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Evaluation of structural dynamic responses by stochastic finite element method

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Abstract. The uncertainties associated with structural parameters and dynamic loading are identified and discussed. Structural parametric uncertainties are treated as random variables and dynamic wind load is simulated as a random process. Dynamic wind-induced responses of structures with parametric uncertainties are investigated by using stochastic finite element method. The formulas for structural dynamic reliability analysis considering the randomness of structural resistance and loading are proposed. Two numerical examples of high-rise structures are presented to illustrate the proposed methodology. The calculated results demonstrate that the variation in structural parameters indeed influences the dynamic response and the first passage probability evaluation of structures.

Key words: dynamic response; finite element method; uncertainty; dynamic reliability.

1. Introduction

Most structures have complex geometrical and material properties and are subjected to complex stochastic environment conditions. The uncertainties in the properties of material, structural damping, geometric parameters and boundary conditions etc. may induce statistical variation in the eigenvalues and eigenvectors and consequently the dynamic response may be affected. Therefore, a realistic analysis and design of structural systems with parametric uncertainties and subjected to stochastic dynamic excitations should take into account for the uncertainties arising from both structural properties and dynamic excitation simultaneously in a consistent and rational manner. However, the uncertainties associated with structural parameters are not usually considered in evaluation of random dynamic response of structures. More work is thus required to study dynamic response of structures with uncertain parameters.

As the name suggests, the stochastic finite element method (SFEM) combines the best features of the finite element methods and the stochastic analysis. Stochastic finite element method has recently become an active area of research. However, it is worth noting that the stochastic finite element method has been mainly applied in structural static analysis and eigenproblems over the last decade (e.g., Spanos and Ghanem 1989, Vanmarcke and Grigoriu 1983). The evaluation of dynamic reliability of structures with parametric uncertainties subjected stochastic dynamic loads

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by SFEM has received relatively little, if any, attention in the literature in the past.

As discussed above, reliable structural design requires correct modelling of the structural parametric uncertainties and considering these uncertainties in the structural analysis. In this paper, the stochastic finite element method is applied for response analysis of structures under stochastic dynamic load actions.

The objective of this paper is to investigate wind-induced vibrations of structures with parametric uncertainties. A probability description of structural response is presented utilising the stochastic finite element method. A reliability analysis procedure is proposed in terms of upcrossing probabilities of wind-induced response. The structural lifetime reliability can be obtained from the conditional reliability through convolution with the probability density function of lifetime extreme wind speed. The probability that a particular response component of a structure will be exceeded in a specified time period can be predicted. In this manner the inherent random nature of the load, structural resistance, and the uncertainties in the description of the wind speed are accounted for. Two numerical examples are presented to illustrate the proposed methodology and the effect of the uncertainties on structural response and dynamic reliability.

2. Dynamic response of structures with parametric uncertainties

In this paper, the uncertainties associated with structural parameters are treated as random variables and dynamic wind load is simulated as a random process, and wind-induced vibrations of structures are evaluated by the stochastic finite element method. According to the probability theory, a stochastic vector $\{Y\}$ can be expressed as

$$\{Y\} = \{\bar{Y}\} + \{\alpha\} \tag{1}$$

in which $\{\bar{Y}\}=E[\{Y\}]$ is the mean of the random vector $\{Y\}$, $\{\alpha\}$ is a random vector with zero mean.

The stochastic finite element method based on the second order perturbation method has shown its accuracy and efficiency (Kareem and Sun 1990, Kleiber and Hien 1992, Li *et al.* 1993a). According to this method, a random variable or process, a random vector or field, Z, can be expressed by the second order Taylor's series expansion at mean value of α as follows:

$$Z = \bar{Z} + \sum_{i=1}^{N} Z_i^{(1)} \alpha_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_{ij}^{(2)} \alpha_i \alpha_j \tag{2}$$

where $\bar{Z}$ represents the mean value of Z, the superscripts (1) and (2) denote the first and second derivatives of Z with respect to α, respectively, and N is the total number of the random variables considered.

The vibration equation of a multi-degree of freedom system is

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{F(t)\} \tag{3}$$

where $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrix, respectively. $\{F(t)\}$ is the vector of random dynamic excitations.

In the following analysis, the structural stiffness matrix $[K]$ and random displacement vector $\{X\}$ are represented by Eq. (2). If these expressions are introduced into the equation of motion (Eq. 3), we obtain the following zeroth-, first- and second-order equations for the dynamic response of the structural system.

Zeroth-order

$$[M]\{\ddot{\bar{X}}\} + [C]\{\dot{\bar{X}}\} + [\bar{K}]\{\bar{X}\} = \{F(t)\} \tag{4}$$

First-order

$$[M]\{\ddot{X}_i^{(1)}\} + [C]\{\dot{X}_i^{(1)}\} + [\bar{K}]\{X_i^{(1)}\} = -[K_i^{(1)}]\{\bar{X}\} \tag{5}$$

Second-order

$$[M]\{\ddot{X}_{ij}^{(2)}\} + [C]\{\dot{X}_{ij}^{(2)}\} + [\bar{K}]\{X_{ij}^{(2)}\} = -[K_{ij}^{(2)}]\{\bar{X}\} - [K_i^{(1)}]\{X_j^{(1)}\} - [K_j^{(1)}]\{X_i^{(1)}\}$$
$$(i, j = 1, 2, ..., N) \tag{6}$$

Because the dynamic load vector $\{F(t)\}$ is a random process vector, the response vectors, $\{\bar{X}\}$, $\{X_i^{(1)}\}$ and $\{X_{ij}^{(2)}\}$ are random process vectors, too. In this paper, the stochastic behaviours of structural stiffness matrix is propagated by means of the stochastic finite element method to demonstrate the methodology, which can be further refined to consider the randomness of structural damping and mass etc., if so desired.

Eqs. (4), (5) and (6) may be solved by the mode superposition method. Let

$$\{X\} = [\phi]\{y\} \tag{7}$$

where $[\phi]$ is the mode shape matrix, $\{y\}$ is the generalised co-ordinate vector which is a process vector and can be also represented by Eq. (2).
Thus, we have

$$\{\bar{X}\} = [\phi]\{\bar{y}\} \tag{8}$$

$$\{X_i^{(1)}\} = [\phi]\{y_i^{(1)}\} \tag{9}$$

$$\{X_{ij}^{(2)}\} = [\phi]\{y_{ij}^{(2)}\} \tag{10}$$

It is assumed that $[C]$ is a uncoupled damping matrix, then,

$$[\phi]^T[M][\phi] = [I] \tag{11}$$

$$[\phi]^T[C][\phi] = [\mathrm{diag}(2\xi_j\omega_j)] = [C^*] \tag{12}$$

$$[\phi]^T[\bar{K}][\phi] = [K^*] = [\mathrm{diag}(\omega_j^2)] \tag{13}$$

in which $[I]$ is a identity matrix.
Then substituting Eqs. (8)-(10) into Eqs. (4)-(6) and using left-handed multiplication by $[\phi]^T$ yield

$$\{\ddot{\bar{y}}\} + [C^*]\{\dot{\bar{y}}\} + [K^*]\{\bar{y}\} = \{f(t)\} \tag{14}$$

$$\{\ddot{y}_i^{(1)}\} + [C^*]\{\dot{y}_i^{(1)}\} + [K^*]\{y_i^{(1)}\} = \{f_{1i}\} \tag{15}$$

$$\{\ddot{y}_{ij}^{(2)}\} + [C^*]\{\dot{y}_{ij}^{(2)}\} + [K^*]\{y_{ij}^{(2)}\} = \{f_{2ij}\} \tag{16}$$

in which

$$\{f(t)\} = [\phi]^T\{F(t)\} \tag{17}$$

$$\{f_{1i}\} = -[\phi]^T[K_i^{(1)}][\bar{X}] \tag{18}$$

$$\{f_{2ij}\} = -[\phi]^T([K_{ij}^{(2)}]\{\bar{X}\} - [K_i^{(1)}]\{X_j^{(1)}\} - [K_j^{(1)}]\{X_i^{(1)}\}) \tag{19}$$

The covariance matrix of displacement response (only considering the first two orders) can be expressed as

$$R_x = R_{\bar{x}} + \sum_{i=1}^{N}\sum_{j=1}^{N} E(\alpha_i\alpha_j) R_{x^{(1)}}(X_i^{(1)}, X_j^{(1)}) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} E(\alpha_i\alpha_j)[R_{\bar{x},x^{(2)}}(\bar{X}, X_{ij}^{(2)}) + R_{x^{(2)},\bar{x}}(X_{ij}^{(2)}, \bar{X})] \tag{20}$$

in which

$$R_{\bar{x}} = E[\{\bar{X}\}\{\bar{X}\}^T] \tag{21}$$

$$R_{x^{(1)}}(X_i^{(1)}, X_j^{(1)}) = E(\{X_i^{(1)}\}\{X_j^{(1)}\}^T] \tag{22}$$

$$R_{\bar{x},x^{(2)}}(\bar{X}, X_{ij}^{(2)}) = E[\{\bar{X}\}\{X_{ij}^{(2)}\}^T] \tag{23}$$

$$R_{x^{(2)},\bar{x}}(X_{ij}^{(2)}, \bar{X}) = E[\{X_{ij}^{(2)}\}\{\bar{X}\}^T] \tag{24}$$

If spectral analysis in the frequency domain is applied, Eq. (20) can be rewritten as

$$\begin{aligned}[S_x(\omega)] = [S_{\bar{x}}(\omega)] + \sum_{i=1}^{N}\sum_{j=1}^{N} E(\alpha_i\alpha_j)[S_{x^{(1)}}(X_i^{(1)}, X_j^{(1)}, \omega)] + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} E(\alpha_i\alpha_j)\{[S_{\bar{x},x^{(2)}}(\bar{X}, X_{ij}^{(2)}, \omega)] \\ + [S_{x^{(2)},\bar{x}}(X_{ij}^{(2)}, \bar{X}, \omega)]\}\end{aligned} \tag{25}$$

Using the matrix form, the spectral density of $\bar{y}$ can be expressed as

$$[S_{\bar{y}}(\omega)] = [H^*(\omega)][S_f(\omega)][H(\omega)] \tag{26}$$

in which $H_j(\omega)$ is the mechanical admittance function, it can be determined by use of Eq. (14),

$$[H(\omega)] = \mathrm{diag}\{H_j(\omega)\}] \tag{27}$$

Similarly, it can be derived from Eq. (17) that

$$[S_f(\omega)] = [\phi]^T[S_F(\omega)][\phi] \tag{28}$$

From Eq. (8), we can obtain

$$[S_{\bar{X}}(\omega)] = [\phi][S_{\bar{y}}(\omega)][\phi]^T = [\phi][H^*(\omega)][\phi]^T[S_F(\omega)][\phi][H(\omega)][\phi]^T \tag{29}$$

The second term, $S_{X^{(1)}}(X_i^{(1)}, X_j^{(1)}, \omega)$ in Eq. (25) can be similarly derived from Eq. (18) as

$$\begin{aligned}[S_{X^{(1)}}(X_i^{(1)}, X_j^{(1)}, \omega)] &= [\phi][S_{y^{(1)}}(y_i^{(1)}, y_j^{(1)}, \omega)][\phi]^T = [\phi][H^*(\omega)][S_{f_1}(f_{1i}, f_{1j}, \omega)][H(\omega)][\phi]^T \\ &= [\phi][H^*(\omega)][\phi]^T[K_i^{(1)}][S_{\bar{X}}(\omega)][K_j^{(1)}]^T[\phi][H(\omega)][\phi]^T\end{aligned} \tag{30}$$

Similarly, $S_{\bar{X},X^{(2)}}(\bar{X}, X_{ij}^{(2)}, \omega)$ can be also derived as

$$[S_{\bar{X},X^{(2)}}(\bar{X}, X_{ij}^{(2)}, \omega)] = [\phi][S_{\bar{y},y^{(2)}}(\bar{y}, y_{ij}^{(2)}, \omega)][\phi]^T = [\phi][H^*(\omega)][S_{f,f_2}(f, f_{2ij}, \omega)][H(\omega)][\phi]^T \tag{31}$$

Using Eqs. (17) and (19) leads to,

$$[S_{f,f_2}(f, f_{2ij}, \omega)] = -[S_{f,\bar{x}}(\omega)][K_{ij}^{(2)}]^T[\phi] - [S_{f,X^{(1)}}(f, X_j^{(1)}, (\omega)][K_i^{(1)}]^T[\phi]$$

$$-[S_{f,X^{(1)}}(f,X_i^{(1)},\omega)][K_j^{(1)}]^T[\phi] \tag{32}$$

in which

$$[S_{f,\bar{X}}(\omega)]=[S_{f,\bar{y}}(\omega)][\phi]^T=[S_f(\omega)][H(\omega)][\phi]^T=[\phi]^T[S_F(\omega)][\phi][H(\omega)][\phi]^T \tag{33}$$

$$\begin{aligned}[S_{f,X^{(1)}}(f,X_j^{(1)},\omega)] &= [S_{f,y^{(1)}}(f,X_j^{(1)},\omega)][\phi]^T=[S_{f,f_{1i}}(f,f_{1i},\omega)][H(\omega)][\phi]^T\\ &=-[S_{f,\bar{X}}(\omega)][K_i^{(1)}]^T[\phi][H(\omega)[\phi]^T=-[\phi]^T[S_F(\omega)][\phi][H(\omega)][\phi]^T[K_i^{(1)}]^T[\phi][H(\omega)][\phi]^T\end{aligned} \tag{34}$$

The characteristic equation is given by

$$([K]-\lambda[M])\{\phi\}=0 \tag{35}$$

Solving the characteristic equation yields the natural frequencies and the corresponding mode shapes. As discussed above, the stiffness of structures is represented by a random variable field in this paper, the natural frequencies and mode shapes of the structures are also random variables. Let

$$\lambda=\bar{\lambda}+\sum_{i=1}^{N}\lambda_i^{(1)}\alpha_i+\frac{1}{2}\sum_{i=1}^{N}\sum_{i=1}^{N}\lambda_{ij}^{(2)}\alpha_i\alpha_j \tag{36}$$

and

$$\{\phi\}=\{\bar{\phi}\}+\sum_{i=1}^{N}\{\phi\}_i^{(1)}\alpha_i+\frac{1}{2}\sum_{i=1}^{N}\sum_{i=1}^{N}\{\phi\}_{ij}^{(2)}\alpha_i\alpha_j \tag{37}$$

Substituting Eq. (36) and Eq. (37) into Eq. (35) leads to,

$$\begin{aligned}&([\bar{K}]+\sum_{i=1}^{N}[K_i^{(1)}]\alpha_i+\frac{1}{2}\sum_{i=1}^{N}\sum_{i=1}^{N}[K_{ij}^{(2)}]\alpha_i\alpha_j-\bar{\lambda}[M]-\sum_{i=1}^{N}[M]\alpha_i\lambda_i^{(1)}-\\ &\frac{1}{2}\sum_{i=1}^{N}\sum_{i=1}^{N}[M]\lambda_{ij}^{(2)}\alpha_i\alpha_j)(\{\bar{\phi}\}+\sum_{i=1}^{N}\{\phi_i^{(1)}\}\alpha_i+\frac{1}{2}\sum_{i=1}^{N}\sum_{i=1}^{N}\{\phi_{ij}^{(2)}\}\alpha_i\alpha_j)=\{0\}\end{aligned} \tag{38}$$

The above equation can be rewritten as

$$([\bar{K}]-\bar{\lambda}[M])\{\bar{\phi}\}=0 \tag{39}$$

$$([\bar{K}]-\bar{\lambda}[M])\{\phi_i^{(1)}\}=-([K_i^{(1)}]-\lambda_i^{(1)}[M])\{\bar{\phi}\} \tag{40}$$

$$\begin{aligned}([\bar{K}]-\bar{\lambda}[M])\{\phi_{ij}^{(2)}\} &= -([K_{ij}^{(2)}]-\lambda_{ij}^{(2)}[M])\{\bar{\phi}\}-([K_i^{(1)}]-\lambda_i^{(1)}[M])\{\phi_j^{(1)}\}\\ &\quad-([K_j^{(1)}]-\lambda_j^{(1)}[M])\{\phi_i^{(1)}\}\end{aligned} \tag{41}$$

Because Eq. (39) is a deterministic equation, the eigenvalue $\bar{\lambda}_i$ and the corresponding eigenvector $\{\bar{\phi}\}_i$ can be determined directly by using conventional eigensolution procedure (e.g., Wang 1978, Li *et al.* 1994, 1996). The solutions of Eq. (40) and Eq. (41) can be found in the Appendix of this paper.

In the following analysis, for the sake of illustration, the formulation is restricted to dynamic wind loading.

If the dynamic load is wind action, and the fluctuating drag is

$$\{F(t)\}=[B]\{p(t)\} \tag{42}$$

where $[B]=[\mathrm{diag}(C_D A_i)]$ is the product of the drag coefficient of the structure, C_D, and windward area of the structure A_i. $\{p(t)\}$ can be expressed as follows

$$\{p(t)\} = \rho[\bar{V}]\{v(t)\} \tag{43}$$

where $[\bar{V}] = [\mathrm{diag}(\bar{V}_i)]$, $\bar{V}_i$ and $v_i(t)$ are the mean wind velocity and the fluctuating components of wind velocity, respectively, at the lumped mass point i. ρ is density of air.

The following equation can be derived from Eqs. (42) and (43)

$$[S_F(\omega)] = \rho^2[B][\bar{V}][S_v(\omega)][\bar{V}]^T[B]^T \tag{44}$$

It is assumed that each term of the matrix, $[S_v(\omega)]$, can be expressed as

$$S_{v_i,v_j}(\omega) = \rho(z_i, z_j, \omega)\, S_v(\omega) \tag{45}$$

where $s_v(\omega)$ is the gust spectrum, and $\rho(z_i, z_j, \omega)$ is the coherence of gust, as suggested by Davenport (1962); it can be taken as,

$$\rho(z_i, z_j, \omega) = \exp\left[-\frac{4\omega|z_i - z_j|}{\pi\bar{V}}\right] \tag{46}$$

In this paper, the Davenport spectrum of wind speed is adopted,

$$s_v(\omega) = \frac{4K\bar{V}_{10}^2}{\omega}\frac{x^2}{(1+x^2)^{4/3}} \tag{47}$$

in which

$$x = \frac{600\omega}{\pi\bar{V}_{10}} \tag{48}$$

where K is the coefficient of ground roughness, $\bar{V}_{10}$ is the mean wind speed at 10m height.

The variances of displacement and velocity responses can be determined by the following equations

$$\sigma_X^2 = \int_0^\infty S_X(\omega)\, d\omega \tag{49}$$

$$\sigma_{\dot{X}}^2 = \int_0^\infty \omega^2 S_X(\omega)\, d\omega \tag{50}$$

3. Dynamic reliability analysis of structures

The structural dynamic reliability is the probability that the structure under the action of random dynamic loads will fulfil its design purpose during a specified period.

If it is assumed that the lifetime of a structure is n years, in general, n is taken as 50, and the probability density function of maximum wind velocity in the n years is $f(\bar{v})$ which can be derived from a statistical analysis of successive years of climatological data, then, the reliability of the structure in its lifetime can be expressed as

$$P_s = \int_0^\infty P(S \le R \mid \bar{V} = \bar{v})\, f(\bar{v})\, d\bar{v} \tag{51}$$

in which S is the structural response quality of interest, R is the corresponding structural resistance. $P(S \le R \mid \bar{V} = \bar{v})$ is called the conditional reliability of the structure given $\bar{V} = \bar{v}$.

Eq. (51) can be written in a discrete form so that it is convenient to be calculated.

$$P_s = \sum_k P(S \le R \mid \bar{V} = \bar{v})[F(\bar{v}_k) - F(\bar{v}_{k-1})] \tag{52}$$

where $F(\bar{v})$ is probability distribution function of the maximum wind speed in the n years.

Analysis of data at various locations with well-behaved wind climates has suggested that the extreme value Type I distribution in general provides a good fit to the extreme yearly wind speed (Simiu 1976, Li 1986, 1988, 1990). A maximum probability plot correlation coefficient criterion has been employed in a study (Kareem and Hseih 1986) for modelling annual extreme winds and also confirm Simiu's conclusion.

Assuming that yearly maximum wind speeds over a period of n years are independent, the PDF of the n years extreme wind speed can be expressed as

$$f(\bar{v}) = na \exp\{-(\bar{v} - a)b - n \exp[-(\bar{v} - a)b]\} \tag{53}$$

in which parameters a and b can be determined from a large sample of annual extreme mean wind speed data by the method of moments.

A typical upcrossing problem for dynamic reliability is illustrated in Fig. 1. If the deterioration of structural resistance with time is considered, then, the reliability bound is a function of time t. The upcrossing rate per unit time, $v(t)$ can be expressed as

$$v(t) = \int_0^\infty v_r(t)\, f_R(r)\, dr \tag{54}$$

where (Rice 1944)

$$v_r(t) = \int_r^\infty (\dot{s} - \dot{r})\, f_{s,\dot{s}}(r, \dot{s})\, d\dot{s} \tag{55}$$

in which $v_r(t)$ is the upcrossing rate per unit time for the barrier R=r with slope $\dot{r}$, $f_R(r)$ is the

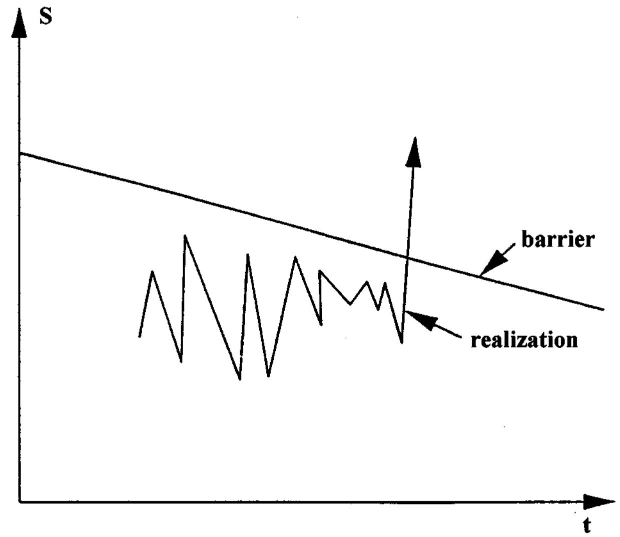

Fig. 1 Typical upcrossing problem for dynamic reliability analysis

probability density function of structural resistance. $\dot{s}$ is a state variable of $\dot{S}(t)$ which is the derivative process of $S(t)$ with respect to t.

If it is assumed that $S(t)$ is a Normal stationary process with zero mean, then, $S(t)$ and $\dot{S}(t)$ are independent Normal stationary process and can be estimated by the procedure presented in the preceding section, the joint probability density function of $S(t)$ and $\dot{S}(t)$ is

$$f_{s,\dot{s}}(s,\dot{s}) = \frac{1}{2\pi\sigma_s\sigma_{\dot{s}}} \exp\left[-\frac{1}{2}\left(\frac{s^2}{\sigma_s^2} + \frac{\dot{s}^2}{\sigma_{\dot{s}}^2}\right)\right] \tag{56}$$

Substituting Eq. (56) into Eq. (55) gives

$$v_r(t) = \frac{A}{2\pi}\frac{\sigma_{\dot{s}}}{\sigma_s} \exp\left(-\frac{r^2}{2\sigma_s^2}\right) \tag{57}$$

in which

$$A = \exp\left(\frac{-\dot{r}^2}{2\sigma_{\dot{s}}^2}\right) - \sqrt{2\pi}\frac{\dot{r}}{\sigma_{\dot{s}}}\phi\left(\frac{-\dot{r}}{\sigma_{\dot{s}}}\right) \tag{58}$$

and $\phi(.)$=the standardised Normal distribution function.

If structural resistance obeys the Normal distribution, that is

$$f_R(r) = \frac{1}{2\pi\sigma_R} \exp\left[-\frac{(r-\bar{r})^2}{2\sigma_R^2)}\right] \tag{59}$$

then

$$P(S \le R \mid \bar{V} = \bar{v}) = \exp\left\{-\frac{A\tau\sigma_{\dot{s}}}{2\pi\sqrt{\sigma_s^2+\sigma_R^2}} \exp\left[-\frac{\bar{r}^2}{2(\sigma_s^2+\sigma_R^2)}\right]\right\} \tag{60}$$

in which τ is the duration of dynamic response considered; in general, for wind loading, τ=10 min.

If the upper and lower bounds of structures for dynamic reliability analysis are considered, then,

$$P(S \le R \mid \bar{V} = \bar{v}) = \exp\left\{-\frac{A\tau\sigma_{\dot{s}}}{2\pi\sqrt{\sigma_s^2+\sigma_{R_1}^2}} \exp\left[-\frac{\bar{r}_1^2}{2(\sigma_2^2+\sigma_{R_2}^2)}\right] - \frac{A\tau\sigma_{\dot{s}}}{2\pi\sqrt{\sigma_s^2+\sigma_{R_2}^2}} \exp\left[-\frac{\bar{r}_2^2}{2(\sigma_s^2+\sigma_{R_2}^2)}\right]\right. \tag{61}$$

where $\bar{r}_1, \bar{r}_2$ and $\sigma_{R_1}, \sigma_{R_2}$ are the mean values and standard deviations of R_1 (the upper bound) and R_2 (the lower bound), respectively.

If.R_1=R_2=R (symmetric bound), Eq. (61) becomes

$$P(S \le R \mid \bar{V} = \bar{v}) = \exp\left\{-\frac{A\tau\sigma_{\dot{s}}}{\pi\sqrt{\sigma_s^2+\sigma_R^2}} \exp\left[-\frac{\bar{r}^2}{2(\sigma_s^2+\sigma_R^2)}\right]\right\} \tag{62}$$

If the deterioration of structural resistance is not considered, that is $\dot{r}=0$, it can be obtained from Eq. (58) that $A=1$.

4. Numerical Example 1

Wuhan T.V. Tower located in Wuhan City, P.R. China is analysed here as a numerical example for the present study. As shown in Fig. 2, this tower consists of tower base, tower body, tower building and a wireless mast. The main structure of the T.V. Tower is a reinforced concrete cone shell with the diameter of the cone varying linearly along the height. Its external base section diameter section is 16m and that of the top section is 3.9m. In the analysis of wind-induced vibration of the tower, the structure is treated as an 18 lumped mass system. Li (1995) investigated free vibration of the tower. The geometric dimension, mass and stiffness distributions of the Wuhan T.V. tower are listed in Table 1.

Li *et al.* (1993b) conducted a detailed statistical analysis of successive years of climatological data for the region of Wuhan, P.R. China. They obtained the probability distribution function

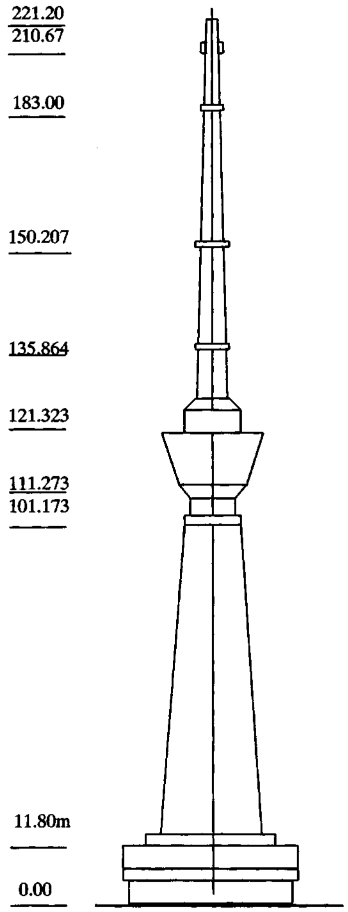

Fig. 2 Wuahn T.V. Tower

(PFD) of the maximum local wind speed (the averaging time is ten minutes) in the n years $[F(\bar{v})]$, which can be adopted for the present dynamic reliability analysis for this T.V. tower. They also found that strong wind in Wuhan City usually occurs in NNE-NE direction, and the corresponding $F(\bar{v})$ in NNE-NE direction was obtained by Li *et al.* (1993b).

As suggested by Li *et al.* (1993b), the wind velocity at the location of the tower is taken as

$$\bar{V}(z) = \bar{V}(10)\left(\frac{z}{10}\right)^{0.16} \tag{63}$$

in which $\bar{V}(z)$ is the mean wind speed at height z.

In the dynamic reliability analysis for the T.V. tower, the top displacement response of this tower is taken as the critical index; Davenport's wind speed spectrum is adopted and the coefficient of ground roughness, K, is taken as 0.003. The calculated results of the dynamic reliability analysis are shown in Fig. 3, in which curve 1 represents the results computed by the Monte Carlo simulation; curve 2 and curve 3 correspond to the dynamic reliability results calculated based on the PDF of the maximum local wind speed for all the wind directions and for NNE-NE direction only, respectively. It can be seen that the calculated results by the proposed procedure are in good agreement with the simulated data by the Monte Carlo method. In particular, when n<10 years, the three sets of data are almost identical, when n=100 years, the difference between them is less than 10 per cent, demonstrating the good applicability of the proposed procedure for the evaluation of structural dynamic reliability.

Table 1. The geometric dimension, mass and stiffness distributions of the T.V. Tower

Lumped Mass No.	Height m	Diameter m	Mass Kg	Stiffness $EJ_i \times 10^7$ KN.m^2
0	0	16		
1	4.975	15	72,571	28,440
2	17.412	12.5	66,739	26,320
3	22.387	11.5	60,849	24,157
4	27.377	10.875	54,192	17,064
5	29.876	10.75	59,317	22,388
6	32.375	10.625	41,658	9,582
7	34.874	10.5	57,256	24,844
8	104.853	7.0	37,208	8,622
9	108.853	7.0	20,735	1,426
10	110.803	7.6	20,735	1,428
11	114.253	7.0	37,935	2,847
12	133.653	7.0	20,735	1,428
13	138.622	5.88	20,735	1,428
14	139.864	5.6	20,281	837
15	142.303	4.5	14,314	374
16	164.203	4.5	14,324	374
17	156.685	3.9	9,759	196
18	187	3.9	9,759	196

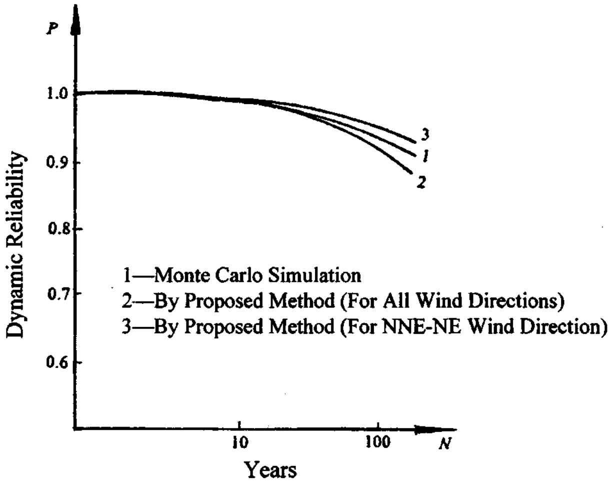

Fig. 3 Dynamic reliability of Wuhan T.V. Tower

5. Numerical Example 2

In the last numerical example, an overall good agreement between the results calculated by the proposed procedure and those simulated by the Monte Carlo method has provided confidence on the proposed computational method and computer programme. Another numerical example utilising a chimney 60m high modelled as a lumped-mass system is analysed here to study the effect of the variation in structural parameters on the structural dynamic reliability. This stack is divided into 8 sections for computation purpose. The Young's modulus of the chimney is 3000 MPa (the coefficient of variation,V=0.01). The structural parameters are listed in Table 2. The top displacement response of this chimney is taken as the critical index in dynamic reliability analysis. The dynamic response of this chimney, considering the randomness of structural stiffness and wind loading, is evaluated according to the procedures proposed in the preceding sections. The results of dynamic reliability analysis of this high-rise structure under wind action computed by the present method are given in Table 3. The effects of structural critical damping ratio ξ and resistance represented by the permissible displacement which is a random variable on dynamic reliability of this structure can be clearly seen through the calculated results presented in Table 3. It is clear that the dynamic reliability of this structure becomes larger as the structural damping and resistance increase.

Table 4 presents the effect of the variation of the structural resistance on the dynamic reliability of this stack. It is clear that the larger the structural resistance variation is, the lower its dynamic reliability is. In particular, this effect becomes more pronounced at large value of V. It should be noted that the variation of structural resistance is inevitable during a long period (e.g., 50 or 100 years) and it is random in nature. The calculated results of this chimney show that the variation in

Table 2 Structural parameters

Section length (meter)	5.0	8.0	7.0	7.5	7.5	7.5	7.5	10.0
Outside diameter (meter)	5.03	4.61	4.31	4.02	3.72	3.42	3.12	2.77
Section weight (KN)	1112	1307	944	899	815	623	577	520

Table 3 The effects of damping ratio ξ and structural resistance (coefficient of variation V=0) on structural dynamic reliability

$\bar{r}$	ξ	10 (year)	20 (year)	30 (year)	40 (year)	50 (year)
H/100	0.02	0.9976	0.9967	0.9954	0.9937	0.9926
H/100	0.03	0.9984	0.9978	0.9971	0.9968	0.9963
H/200	0.02	0.8872	0.7880	0.7700	0.6216	0.5522
H/200	0.03	0.9443	0.8928	0.8442	0.7981	0.7543

Note: H is the chimney height (60m).

Table 4 The effects of structural resistance variation on the structural dynamic reliability

$\bar{r}$	ξ	P (50 Years)
H/100, V=0	0.02	0.9926
H/100, V=0.11	0.02	0.9912
H/100, V=0.25	0.02	0.9322

structural parameters indeed influence the first passage probability of this structure. These parameters such as the structural resistance should be treated as random variables in the evaluation of structural dynamic reliability.

6. Conclusions

In this paper, structural parametric uncertainties and structural resistance were treated as random variables, and dynamic wind load was considered as a random process. Structural dynamic responses under the action of stochastic wind loads are evaluated by the stochastic finite element method. The formulas for structural dynamic reliability analysis considering the randomness of structural resistance and external dynamic loading are proposed. Two numerical examples of high-rise structures are presented to illustrate the proposed methodology. The calculated results demonstrate the good applicability of the proposed procedure and that the variation in structural parameters indeed influences the structural responses and dynamic reliability.

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Appendix

When a structural stiffness matrix is a stochastic matrix, the natural frequency and mode shape of the structure are thus random variables. In this case, the characteristic equations can be expressed as

$$([\bar{K}] - \bar{\lambda}[M])\{\bar{\phi}\} = 0 \tag{A.1}$$

$$([\bar{K}] - \bar{\lambda}[M])\{\phi_i^{(1)}\} = -([K_i^{(1)}] - \lambda_i^{(1)}[M])\{\bar{\phi}\} \tag{A.2}$$

$$([\bar{K}] - \bar{\lambda}[M])\{\phi_{ij}^{(2)}\} = -([K_{ij}^{(2)}] - \lambda_{ij}^{(2)}[M])\{\bar{\phi}\} - ([K_i^{(1)}] - \lambda_i^{(1)}[M])\{\phi_j^{(1)}\} - ([K_j^{(1)}] - \lambda_j^{(1)}[M])\{\phi_i^{(1)}\} \tag{A.3}$$

Because Eq. (A.1) is a deterministic equation, the eigenvalues $\bar{\lambda}_i$ and the corresponding eigenvectors $\{\bar{\phi}\}_i$, (i=1, 2, ...) can be determined by conventional eigensolution procedures. According to the orthogonality properties of mode shapes we assume

$$\{\bar{\phi}\}^T[M]\{\bar{\phi}\} = I \tag{A.4}$$

Using the symmetry behaviour of matrix yields the transposition of Eq. (A.1) as follows

$$\{\bar{\phi}\}^T([\bar{K}] - \bar{\lambda}[M]) = \{0\}^T \tag{A.5}$$

Letting the left-hand of Eq. (A.2) multiplication by $\{\bar{\phi}\}^T$ and according to Eq. (A.5) lead to

$$\{0\} = -\{\bar{\phi}\}^T([K_i^{(1)}] - \lambda_i^{(1)}[M])\{\bar{\phi}\} \tag{A.6}$$

then

$$\lambda_i^{(1)} = \{\bar{\phi}\}^T[K_i^{(1)}]\{\bar{\phi}\} \tag{A.7}$$

Because the coefficient determinant of $\{\phi_i^{(1)}\}$ is equal to zero, we can not directly find $\{\phi_i^{(1)}\}$ in Eq. (A.2). Thus, it is necessary to make the following assumption.

$$\{\bar{\phi}\}^T[M]\{\phi_i^{(1)}\} = 0 \tag{A.8}$$

Then, Eq. (A.2) and Eq. (A.8) can be unified as

$$\begin{bmatrix} [\bar{K}]-\bar{\lambda}[M] \\ \{\bar{\phi}\}^T[M] \end{bmatrix} \{\phi_i^{(1)}\} = -\begin{bmatrix} [k_i^{(1)}]-\lambda_i^{(1)}[M] \\ 0 \end{bmatrix} \{\bar{\phi}\} \tag{A.9}$$

or

$$[C_1]\{\phi_i^{(1)}\} = [D_1]\{\bar{\phi}\} \tag{A.10}$$

in which

$$[C_1] = \begin{bmatrix} [\bar{K}]-\bar{\lambda}[M] \\ \{\bar{\phi}\}^T[M] \end{bmatrix}, \quad [D_1] = -\begin{bmatrix} [K_i^{(1)}]-\lambda_i^{(1)}[M] \\ 0 \end{bmatrix}$$

The matrix $([C_1]^T[C_1])$ is a non-singular matrix. Letting the left-hand of Eq. (A.10) multiplication by $[C_1]^T$, we can obtain

$$\{\phi_i^{(1)}\} = ([C_1]^T[C_1])^{-1}[C_1]^T[D_1]\{\bar{\phi}\} \tag{A.11}$$

The procedure of solving the eigenvalues $\lambda_{ij}^{(2)}$ and eigenvectors $\phi_{ij}^{(2)}$ of Eq. (A.3) is similar to that of Eq. (A.2). Letting left-hand multiplication by $\{\bar{\phi}\}^T$ for Eq. (A.3) yields,

$$\{\bar{\phi}\}^T[K_{ij}^{(2)}]\{\bar{\phi}\} - \lambda_{ij}^{(2)} + \{\bar{\phi}\}^T([K_i^{(1)}]\{\phi_j^{(1)}\} + [K_j^{(1)}]\{\phi_i^{(1)}\}) = 0 \tag{A.12}$$

The assumption given in Eq. (A.8) is adopted in the derivation for the above equation. From the above equation, we obtain

$$\lambda_{ij}^{(2)} = \{\bar{\phi}\}^T([K_{ij}^{(2)}]\{\bar{\phi}\} + [K_i^{(1)}]\{\phi_j^{(1)}\} + [K_j^{(1)}]\{\phi_i^{(1)}\}) \tag{A.13}$$

Similar to Eq. (A.8), the following assumption is made, that is

$$\{\bar{\phi}\}^T[M]\{\phi_{ij}^{(2)}\} + \{\phi_i^{(1)}\}^T[M]\{\phi_j^{(1)}\} = 0 \tag{A.14}$$

Eq. (A.14) and Eq. (A.3) can be unified as

$$[C_1]\{\phi_{ij}^{(2)}\} = -[D_2]\{\bar{\phi}\} + [E_i]\{\phi_j^{(1)}\} + [E_j]\{\phi_i^{(1)}\} \tag{A.15}$$

where

$$[D_2] = -\begin{bmatrix} [K_{ij}^{(2)}]-\lambda_{ij}^{(2)}[M] \\ 0 \end{bmatrix} \quad [E_k] = \begin{bmatrix} [K_k^{(1)}]-\lambda_k^{(1)}[M] \\ \frac{1}{2}\{\phi_k^{(2)}\}[M] \end{bmatrix} \quad (k=i \text{ or } j) \tag{A.16}$$

From Eq. (A.15) we can obtain the following equation

$$\{\phi_{ij}^{(2)}\} = ([C_1]^T[C_1])^{-1}[C_1]^T(-[D_2]\{\bar{\phi}\} + [E_i]\{\phi_j^{(1)}\} + [E_j]\{\phi_i^{(1)}\}) \tag{A.17}$$