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# Mode localization and frequency loci veering in an aircraft with external stores

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**Abstract.** This paper presents an investigation of the mode localization and frequency loci veering phenomena in an aircraft with disordered external stores. Two theoretical analyses are carried out to study the occurring mechanism of the two phenomena: condensation technique in the subspace spanned by modes of interest and geometric mapping theory in the complex plane. Two simple criteria for predicting the occurrence of the mode localization and frequency loci veering are put forward. The prediction of the phenomena by our theoretically proposed criteria is in good agreement with that obtained through numerical calculations of characteristic solutions of the disordered system.

**Key words:** structural dynamics; aircraft; external stores; disorder; mode localization; frequency loci veering; geometric theory; complex plane mapping; condensation; subspace; eigenvalue problem.

## 1. Introduction

Idealized regularities such as perfect periodicity, complete symmetry and substructural identity are a convenient and frequent assumption in structural analysis. The actual structures, however, possess irregularities or disorders to a certain extent owing to tolerances in manufacturing, assemblage and material property. It is well known that the presence of small disorder may localize the vibration modes, inhibit the propagation of vibration within the structure and confine the vibrational energy to regions close to the excitation source. This phenomenon, referred to as mode localization, was first discovered by Anderson (1958) in his study of crystals and excited considerable interest in the field of solid state physics (e.g., Rosenstock and McGill 1968, Mott and Davis 1971). A conclusion was drawn that the electron eigenstates may become localized in a disordered solid.

In recent years there has been tremendous interest amongst the researchers on structural dynamics in the phenomenon of mode localization, where it has been encountered in assemblies

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of weakly coupled components, such as chains of coupled pendulums (Hodges 1982, Pierre and Dowell 1987), blade/disc systems (Valero and Bendiksen 1987), space structures (Bendiksen 1987), multispan beams (Bouzit and Pierre 1992, 1995a, Lust, Friedmann and Bendiksen 1990), and nearly periodic structures (Cai, Cheung and Chan 1995, Cai and Lin 1991, Kissel 1988, Pierre and Cha 1989). Moreover, strikingly few experiments were carried out to verify the existence of localized modes predicted analytically and numerically (Bouzit and Pierre 1995b, Hodges and Woodhouse 1983, Levine-west and Salama 1992, Pierre, Tang and Dowell 1987). These studies led to some general conclusions regarding the influences of disorders. For example, localization was found to be most significant in periodic structures under the joint of small disorder and weak internal coupling. On the other hand, much attention has also been paid to the behaviour of the frequency loci of the system when mode localization occurs. In a pioneer paper, Leissa first discovered a fascinating characteristic: when two frequency loci approach each other and do not cross but rather veer away from each other with high local curvatures (Leissa 1974). This phenomenon, referred to as frequency loci veering, has aroused interest in some novel studies on structural dynamics (Kuttler and Sigillito 1981, Perkins and Mote 1986). Hereafter, investigations have been made on the correlation that the occurrence of frequency loci veering is associated with the occurrence of mode localization (Chen and Ginsberg 1992, Happawana, Bajaj and Nwokah 1993, Natsiavas 1993, Pierre 1988). A conclusion is reached that both mode localization and frequency loci veering are catastrophic type phenomena (Pierre 1988). It is obvious that these researchers focus their studies on nearly periodic or circular structures, e.g., chains of disordered pendulums, mistuned bladed disk assemblies, disordered multispan beams, irregular space structures, and so on.

Generally speaking, aircraft structures are commonly assumed to be symmetrical and only one half is needed to be analyzed theoretically. However, it is frequently not the case in reality because of various non-symmetrical factors. Liu and Zhao studied the mode localization and frequency loci veering of a simplified horizontal tail system (Liu and Zhao 1994). In fact, such system is no more than a model of two coupled beams with a few lumped masses. Up to now, however, few have tried to study the behaviour of an aircraft structure with disorders from external stores. To this end, a more complicated aircraft model with external stores is established in the present paper. This model is carefully chosen until it gives main results that agree with experimental data of the aircraft. The emphasis is on analysing the mechanism of mode localization and frequency loci veering when disordered parameters are introduced into the structure. We cope with such theoretical analysis by two methodologies: one is subspace condensation technique once applied to derivation of perturbation formulas (Chen, Liu and Zhao 1995), the other is geometric method originally put forward (Traintafyllou and Traintafyllou 1991), but in this paper the later method is improved and widely extended to demonstrate the existence of mode localization and frequency loci veering, thus allowing one to gain an insight into the two phenomena with less difficulties. From the proposed procedures, two simple criteria are obtained which can be used to find the conditions that mode localization and frequency loci veering occur when disorder of mass or stiffness of the model exists. Numerical calculation results are given to verify the theoretical predictions and excellent agreement is observed. As a result, the effectiveness of the derived criteria and the existence of the localized modes and veered frequency loci in the aircraft have been confirmed. The results obtained are valuable in the analysis of some non-symmetric effects in aircraft design.

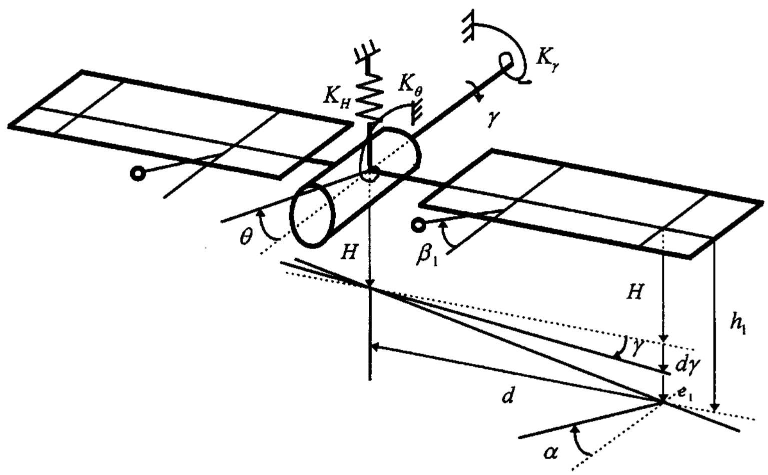


Fig. 1 Analytical model of an aircraft with external stores

## 2. Analytical model

Based upon the designed data, a simplified model of an aircraft with external stores having 8 degrees of freedom ($h_1$, $h_2$, $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $H$, $\gamma$) is established as shown in Fig. 1. The absolute displacements of the left and right representative wing sections (their positions relative to the fuselage are denoted by $d$) are $h_1$ and $h_2$ respectively. The elastic torsional angles of the two wing sections are $\alpha_1$ and $\alpha_2$ respectively. The relative vertical displacements of the two wing sections are $e_1$ and $e_2$ respectively. The pitching angles relative to wing sections of the left and right external stores are $\beta_1$ and $\beta_2$ respectively. The vertical displacement, rolling angle and pitching angle of the fuselage are $H$, $\gamma$ and $\theta$ respectively. It is obvious $h_1=H+d\gamma+e_1$. According to Lagrange's equations and relationships of the relevant design parameters, the non-dimensional equations of free vibrations have been derived by Liu (1993) in his study of asymmetric store flutter. The non-dimensional mass and stiffness matrices are $M=[m_{ij}]$, $K=[k_{ij}]$ respectively, where $m_{ij}$, $k_{ij}$ are given in Appendix A.

When a disorder exists, the left and right structures are asymmetric. Let $\mu_{\beta 1}\neq\mu_{\beta 2}$ denote the mass asymmetry of the left and right external stores, and $\omega_{\beta 1}\neq\omega_{\beta 2}$ the stiffness asymmetry. Without loss of generality, we appoint

$$\mu_{\beta_1}=(1+\varepsilon_1)\,\mu_{\beta_2},\quad \omega_{\beta_1}=(1+\varepsilon_2)\,\omega_{\beta_2} \tag{1}$$

where $\varepsilon_1$ and $\varepsilon_2$ are mass and stiffness disorders respectively. In the following analysis, the variations of $M$ and $K$ for $\varepsilon_1$ or/and $\varepsilon_2$ are all expressed by $\varepsilon M_1$ and $\varepsilon K_1$ respectively, where $\varepsilon$ is a first order parameter.

## 3. Analysis by condensation technique

The original eigenvalue problem and corresponding normalized condition are expressed by (Chen, Liu and Zhao 1995)

$$K\varphi = \lambda M\varphi \tag{2}$$

$$\varphi^T M\varphi = 1 \tag{3}$$

where the eigenvalues can be written as $\lambda_1 < \lambda_2 < \cdots < \lambda_i \cong \lambda_j < \cdots < \lambda_n$, $n$ is the order of $M$ or $K$. This notation indicates the system has $j-i+1$ repeated or nearly equal (closely spaced) eigen frequencies. The corresponding eigen modes are $\varphi_1, \varphi_2, \cdots, \varphi_n$.

The perturbed eigenvalue problem is

$$(K + \varepsilon K_1)\psi = \eta(M + \varepsilon M_1)\psi \tag{4}$$

First, we choose several modes to span an eigensubspace $\phi$, for practical use, the modes corresponding to the equal or nearly equal frequencies are used. Here, without loss of generality, the first and second eigenpairs are chosen, i.e., $\lambda_1$, $\varphi_1$ and $\lambda_2$, $\varphi_2$, $\phi = [\varphi_1, \varphi_1]$, in which the perturbed eigensolutions are to be found.

Performing condensation to Eq. (4), we have

$$\phi^T(K + \varepsilon K_1)\phi q = \mu\phi^T(M + \varepsilon M_1)\phi q \tag{5}$$

where $q$ is two dimensional generalized vector, $\mu$ is the approximate value of $\eta$ in Eq. (4).

The above two dimensional eigenproblem with normalized condition

$$q^T\phi^T(M + \varepsilon M_1)\phi q = I \tag{6}$$

can be solved to obtain two eigenpairs $(\mu_1, q_1)$, $(\mu_2, \theta_2)$, where $I$ is unit matrix.

Then, the first two modes of Eq. (4) are put into an expansion form with respect to $\varepsilon$

$$\psi_i = \phi q_i + \delta\psi_i \quad (i = 1, 2) \tag{7}$$

where $\delta\psi_i$ is a small vector of the same order as $\varepsilon$, $q_i = (q_{1i}, q_{2i})^T$. So the two eigenvectors of the perturbed system (4) can be written as

$$\begin{cases} \psi_1 = q_{11}\varphi_1 + q_{21}\varphi_2 + O(\varepsilon) \\ \psi_2 = q_{12}\varphi_1 + q_{22}\varphi_2 + O(\varepsilon) \end{cases} \tag{8}$$

Because of the symmetry of the original system, its eigenvectors $\varphi_1$ and $\varphi_2$ are either symmetric or antisymmetric. Therefore, if one of $\lambda_1$ and $\lambda_2$ corresponds to symmetric (antisymmetric) mode, the other corresponds to antisymmetric (symmetric) mode, and the absolute value of $q_{21}/q_{11}$ or $q_{22}/q_{12}$ approaches 1, then the linear combination of $\varphi_1$ and $\varphi_2$, see Eq. (8), will certainly lead to an outcome that in the mode vibration the amplitudes of one half structure are inevitably larger, meanwhile those of the other half structure are relatively smaller. Moreover, the loci of $\mu_1$ and $\mu_2$ versus $\varepsilon$ veer with each other. In actual symmetric structures, fortunately, two equal or nearly equal frequencies are frequently bound up with a symmetric mode and an antisymmetric one. This fact will be found in the following analysis.

From the above physically intrinsic quality exploration, a criterion determining whether or not the mode localization and frequency loci veering phenomena occur is established, i.e., if

$$B = |q_{21}/q_{11}| \to 1, \quad \text{or} \quad B' = |q_{22}/q_{12}| \to 1 \tag{9}$$

then the two phenomena may occur.

Now, the condensation technique is used to analyze the simplified aircraft model with stores in

Table 1 Eigen frequencies (rad/s) and symmetry of modes in system (2)

| Order | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Frequency | 27.8897 | 36.8450 | 42.1720 | 42.2185 | 157.3689 | 382.8259 | 383.1890 | 555.3913 |
| Symmetry | S | A | A | S | A | S | A | S |

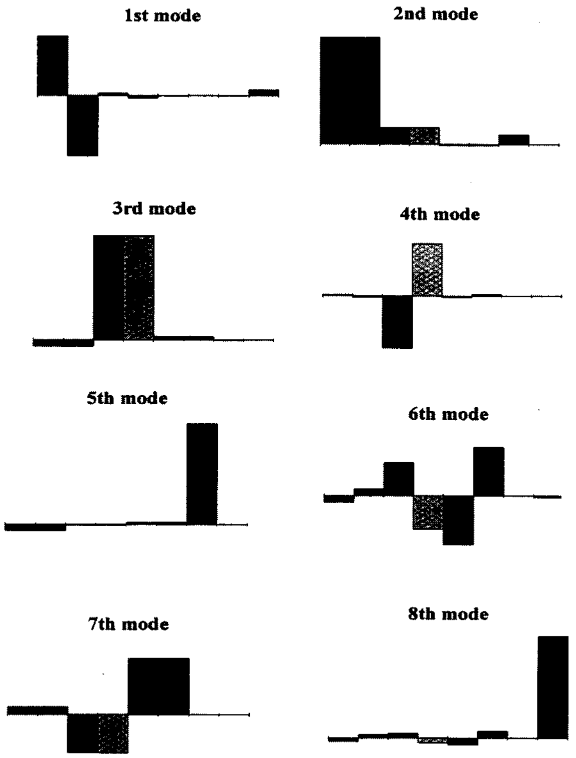


Fig. 2 Eight modes of the original symmetric system ($\varepsilon_1$=0). The corresponding eight frequencies are 27.8897, 36.8450, 42.1720, 42.2185, 157.3689, 382.8259, 383.1890, 555.3913 (rad/s), respectively

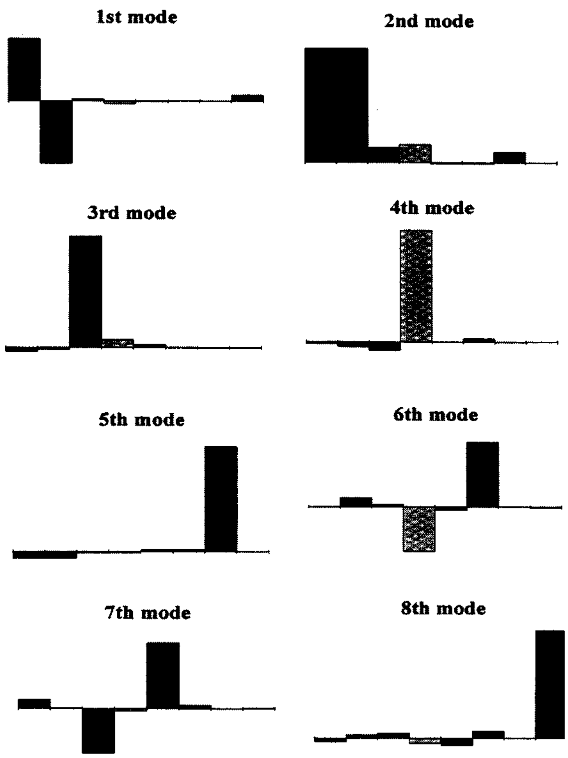


Fig. 3 Eight modes of the disordered system ($\varepsilon_1$= 0.03). The corresponding eight frequencies are 27.8782, 36.8350, 41.8316, 42.1971, 157.3685, 382.9981, 386.5931, 555.3941 (rad/s), respectively

Fig. 1. When $\varepsilon$=0, the frequencies and the symmetries of corresponding modes of the original symmetric system (2) are listed in Table 1, in which "S" denotes the symmetric modes and "A" denotes the antisymmetric modes. The eight modes are plotted in Fig. 2, where each volume strip from the left to the right indicates the mode amplitudes of the eight degrees of freedom, i.e., $h_1$, $h_2$, $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $H$, $\gamma$ respectively.

When a disorder of store is introduced, the mass and stiffness of the left and right external stores are no longer symmetric. This is to say, the symmetry of the system is violated. As an example, let $\varepsilon_1$=0.03 in Eq. (1), $B$ in Eq. (9) can be calculated for different combinations of modes. The obtained results are listed in Table 2, in which "2-3" denotes combination of the second and third order modes, i.e., $\phi=[\varphi_2, \varphi_3]$, and so on.

From Table 2, the values of $B$ for only two combinations (3-4 and 6-7) approach 1, in fact,

Table 2 Values of $B$ for different mode combinations

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1-2 | .00086 | 2-3 | .00702 | 3-5 | .00001 | 4-8 | .00001 |
| 1-3 | .00208 | 2-4 | .00683 | 3-6 | .00015 | 5-6 | .00008 |
| 1-4 | .00204 | 2-5 | .00007 | 3-7 | .00015 | 5-7 | .00008 |
| 1-5 | .00009 | 2-6 | .00003 | 3-8 | .00001 | 5-8 | .00001 |
| 1-6 | .00002 | 2-7 | .00003 | 4-5 | .00001 | 6-7 | .90236 |
| 1-7 | .00002 | 2-8 | .00003 | 4-6 | .00015 | 6-8 | .00046 |
| 1-8 | .00004 | 3-4 | .86464 | 4-7 | .00015 | 7-8 | .00046 |

they exactly are two groups of nearly equal frequencies corresponding to a symmetric mode and an antisymmetric one in each group (see Table 1). The others of $B$ are much less than 1. Therefore, for $\varepsilon_1$=0.03 the four modes (3, 4, 6, 7) undergo localization, the corresponding frequency loci (3, 4 and 6, 7) veer with each other. In order to verify these findings, the eight modes for $\varepsilon_1$=0.03 are plotted in Fig. 3, from which we can observe that the 1st, 2nd, 5th, 8th order modes remain nearly unchanged relative to the corresponding four modes plotted in Fig. 2, however, marked differences occur for the 3rd, 4th, 6th, 7th order modes between Fig. 3 and Fig. 2. In fact, the four (3, 4, 6, 7) modes in Fig. 3 are localized, but the other four (1, 2, 5, 8) modes are not. Meanwhile, the eight frequency loci are plotted in Figs. 4(a), (b), from which the veering for the 3-4 and 6-7 frequency loci is clearly observed, but no veering occurs in the other four frequency loci. To be seen more clearly, the 3-4 and 6-7 frequency loci are again plotted in Figs.

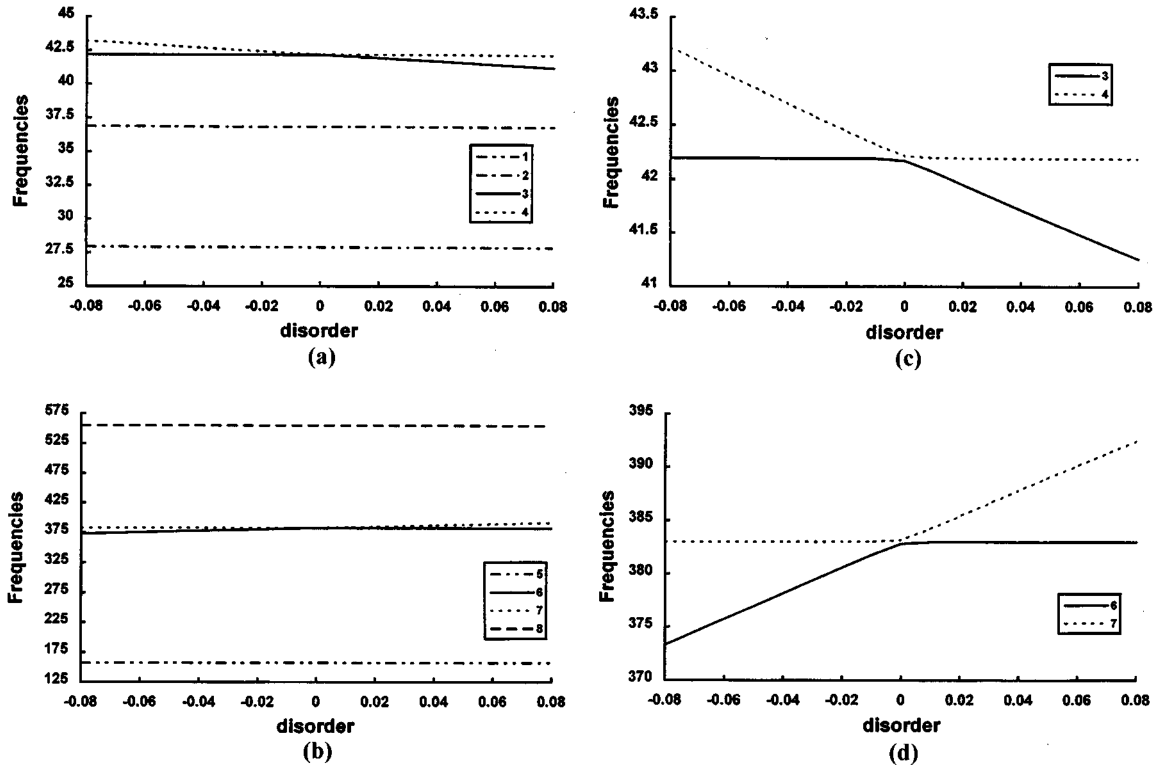


Fig. 4 (a) 1st, 2nd, 3rd, 4th frequency loci vs. disorder $\varepsilon_1$, (b) 5th, 6th, 7th, 8th frequency loci vs. disorder $\varepsilon_1$, (c) 3rd, 4th frequency loci vs. disorder $\varepsilon_1$, (d) 6th, 7th frequency loci vs. disorder $\varepsilon_1$

4(c), (d) respectively. These results have completely verified the validity of the criterion (9). For other values of $\varepsilon_1$ or/and $\varepsilon_2$ in Eq. (1), the same analyses can be made.

## 4. Analysis by geometric theory

Triantafyllou *et al.* developed a geometric theory for the analysis of mode localization and frequency coalescence (Triantafyllou and Triantafyllou 1991). In this section, we will improve and extend this theory in order to apply it to analyze mode localization and frequency loci veering in multi-degree-of-freedom systems. To this purpose, we introduce the so called condensation technique in section 3 into the geometric theory. First, an eigensubspace is spanned by the eigenvectors corresponding to the several eigenvalues of interest. Then, by using the geometric method, the analysis can be easily carried out in the subspace. Again, without loss of generality, the subspace is spanned by the first two eigenvectors. Considering that

$$\begin{cases} \phi^T(K+\varepsilon K_1)\phi = \Lambda + \varepsilon H \equiv \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + \varepsilon \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \\ \phi^T(M+\varepsilon M_1)\phi = I + \varepsilon G \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \varepsilon \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \end{cases} \tag{10}$$

where $\Lambda$=diag. $(\lambda_1, \lambda_2)$, $I$=diag. (1, 1), $H=\phi^T K_1 \phi$, $G=\phi^T M_1 \phi$, $h_{21}=h_{12}$, $g_{21}=g_{12}$ as a result of the symmetry of the mass and stiffness matrices. Therefore, one can rewrite Eq. (5) as

$$\left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + \varepsilon \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix}\right) q = \mu \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \varepsilon \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix}\right) q \tag{11}$$

By expanding the eigen equation corresponding to Eq. (11) and neglecting terms higher than $O(\varepsilon^2)$, the following equation can be obtained

$$(1+\varepsilon p_1)\mu^2 - (p_2+\varepsilon p_3)\mu + (p_4+\varepsilon p_5) = 0 \tag{12}$$

Solving Eq. (12) yields

$$\mu = [(p_2+\varepsilon p_3) \pm \sqrt{p_9}]/[2(1+\varepsilon p_1)] \tag{13}$$

where $p_i$'s are given in Appendix B.

On the basis of Eq. (13), it can be easily concluded that $\mu$ has a double root if and only if $p_9$=0, then $\varepsilon$ can be readily obtained as

$$\varepsilon = [-p_7 \pm \sqrt{p_7^2 - 4p_6 p_8}]/(2p_6) \tag{14}$$

By the geometric theory, if $\varepsilon$ determined by Eq. (14) is real, a real double root can be obtained, the loci are bound to crossover; if $\varepsilon$ is complex, a complex double root can be obtained, then the geometric structure of a saddle/branch point can be observed. In addition, if the module of $\varepsilon$ is

Table 3 Values of $\varepsilon$ for different mode combinations

| | | | | | |
|---|---|---|---|---|---|
| 1-2 | 1-3 | 1-4 | 1-5 | 1-6 | 1-7 |
| −5.510±i10.07 | 2.316±i2.864 | 2.348±i2.990 | −34.82±i0.9397 | −1.979/−1.879 | −1.980/−1.880 |
| 1-8 | 2-3 | 2-4 | 2-5 | 2-6 | 2-7 |
| −35.91/−35.15 | 0.8429±i0.5073 | 0.8731±i0.5370 | −53.60±i2.387 | −2.018/−1.908 | −2.018/−1.909 |
| 2-8 | 3-4 | 3-5 | 3-6 | 3-7 | 3-8 |
| −56.33/−54.66 | −6.962e-5±i3.792e-3 | -3.309±i3.738e-2 | −1.361/−1.149 | −1.361/−1.149 | −3.262/−3.230 |
| 4-5 | 4-6 | 4-7 | 4-8 | 5-6 | 5-7 |
| −3.145±i3.917e-2 | −1.378/−1.164 | −1.378/−1.164 | −3.376/−3.342 | −1.831/−1.785 | −1.832/−1.786 |
| 5-8 | 6-7 | 6-8 | 7-8 | | |
| −1633/−1109 | −7.40e-6±i3.03e-3 | 8.886±i2.260 | 8.763±i2.201 | | |

infinitely small, the mode localization and loci veering phenomena occur. From the analysis, it can be seen that, on the practical side, to achieve localization and frequency loci veering, one must consider a system having closely spaced eigenvalues to start with, and then study conditions of obtaining a non-degenerate saddle point, and as close to the real parameter axis as possible, i.e., if

$$\varepsilon = \varepsilon_R + i\varepsilon_I, \quad |\varepsilon| \ll 1 \tag{15a, b}$$

where $i=\sqrt{-1}$, then mode localization and frequency loci veering may occur. Conditions (15) can be regarded as another criterion relative to criterion (9).

The above mentioned procedure is also used to analyze the mode localization and frequency loci veering of the aircraft model in section 2. $\varepsilon$ in Eq. (14) is calculated for all of the different combinations of modes. The obtained results are listed in Table 3.

From Table 3, real values of $\varepsilon$ (for combinations of 1-6, 1-7, 1-8, 2-6, 2-7, 2-8, 3-6, 3-7, 3-8, 4-6, 4-7, 4-8, 5-6, 5-7, 5-8) correspond to the crossover of loci, so no catastrophic phenomena may occur. Complex values of $\varepsilon$ (for combinations of 1-2, 1-3, 1-4, 1-5, 2-3, 2-4, 2-5, 3-4, 3-5, 4-5, 6-7, 6-8, 7-8) satisfy Eq. (15a), but only four of which for combinations of 3-4 and 6-7 satisfy Eq. (15b), too. Therefore, we can deduce that the 3rd, 4th, 6th and 7th order modes are localized and 3-4, 6-7 frequency loci veer when small disorder is introduced. These conclusions are exactly in full agreement with that obtained by the condensation technique and the numerical calculation in section 3, thus also confirming the validity of criterion (15).

## 5. Conclusions

The aircraft model with external stores has been analyzed. The condensation technique and improved geometric theory have been used to obtain two criteria for predicting the occurrence of mode localization and frequency loci veering. From this study, the following main conclusions may be drawn.

(1) In the case of small disorder of the left and right external stores, i.e., the case of asymmetry of stores, mode localization and frequency loci veering may occur in the model. It is worthwhile paying special attention to the influence of disorder on dynamics in the field of aircraft design.

(2) The two criteria are very simple in form and can be used conveniently with less computation by applying the condensation technique, in which only the eigenpair results of the

original symmetric system are needed to allow one to perform the analysis.

(3) The main advantage of the geometric method is its conceptual simplicity. However, its application is somewhat confined in the analysis of systems having many degrees of freedom. This paper has demonstrated that it is possible to cope with such limitation and extend the application of this methodology by combining with the condensation procedure.

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## Appendix A

$$m_{11}=\mu_w+0.25\mu_{\beta_1},\quad m_{13}=\mu_w\chi_\alpha+0.25\mu_{\beta_1}(\chi_{\beta_1}-\bar{l}_1),\quad m_{15}=0.5\mu_{\beta_1}\chi_{\beta_1},$$

$$m_{22}=\mu_w+0.25\mu_{\beta_2},\quad m_{24}=\mu_w\chi_\alpha+0.25\mu_{\beta_2}(\chi_{\beta_2}-\bar{l}_2),\quad m_{26}=0.5\mu_{\beta_2}\chi_{\beta_2},$$

$$m_{33}=\mu_w r_\alpha^2+0.25\mu_{\beta_1}(r_{\beta_1}^2+\bar{l}_1^2-2\chi_{\beta_1}\bar{l}_1),\quad m_{35}=0.5\mu_{\beta_1}(r_{\beta_1}^2-\chi_{\beta_1}\bar{l}_1),$$

$$m_{44}=\mu_w r_\alpha^2+0.25\mu_{\beta_2}(r_{\beta_2}^2+\bar{l}_2^2-2\chi_{\beta_2}\bar{l}_2),\quad m_{46}=0.5\mu_{\beta_2}(r_{\beta_2}^2-\chi_{\beta_2}\bar{l}_2),$$

$$m_{55}=\mu_{\beta_1}r_{\beta_1}^2,\quad m_{66}=\mu_{\beta_2}r_{\beta_2}^2,\quad m_{77}=\mu_H,\quad m_{88}=\mu_\gamma,$$

$$m_{ij}=0(\text{other } i,\, j),\quad m_{ij}=m_{ji}(i,\, j=1\sim8)$$

$$k_{11}=\mu_w\left(\frac{\omega_{h_1}}{\omega_{\alpha_1}}\right)^2,\quad k_{17}=-k_{11},\quad k_{22}=\mu_w\left(\frac{\omega_{h_2}}{\omega_{\alpha_1}}\right)^2,\quad k_{27}=-k_{22},\quad k_{18}=-4.8k_{11},$$

$$k_{28}=4.8k_{22},\quad k_{33}=c_1\mu_w r_\alpha^2,\quad c_1=\frac{\omega_\theta^2+\omega_{\alpha_2}^2}{\Psi},\quad k_{34}=c\,\mu_w r_\alpha^2,\quad c=\frac{\omega_{\alpha_2}^2}{\Psi},$$

$$k_{44}=c_2\mu_w r_\alpha^2\left(\frac{\omega_{\alpha_2}}{\omega_{\alpha_1}}\right)^2,\quad c_2=\frac{\omega_\theta^2+\omega_{\alpha_1}^2}{\Psi},\quad \Psi=\omega_\theta^2+\omega_{\alpha_1}^2+\omega_{\alpha_2}^2,$$

$$k_{55}=m_{55}\left(\frac{\omega_{\beta_1}}{\omega_{\alpha_1}}\right)^2,\quad k_{66}=m_{66}\left(\frac{\omega_{\beta_2}}{\omega_{\alpha_1}}\right)^2,\quad k_{77}=\frac{\mu_H\omega_H^2+\mu_w(\omega_{h_1}^2+\omega_{h_2}^2)}{\omega_{\alpha_1}^2},$$

$$k_{88}=\frac{\mu_\gamma\omega_\gamma^2+4.8^2\mu_w(\omega_{h_1}^2+\omega_{h_2}^2)}{\omega_{\alpha_1}^2},\quad k_{78}=\frac{4.8\mu_w(\omega_{h_1}^2-\omega_{h_2}^2)}{\omega_{\alpha_1}^2},$$

$$k_{ij}=0(\text{other } i,\, j),\quad k_{ji}=k_{ij}(i,\, j=1\sim8)$$

where $\mu_w$, $\mu_\beta$, $\mu_H$ are the nondimensional masses of wing, store and fuselage divided by $\pi\rho b^3$ respectively, $\rho$ is the air density, $b$ is the length of wing semichord, $\mu_\gamma$ is the nondimensional rolling moments of inertia of fuselage divided by $\pi\rho b^5$, $\chi_\alpha b$ and $\chi_\beta b$ are the distances from the centres of gravity of the wing section and the store respectively to the rotation axis of wing, $r_\alpha b$ and $r_\beta b$ are the gyroscopic radii of the wing section and the store respectively, $\bar{l}b$ is the distance from the hinged point of the store to the rotation axis of wing, subscript "1" and "2" denote the left and right wings or stores respectively, Each $\omega$ (rad/s) denotes the branch frequency corresponding its subscript, $ab$ is the distance from the midpoint of wing chord to the rotation axis of wing. The following specific parameter values obtained from the simplified designed data in the case of symmetric structural model are used for the analysis:

$$\mu_w = 200, \quad \mu_H = 300, \quad \mu_\gamma = 100, \quad \mu_{\beta_1} = \mu_{\beta_2} = 53, \quad a = -0.33, \quad \chi_\alpha = 0.13,$$
$$r_\alpha^2 = 0.147, \quad \bar{l}_1 = \bar{l}_2 = 0.67, \quad \chi_{\beta_1} = \chi_{\beta_2} = -1, \quad r_{\beta_1}^2 = r_{\beta_2}^2 = 1, \quad b = 1, \quad \omega_{h_1} = \omega_{h_2} = 40,$$
$$\omega_{\alpha_1} = \omega_{\alpha_2} = 64, \quad \omega_{\beta_1} = \omega_{\beta_2} = 230, \quad \omega_H = 150, \quad \omega_\theta = 1450, \quad \omega_\gamma = 400$$

## Appendix B

$$p_1 = g_{11} + g_{22}, \quad p_2 = \lambda_{10} + \lambda_{20}, \quad p_3 = \lambda_{10} g_{22} + \lambda_{20} g_{11} + h_{11} + h_{22}, \quad p_4 = \lambda_{10}\lambda_{20},$$
$$p_5 = \lambda_{10} h_{22} + \lambda_{20} h_{11}, \quad p_6 = p_3^2 - 4p_1 p_5, \quad p_7 = 2p_2 p_3 - 4p_1 p_5 - 4p_5, \quad p_8 = p_2^2 - 4p_4,$$
$$p_9 = p_6 \varepsilon^2 + p_7 \varepsilon + p_8$$