mathcal{L}(\beta_0)=0$ and $\mathcal{L}(\beta_2\,\rho^2)=0$, – two independent series solutions can be expressed respectively in the following

$$V(\rho) = \sum_{k=0}^{\infty} b_k \rho^{4k}$$

and

$$V(\rho) = \sum_{k=0}^{\infty} c_k \rho^{4k+2}$$

where

$$b_0 = c_0 = 1$$

$$b_k = \beta_{4k} = \alpha^k g_k,\ c_k = \beta_{4k+2} = \alpha^k f_k \quad (k=1,\ 2,\ \cdots)$$

$$b_k = (2k+1)^2 c_k \quad (k=1,\ 2,\ \cdots)$$

$$f_k = \frac{1}{16^k[(2k+1)!]^2},\ g_k = \frac{1}{16^k[(2k)!]^2} \quad (k=1,\ 2,\ \cdots)$$

(b) The Particular Solution for the Equation $\mathcal{L}[V(\rho)]-\alpha V(\rho)=\rho^m$

Case(i): $m=0$

The particular solution $V^*(\rho)$ is

$$V^*(\rho) = -\frac{1}{\alpha}$$

Case(ii): $m=2$

$$V^*(\rho) = -\frac{1}{\alpha}\rho^2$$

Case(iii): $m=4n$ ($n=1,\ 2,\ \cdots$)

$$V^*(\rho) = -\frac{1}{\alpha b_n}\sum_{k=0}^{n} b_k \rho^{4k}$$

Case(iv): $m=4n+2$ ($n=1,\ 2,\ \cdots$)

$$V^*(\rho) = -\frac{1}{\alpha c_n}\sum_{k=0}^{n} c_k \rho^{4k+2}$$

## Notations

| | |
|---|---|
| $L$ | = length of each beam member |
| $A$ | = area of cross-section of each beam member |
| $a$ | = radius of the plate |
| $m_0$ | = mass density |
| $EI$ | = transverse bending stiffness |
| $EI_0$ | = lateral bending stiffness |
| $GJ$ | = twisting stiffness |
| $t$ | = time |

| | |
|---|---|
| $\tau$ | = nondimensional time $\sqrt{\dfrac{3EI+GJ}{4Lm_0a^4}}\,t$ |
| $w$ | = transverse (joint) displacement (deflection) |
| $W$ | = nondimensional transverse (joint) displacement (deflection) |
| $W_c$ | = nondimensional amplitude of the plate |
| $\omega_0$ | = nondimensional linear frequency |
| $\omega$ | = nondimensional non-linear frequency |
| $b_k$, $c_k$ | = coefficients in power series |