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Free vibration analysis of functionally graded beams with variable cross-section by the differential quadrature method based on the nonlocal theory

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Abstract. This paper attempts to investigate the free vibration of functionally graded material beams with nonuniform width based on the nonlocal elasticity theory. The theoretical formulations are established following the Euler–Bernoulli beam theory, and the governing equations of motion of the system are derived from the minimum total potential energy principle using the nonlocal elasticity theory. In addition, the Differential Quadrature Method (DQM) is applied, along with the Chebyshev-Gauss-Lobatto polynomials, in order to determine the weighting coefficient matrices. Furthermore, the effects of the nonlocal parameter, cross-section area of the functionally graded material (FGM) beam and various boundary conditions on the natural frequencies are examined. It is observed that the nonlocal parameter and boundary conditions significantly influence the natural frequencies of the functionally graded material beam cross-section. The results obtained, using the Differential Quadrature Method (DQM) under various boundary conditions, are found in good agreement with analytical and numerical results available in the literature.

Keywords: free vibration; nonuniform width; Euler-Bernoulli beam; Nonlocal theory; Differential Quadrature Method; Functionally Graded Material

1. Introduction

Over the last few years, the usage of composite materials has been rapidly increasing due to their beneficial properties, such as their high specific strength, high specific modulus, greater corrosion resistance and longer fatigue life. The use of composite materials in several industrial applications, i.e. automotive, aeronautical, marine, railway and civil engineering, has been growing over the past few years. Functionally graded materials (FGMs) are composite materials that result from the combination of two or more distinct components in a way to attain optimum physical and mechanical properties. In addition, some of these materials possess a number of advantages that make them attractive in potential applications. In particular, their electrical conductivity and thermal properties make them suitable as multifunctional materials. The development of multifunctional composite materials and structures is aimed at providing innovative functionalities to structures in addition to their load carrying capacity (Koizumi 1997).

Nanostructured elements, which are emerging as a new class of materials, have attracted great attention in the scientific community due to their attractive properties. Conducting experiments on nanoscale specimens turns out to be difficult and expensive (Eringen 1983, 2002). Therefore, the development of appropriate mathematical models for nanostructures is an important issue with regard to the applications of nanotechnology. The nonlocal mechanics theory has been developed to take into account details of nano/micro-structure in an implicit way. In this theory, a nonlocal parameter, which characterizes the internal length of such nanostructured objects, is introduced. During the past few years, a great deal of research has been carried out on the topic of vibration analysis of isotropic, orthotropic and FGM structures due to their significant importance in the field of in nanoscale engineering. Recently, several theories and models, in which beams are generally subjected to various types of mechanical loads, have been developed (Aydogdu 2009, Reddy 2007). A study realized by (Tahouneh *et al*. 2018) treated the effects of agglomeration, geometrical, and material parameters on the frequency parameters of the sandwich functionally graded nanocomposite plate. (Tahouneh *et al*. 2019) used an analysis technique concerning to vibration analysis of a single layered graphene sheet (SLGS) with corner cutout based on the nonlocal elasticity model framework of classical Kirchhoff thin plate. The buckling, bending and vibration behaviours of functionally graded nanobeams with nonuniform thickness are investigated in (Rajasekaran and Khaniki 2018) with nonlocal Eringen theory. For molecular study is introduced by (Tahouneh *et al*. 2020) to study the vibration

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analysis of vacancy defected graphene sheet as a nonisotropic structure via molecular dynamic and continuum approaches.

The Differential Quadrature Method (DQM) proved to be a simple and efficient numerical technique for solving linear and nonlinear partial differential equations by the discretization of any derivative at a point with a weighted linear sum of the functional values of its neighboring points, as reported in (Bellman *et al.* 1972). This method addresses complex problems for the static and dynamic analysis of various structural components, such as beams, plates, and cylinders. (Tahouneh 2014) studied the free vibrations of bidirectional functionally graded annular plates lying on a two-parameter elastic foundation. The formulations, which are based on bi-dimensional and three-dimensional elasticity theories, are solved using numerical methods based on the differential quadrature method (DQM) for the analysis of structural and dynamical systems (Bozdogan 2012, Yas *et al.* 2011, Bambill *et al*. 2010). (Rajasekaran *et al*. 2009) presented a unified solution method for the classical beam theory. According to the classical approach used in the field of strength of materials, the system of external forces, geometry, mechanical characteristics of materials are well known, which makes it possible to determine the internal stresses by using a differential analysis of the conservative kinematic laws. The free vibration and buckling analysis of beams, using Eringen's theory for nonlocal elasticity, was carried out using the modified differential quadrature method developed by (Murmu and Pradham 2008). (Ghazaryan *et al.* 2017) studied the free vibrations of non-uniform cross-section and axially functionally graded Euler–Bernoulli beams with various boundary conditions by differential transform method, (Garijo 2015) dealt with the eigenvalues problem related to the free vibration of Euler–Bernoulli beams of variable cross-section issolved using a collocation technique based on Bernstein polynomials. (Mechab *et al*. 2016) proposed the solution for free vibration analysis of orthotropic beams with local and nonlocal formulation using the high-order theory including the Poisson effect. (Nedri *et al*. 2014) examined the free vibration of laminated composite plates on elastic foundations with a refined hyperbolic shear deformation theory.

The present investigation aims at deriving the nonlocal elasticity for modeling nano E-FGM beams based on Euler-Bernoulli beam theory. Various boundary conditions and non-uniform cross sections are considered for obtaining the nano E-FGM beam models. For this, the nonlocal analytical model is applied to simply supported, cantilever, propped cantilever and clamped beams. The effects of small-scale parameters on the deflections and bending moments in beams are examined. In addition, the free vibration numerical results are obtained by solving the beam bending differential equation with variable coefficients, using the differential quadrature method (DQM).

2. Mathematical formulation

Consider a straight uniform beam with length L and thickness h. A coordinate system (x, z) is introduced on the central axis of the beam, with the non-uniform variation of the width *b(x)*, and the z axis is along the thickness direction; the origin of the coordinate system is placed at the left end of the beam. Moreover, it is assumed that the beam deformations take place in the (x,z) plane, and therefore the displacement components (u,w) along the x and z directions depend solely on the coordinates x and z and time *t*. The general form of the following displacement field can be written as:

$$\begin{cases} u(x,t)=u_0(x,t)-z\dfrac{\partial w_0(x,t)}{\partial x} \\ w(x,t)=w_0(x,t) \end{cases} \tag{1}$$

Here *u* and *w* are the axial and transverse displacements of the beam center line in the *x* and *z* directions, respectively, and *t* denotes time.

$$\varepsilon_x=\frac{\partial u}{\partial x}=\frac{\partial u_0(x,t)}{\partial x}-z\frac{\partial^2 w_0(x,t)}{\partial x^2} \tag{2}$$

$$\sigma_x=Q_{11}(z)\varepsilon_x=Q_{11}(z)\left(\frac{\partial u_0(x,t)}{\partial x}-z\frac{\partial^2 w_0(x,t)}{\partial x^2}\right) \tag{3}$$

2.1 Nonlocal model for free vibration of E-FGM beams

Consider a linear homogenous nonlocal elastic body. The stress components, while neglecting the body forces, can be expressed as (Eringen 1983):

$$\sigma_{ij}(x)=\int_V \lambda(|x'-x|,\alpha)\,C_{ijkl}\,\varepsilon_{kl}(x')\,dv(x') \tag{4}$$

where σ_{ij}, ε_{kl} and C_{ijkl} are the stress, strain and fourth-order elasticity tensors, respectively. The term $\lambda(|x'-x|,\alpha)$ is the nonlocal modulus, and the attenuation function is the kernel function which is included into the constitutive equations to measure the nonlocal effects at the point x induced by the local strains at any point x'; the value of $|x'-x|$ is the Euclidean distance. Also, α is the scale coefficient or nonlocal unit length scale parameter describing the effect of the micro- and nanoscale on the mechanical behavior. The term α depends on the internal characteristic lengths (lattice parameter, granular size, distance between C–C bonds), ℓ_i , and the external characteristic lengths (crack length, wave length), l_e. It is expressed as $\alpha=\frac{e_0\ell_i}{\ell_e}$, where the parameter e_0 is estimated so as the relations of the nonlocal elasticity model could provide satisfactory approximations of the atomic dispersion curves of plane waves with those of the atomic lattice dynamics. Due to the difficulty of solving the integral constitutive relation, it was decided to use the Eringen simplified equation in its differential form, as given by Equation (4), as a basis for the formulation of the nonlocal constitutive equation (5):

$$\left(1-(e_0a)^2\nabla^2\right)\sigma_{ij}=C(z):\varepsilon \tag{5}$$

where ':' represents the double dot product; ∇^2 is the Laplacian operator, expressed as: $\frac{\partial^2}{\partial x^2}$

Thus, using the Laplacian operator, the nonlocal constitutive relations can be expressed as:

$$\sigma_{xx} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = Q_{11}(z)\varepsilon_{xx} \tag{6}$$

where

$$Q_{11}(z) = \frac{E(z)}{1-\upsilon^2} \tag{7}$$

where σ_x and ε_x are the stress and strain components, respectively. In addition, the elastic constants Q_{11} are expressed in terms of Young's modulus E and Poisson's ratio ν.

2.2 Governing equations and boundary conditions

Using equations 1, 2, and 3 for strains and stresses as well as the dynamic version of the principle of virtual work, variationally consistent governing differential equations and the corresponding boundary conditions for the beam under consideration are obtained. When the principle of virtual work is applied to the beam, the following equation is obtained:

$$\left[\int_{-h/2}^{h/2}\int_S [\sigma_x \delta\, \varepsilon_x] dS\, dz - \int_{-h/2}^{h/2}\int_S \rho\left[\ddot{W}\,\delta W\right] dS\, dz\right] dt = 0 \tag{8}$$

Here the quantity δ denotes the variational operator. Employing Green's theorem in Equation (8) allows obtaining the coupled Euler–Lagrange equations, which represent the governing differential equations of the beam, along with its associated boundary conditions. The resulting governing differential equations are as follows:

$$\begin{cases} \dfrac{\partial N_{xx}}{\partial x} = 0 \\ \dfrac{\partial^2 M_{xx}}{\partial x^2} - I_{11} b(x) \dfrac{\partial^2 w_0(x,t)}{\partial t^2} = 0 \end{cases} \tag{9}$$

$$\begin{Bmatrix} N_{xx} \\ M_{xx} \end{Bmatrix}^L = b(x) \begin{bmatrix} \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}(z) dz & \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}(z) z dz \\ \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}(z) z dz & \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}(z) z^2 dz \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_0(x,t)}{\partial x} \\ \dfrac{\partial^2 w_0(x,t)}{\partial x^2} \end{Bmatrix} \tag{10}$$

Where A_{ij},B_{ij}, D_{ij} and I_{11}, are the beam stiffness, as defined by

$$\begin{gathered} A_{11} = \int_{\frac{h}{2}}^{\frac{h}{2}} Q_{11}(z) dz\,,\ B_{11} = \int_{\frac{h}{2}}^{\frac{h}{2}} Q_{11}(z) z dz\,, \\ D_{11} = \int_{\frac{h}{2}}^{\frac{h}{2}} Q_{11}(z) z^2 dz, I_{11} = \int_{\frac{h}{2}}^{\frac{h}{2}} \rho(z) dz \end{gathered} \tag{11}$$

Here the local stresses N and M are defined by

$$\begin{Bmatrix} N_{xx} \\ M_{xx} \end{Bmatrix}^L = b(x) \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_0(x,t)}{\partial x} \\ \dfrac{\partial^2 w_0(x,t)}{\partial x^2} \end{Bmatrix} \tag{12}$$

The nonlocal higher-order beam generalized constitutive law can be written as:

$$\begin{Bmatrix} N_{xx} \\ M_{xx} \end{Bmatrix} - (ea_0)^2 \frac{\partial^2}{\partial x^2} \begin{Bmatrix} N_{xx} \\ M_{xx} \end{Bmatrix} = b(x) \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_0(x,t)}{\partial x} \\ \dfrac{\partial^2 w_0(x,t)}{\partial x^2} \end{Bmatrix} \tag{13}$$

The bending moment may then be expressed as:

$$\begin{gathered} N_{xx} = (ea_0)^2 I_{11} b(x) \frac{\partial^2 w_0(x,t)}{\partial t^2} + b(x) A_{11} \frac{\partial u_0(x,t)}{\partial x} \\ - b(x) B_{11} \frac{\partial^2 w_0(x,t)}{\partial x^2} \end{gathered} \tag{14a}$$

$$\begin{gathered} M_{xx} = (ea_0)^2 I_{11} b(x) \frac{\partial^2 w_0(x,t)}{\partial t^2} + b(x) B_{11} \frac{\partial u_0(x,t)}{\partial x} \\ - b(x) D_{11} \frac{\partial^2 w_0(x,t)}{\partial x^2} \end{gathered} \tag{14b}$$

Substitution of Equation (13) into Equation (10) leads to the governing equations given below

$$\begin{cases} \dfrac{\partial}{\partial x}\left(b(x)\left(A_{11} \dfrac{\partial u_0(x,t)}{\partial x} + B_{11} \dfrac{\partial^2 w_0(x,t)}{\partial x^2} \right)\right) = 0 \\ \dfrac{\partial^2}{\partial x^2}\left(b(x) \left(\begin{gathered} (ea_0)^2 I_{11} \dfrac{\partial^2 w_0(x,t)}{\partial t^2} + B_{11} \dfrac{\partial u_0(x,t)}{\partial x} \\ - D_{11} \dfrac{\partial^2 w_0(x,t)}{\partial x^2} \end{gathered} \right)\right) \\ - I_{11} b(x) \dfrac{\partial^2 w_0(x,t)}{\partial t^2} = 0 \end{cases} \tag{15}$$

Furthermore, multiplication of Equation (15a) by B_{11} and Equation (15b) by A_{11} allows obtaining

$$\begin{cases} B_{11} \dfrac{\partial^2}{\partial x^2}\left(b(x)\left(A_{11} \dfrac{\partial u_0(x,t)}{\partial x} + B_{11} \dfrac{\partial^2 w_0(x,t)}{\partial x^2} \right)\right) = 0 \\ A_{11} \dfrac{\partial^2}{\partial x^2}\left(b(x) \left(\begin{gathered} (ea_0)^2 I_{11} \dfrac{\partial^2 w_0(x,t)}{\partial t^2} + B_{11} \dfrac{\partial u_0(x,t)}{\partial x} \\ - D_{11} \dfrac{\partial^2 w_0(x,t)}{\partial x^2} \end{gathered} \right)\right) \\ - I_{11} b(x) \dfrac{\partial^2 w_0(x,t)}{\partial t^2} = 0 \end{cases} \tag{16}$$

Subtracting Equation (16a) from Equation (16b) gives one single differential equation as a function of $w_0(x,t)$:

$$b(x)\left(B_{11}^2 - A_{11}D_{11}\right)\frac{\partial^4 w_0(x,t)}{\partial x^4} + 2\frac{\partial b(x)}{\partial x}\left(B_{11}^2 - A_{11}D_{11}\right)\frac{\partial^3 w_0(x,t)}{\partial x^3} + \frac{\partial^2 b(x)}{\partial x^2}\left(B_{11}^2 - A_{11}D_{11}\right)\frac{\partial^2 w_0(x,t)}{\partial x^2} + A_{11}I_{11}\left((ea_0)^2\frac{\partial^2 b(x)}{\partial x^2} - b(x)\right)\frac{\partial^2 w_0(x,t)}{\partial t^2} + 2(ea_0)^2 A_{11}I_{11}\frac{\partial b(x)}{\partial x}\frac{\partial^3 w_0(x,t)}{\partial x \partial t^2} + (ea_0)^2 A_{11}I_{11}b(x)\frac{\partial^4 w_0(x,t)}{\partial x^2 \partial t^2} = 0 \tag{17}$$

For harmonic vibrations, the transverse displacement can be expressed as:

$$w_0(x,t) = w(x)e^{\,i\omega t} \tag{18}$$

$$b(x)\frac{\partial^4 w(x)}{\partial x^4} + 2\frac{\partial b(x)}{\partial x}\frac{\partial^3 w(x)}{\partial x^3} + \left(\frac{\partial^2 b(x)}{\partial x^2} - \frac{(ea_0)^2 A_{11}I_{11}\omega^2 b(x)}{\left(B_{11}{}^2 - A_{11}D_{11}\right)}\right)\frac{\partial^2 w(x)}{\partial x^2} - \frac{2(ea_0)^2 A_{11}I_{11}\omega^2}{\left(B_{11}{}^2 - A_{11}D_{11}\right)}\frac{\partial b(x)}{\partial x}\frac{\partial w(x)}{\partial x} - \frac{A_{11}I_{11}\omega^2}{\left(B_{11}{}^2 - A_{11}D_{11}\right)}.\left((ea_0)^2\frac{\partial^2 b(x)}{\partial x^2} - b(x)\right)w(x) = 0 \tag{19}$$

Let us introduce the dimensionless formulation, based on the following dimensionless parameters that are given by:

$$\zeta = \frac{A_{11}D_{11} - B_{11}{}^2}{A_{11}I_{11}};\quad \varpi^2 = \frac{\omega^2 L^4}{\zeta};\quad W(\xi) = \frac{w(x)}{L}$$
$$\xi = \frac{x}{L};\quad g = \frac{ea_0}{L} \tag{20}$$

$$\frac{\partial^4 W(\xi)}{\partial \xi^4} + 2\frac{b'(\xi)}{b(\xi)}\frac{\partial^3 W(\xi)}{\partial \xi^3} + \left(\frac{b''(\xi)}{b(\xi)} + g^2\varpi^2\right)\frac{\partial^2 W(\xi)}{\partial \xi^2} + \left(2g^2\varpi^2\frac{b'(\xi)}{b(\xi)}\right)\frac{\partial W(\xi)}{\partial \xi} + \left(\varpi^2\left(g^2\frac{b''(\xi)}{b(\xi)} - 1\right)\right)W(\xi) = 0 \tag{21}$$

where ζ_0 represents the value of ζ of an isotropic homogeneous beam ($E_2 / E_1 = 1$).

2.3 Solution procedure using the differential quadrature method

In the differential quadrature method (DQM), the partial derivatives, appearing in the partial differential equation of a function, with respect to a space variable at a given interpolation point is approximated as a weighted linear sum of the function values at all chosen interpolation points. Thus, the differential quadrature method allows transforming the governing differential equation into a set of equivalent simultaneous equations. This is done by replacing the partial derivative by equivalent weighting coefficients.

The first partial derivative is equivalent to a weighting coefficient matrix,

$$\frac{\partial}{\partial x} \equiv [A]_x = \left[\frac{\partial}{\partial x}N_0\right].[N_0]^{-1} \tag{22}$$

Similarly, the second, third and fourth order partial derivatives are expressed as:

$$\frac{\partial^2}{\partial x^2} \equiv \frac{\partial}{\partial x}\frac{\partial}{\partial x} = [A]_x[A]_x = [B]_x$$
$$\frac{\partial^3}{\partial x^3} \equiv \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x} = [A]_x[A]_x[A]_x = [C]_x \tag{23}$$
$$\frac{\partial^4}{\partial x^4} \equiv \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x} = [A]_x[A]_x[A]_x[A]_x = [D]_x$$

Where, $[N_0]$ is a matrix developed from Chebyshev polynomials with interpolation points defined as:

$$T_n(x_i) = \cos\left(\frac{n}{\cos(x_i)}\right) \tag{24}$$

The discretization points are obtained from the Chebyshev-Gauss-Lobatto interpolation points which are located in the local coordinate system within the interval [-1, 1]

$$x_i = \frac{1}{2}\left[1 - \cos\left(\pi\frac{(i-1)}{(n-1)}\right)\right],\quad i = 1,2,3,\ldots.,n \tag{25}$$

Note that *n is* the number of interpolation points. The differential vibration equation in discrete form, with dimensional parameters, is given by the expression:

$$\sum_{j=1}^{n} D_{ij}w_j + 2\frac{b'(x)}{b(x)}\sum_{j=1}^{n} C_{ij}w_j + \left(\frac{b''(x)}{b(x)} + \frac{(ea_0)^2\omega^2}{\zeta}\right)\sum_{j=1}^{n} B_{ij}w_j + \left(\frac{2(ea_0)^2\omega^2}{\zeta}\frac{b'(x)}{b(x)}\right)\sum_{j=1}^{n} A_{ij}w_j + \left(\frac{\omega^2}{\zeta}\left((ea_0)^2\frac{b''(x)}{b(x)} - 1\right)\right)w_j = 0;\quad i = 1,2,3\ldots\ldots,n \tag{26}$$

With nondimensional formulation parameters:

$$\sum_{j=1}^{n} D_{ij}W_j + 2\frac{b'(\xi)}{b(\xi)}\sum_{j=1}^{n} C_{ij}W_j + \left(\frac{b''(\xi)}{b(\xi)} + g^2\varpi^2\right)\sum_{j=1}^{n} B_{ij}W_j + \left(2g^2\varpi^2\frac{b'(\xi)}{b(\xi)}\right)\sum_{j=1}^{n} A_{ij}W_j + \left(\varpi^2\left(g^2\frac{b''(\xi)}{b(\xi)} - 1\right)\right)W_j = 0;\quad i = 1,2,3\ldots\ldots,n \tag{27}$$

In order to implement the various boundary conditions in Equations 21 and 27, the governing equations are rewritten in Table 1.

Table 1 Different boundary conditions of E-FGM beam with the nondimensional formulation

Simply supported - simply supported (S–S)

$$W=0\ ,\ \left(d^2W/d\xi^2\right)=0\ \ at\ \ \xi=0$$
$$W=0\ ,\ \left(d^2W/d\xi^2\right)=0\ \ at\ \ \xi=1$$
$$W_{1j}=0\ ,\ and\ \ \sum_{j=1}^{n}B_{1j}W_j=0$$
$$W_{nj}=0\ ,\ and\ \ \sum_{j=1}^{n}B_{nj}W_j=0$$

Simply supported - clamped supported (S–C)

$$W=0\ ,\ \left(d^2W/d\xi^2\right)=0\ \ at\ \ \xi=0$$
$$W=0\ ,\ \left(dW/d\xi\right)=0\ \ \ at\ \ \xi=1$$
$$W_{1j}=0\ ,\ and\ \ \sum_{j=1}^{n}B_{1j}W_j=0$$
$$W_{nj}=0\ ,\ and\ \ \sum_{j=1}^{n}A_{nj}W_j=0$$

Clamped - clamped (C–C)

$$W=0\ ,\ \left(dW/d\xi\right)=0\ \ at\ \ \xi=0$$
$$W=0\ ,\ \left(dW/d\xi\right)=0\ \ at\ \ \xi=1$$
$$W_{1j}=0\ ,\ and\ \ \sum_{j=1}^{n}A_{1j}W_j=0$$
$$W_{nj}=0\ ,\ and\ \ \sum_{j=1}^{n}A_{nj}W_j=0$$

Clamped – Free (C–F)

$$W=0\ ,\ \left(dW/d\xi\right)=0\ \ at\ \ \xi=0$$
$$\left(d^2W/d\xi^2\right)=0\ \ and\ \ \left(d^3W/d\xi^3\right)=0\ \ \ at\ \ \xi=1$$
$$W_{1j}=0\ ,\ and\ \ \sum_{j=1}^{n}A_{1j}W_j=0$$
$$\sum_{j=1}^{n}B_{nj}W_j=0\ ,\ and\ \ \sum_{j=1}^{n}C_{nj}W_j=0$$

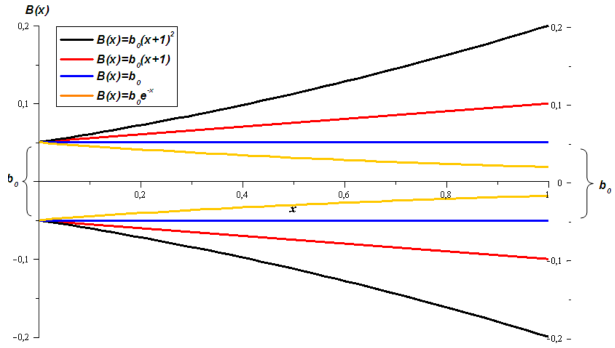


Fig. 1 Variation of uniform and nonuniform width of beam for the different shape function

Table 2 First three Nondimensional natural frequencies $\overline{\omega}=\frac{\bar{\omega}_n}{\sqrt{\zeta_0}}$ of nonuniform beam with exponential width for $B(\xi)=b_0e^{-\delta\xi}$

E_1 / E_2	δ	Mode number	S-S Present DQM (N=11)	S-S Cem Ece *et al.* 2007	C-C Present DQM (N=55)	C-C CemEce *et al.* 2007
1	0	1	9.869	9.869	22.373	22.373
		2	39.478	39.478	61.671	61.673
		3	88.850	88.826	120.900	120.903
	1	1	9.773	9.773	22.511	22.512
		2	39.570	39.570	61.857	61.860
		3	88.968	88.970	121.104	121.108
	2	1	9.487	9.487	22.936	22.938
		2	39.852	39.852	62.419	62.423
		3	89.405	89.405	121.722	121.723

Table 3 First three Nondimensional natural frequencies ω_n of uniform width $B(\xi)=b_0$ with E-FGM beams

E_1 / E_2	Mode number	S-S Present DQM (N=11)	S-S Yang *et al.* 2008	C-C Present DQM (N=55)	C-C Yang *et al.* 2008
0.2	1	9.273	9.270	21.094	21.020
	2	37.091	37.090	57.941	57.940
	3	83.454	84.280	113.587	113.590
1	1	9.869	9.870	22.373	22.370
	2	39.478	39.480	61.671	61.670
	3	88.826	88.830	120.900	120.900
5	1	9.273	9.270	21.094	21.020
	2	37.091	37.090	57.941	57.940
	3	83.451	84.280	113.587	113.590

3. Material gradient of E-FGM beams

Consider an elastic E-FGM beam with the different shape function of width. The Young's modulus, the Poisson's ratio and mass density of the beams vary only in the thickness direction with exponential function (E-FGM) as follow:

$$E(z)=E_0e^{\alpha z}\ ;\ \rho(z)=\rho_0e^{\alpha z}\ ;\ \mu(z)=\mu_0e^{\alpha z} \tag{28}$$

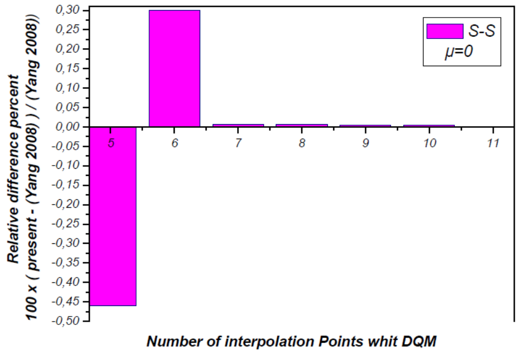


Fig. 2 Relative errors for nondimensional frequency parameters with number of interpolation points for a simply-simply (S-S) boundary condition

Table 4 Dimensionless fundamental frequency for simply supported-simply supported (S- S) beams for different non-local parameters and various number of interpolation points

	Number of interpolation points				
g/100	DQM (N=4)	DQM (N=5)	DQM (N=7)	DQM (N=10)	(Reddy 2007) (Pradhan 2009)
0	10.6667 (+8.08%)	9.8240 (-0.46%)	9.8697 (+0.007%)	9.8696 (+0.006%)	9.869
0.5	10.3931 (+7.88%)	9.5912 (-0.44%)	9.6348 (+0.008%)	9.6347 (+0.007%)	9.634
1	10.1396 (+7.70%)	9.3743 (-0.43%)	9.4160 (+0.011%)	9.4159 (+0.009%)	9.415
1.5	9.9038 (+7.52%)	9.1714 (-0.43%)	9.2114 (+0.004%)	9.2113 (+0.003%)	9.211
2	9.6836 (+7.37%)	8.9812 (-0.42%)	9.0195 (+0.005%)	9.0195 (+0.005%)	9.019
2.5	9.4776 (+6.60%)	8.8023 (-0.42%)	8.8392 (+0.002%)	8.8392 (+0.002%)	8.839
3	9.2841 (+6.60%)	8.6338 (-0.40%)	8.6693 (+0.003%)	8.6693 (+0.003%)	8.669
3.5	9.1021 (+6.98%)	8.4745 (-0.39%)	8.5089 (+0.011%)	8.5088 (+0.009%)	8.508
4	8.9303 (+6.87%)	8.3237 (-0.39%)	8.3570 (+0.011%)	8.3569 (+0.010%)	8.356
4.5	8.7679 (+6.77%)	8.1807 (-0.38%)	8.2130 (+0.012%)	8.2129 (+0.011%)	8.212
5	8.6141 (+6.66%)	8.0449 (-0.38%)	8.0761 (+0.001%)	8.0761 (+0.001%)	8.076

where, E_0, ρ_0 and μ_0 are the values of the Young's modulus, mass density, and the Poisson's ratio at the midplane of the beam. α, is a constant defining the material property variation along the thickness direction, for isotropic homogeneous beam $\alpha = 0$. The top surface of the beam is aluminium with the material parameters: $E_1 = 70GPa, \mu_1 = 0.33, \rho_1 = 2780kg/m^3$.

The results presented in Table 2 allow us to validate the first three normalized natural frequencies of isotropic beam with uniform and exponential width, see Figure 1.

The natural frequencies of isotropic and E-FGM beam without nonlocal effects, obtained in the present study, proved to be in excellent agreement with those given by (Cem Ece *et al.* 2007) in Table 2 and (Yang *et al.* 2008) in Table 3, for cases of simply-supported (S-S), clamped-clamped (C-C) beams.

The results in Table 4 of the natural frequencies with nonlocal effects for simply-supported (S-S) beam converge at ten interpolation points (N = 10) with a relative error of 0.7% with respect to the results of (Reddy 2007) and (Pradhan and Murmu 2009). These results are similar with the nonlocal parameter between 0 and 0.05. The convergence of the natural frequencies without nonlocal effects for case simply-supported (S-S) beam is provided at N = 10 grid interpolation points with relative errors equal to 0.01% with Yang *et al.* 2008) see Figure 2. For cases of clamped-clamped (C-C), clamped-free (C-F) and simply supported- clamped (S-C) beams, the convergence is provided at N = 55 grid interpolation points with relative errors equal to -2.8710-4, 0.15 and 0.029 %, respectively, as shown in Figure 3

Increasing the nonlocal parameter in Table 5 to 8 leads to a considerable decrease in the free vibration frequency.

These conclusions are obtained for different beam widths and mechanical properties of E-FGM materials, and also for different boundary conditions, except for the clamped-free (C-F) case where this variation is linear and weak. For the boundary condition of a simply-supported (S-S) beam, the width variation does not affect the free vibrations; the results obtained by varying the width are almost identical, as shown in Figure (4.a). With regard to the other boundary conditions (S-C), (C-C) and (C-F), the difference is found to be significant see Figures 4b, 4c and 4d

Figure 5 illustrates the variation of the nondimensional fundamental frequency as a function of the modulus ratio, for various nonlocal parameters and boundary conditions. It is clearly noted that as the modulus ratio of the E-FGM beam increases, the nondimensional fundamental frequency, for the different nonuniform beam widths, goes up to eventually reach a maximum value in this isotropic material ($E_2//E_1=1$). Beyond this value, the vibrational frequency starts decreasing gradually for all boundary conditions. Therefore, increasing the nonlocal parameter causes the vibrational frequency to decrease under all boundary conditions.

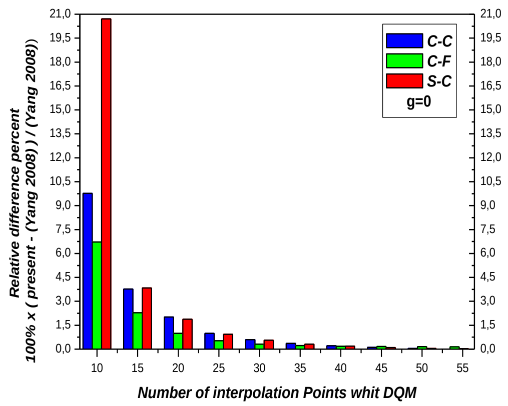


Fig. 3 Relative errors for nondimensional frequency parameters with number of interpolation points for the boundary conditions: clamped – clamped(C-C), clamped – free (C-F) and simply supported- clamped (S-C)

Table 5 Nondimensional fundamental frequencies of nonuniform E-FGM beam with non-local parameters and Simply supported-Simply supported (S-S) boundary conditions. Interpolation points N=11

E_2/E_1	Nonlocal parameter g^2	$B(\xi)=b_0$	$B(\xi)=b_0.e^{-\xi}$	$B(\xi)=b_0.(\xi+1)$	$B(\xi)=b_0.(\xi+1)^2$
0.2	0	9.2726	9.1817	9.2309	9.1438
	1/100	8.8463	8.8018	8.8093	8.7346
	2/100	8.4739	8.4361	8.4407	8.3757
	5/100	7.5875	7.5630	7.5621	7.5165
1	0	9.8696	9.7729	9.8251	9.6927
	1/100	9.4159	9.3301	9.3765	9.2589
	2/100	9.0195	8.9425	8.9841	8.8784
	5/100	8.0761	8.0169	8.0489	7.9676
2	0	9.7526	9.1817	9.2309	9.1438
	1/100	8.8463	8.8018	8.8093	8.7346
	2/100	8.4739	8.4361	8.4407	8.3757
	5/100	7.5875	7.5630	7.5621	7.5165

Table 6 Nondimensional fundamental frequencies of nonuniform E-FGM beam with non-local parameters and Simply Supported-Clamped (S-C) boundary conditions. Interpolation points N=55

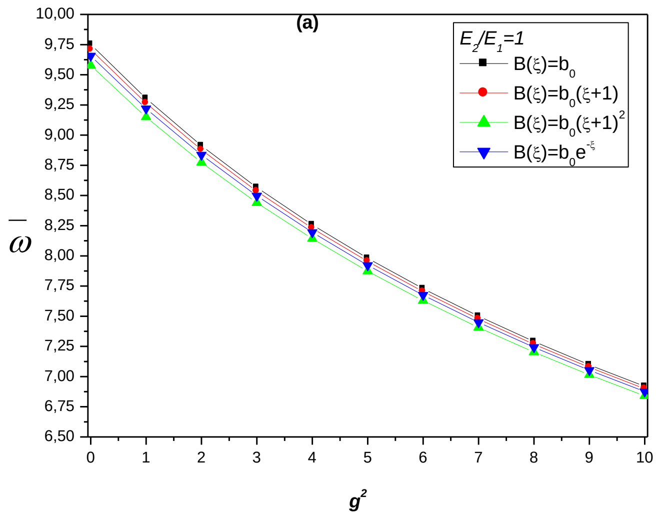
E_2/E_1	Nonlocal parameter g^2	$B(\xi)=b_0$	$B(\xi)=b_0.e^{-\xi}$	$B(\xi)=b_0.(\xi+1)$	$B(\xi)=b_0.(\xi+1)^2$
0.2	0	14.5733	13.5924	15.1018	15.7637
	1/100	13.7978	12.8707	14.3134	14.9706
	2/100	13.1323	12.2508	13.6350	14.2846
	5/100	11.5906	10.8137	12.0576	12.6771
1	0	15.4182	14.3783	16.0441	16.6796
	1/100	14.5992	13.6163	15.2080	15.8420
	2/100	13.8962	12.9616	14.4883	15.1172
	5/100	12.2671	11.4432	12.8144	13.4183
5	0	14.5733	13.5924	15.1018	15.7637
	1/100	13.7978	12.8707	14.3134	14.9706
	2/100	13.1323	12.2508	13.6350	14.2846
	5/100	11.5906	10.8137	12.0576	12.6771

Table 7 Nondimensional fundamental frequencies of nonuniform E-FGM beam with non-local parameters And Clamped-Clamped (C-C) boundary conditions. Integration points N=55

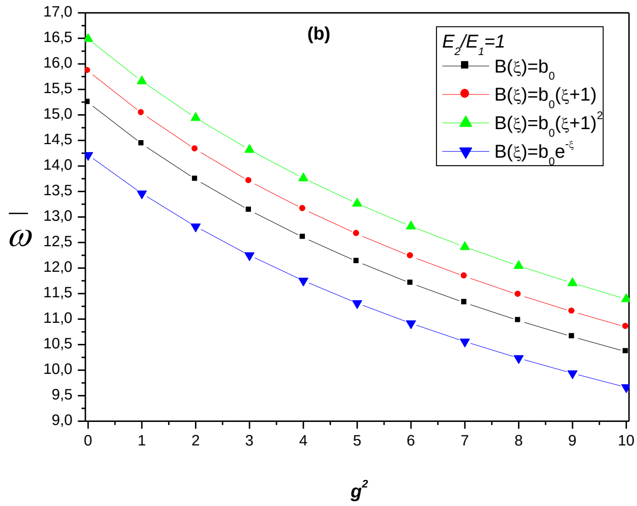
E_2/E_1	Nonlocal parameter g^2	$B(\xi)=b_0$	$B(\xi)=b_0.e^{-\xi}$	$B(\xi)=b_0.(\xi+1)$	$B(\xi)=b_0.(\xi+1)^2$
0.2	0	21.0940	21.2797	20.8797	20.8196
	1/100	20.1306	19.9624	19.7383	19.1851
	2/100	19.0532	18.9610	18.6522	18.1583
	5/100	16.6308	16.6721	16.2282	15.8497
1	0	22 .3739	22.5117	22.3938	22.1186
	1/100	21.4266	21.2476	20.2109	20.4203
	2/100	20.2798	20.1818	19.8531	19.3273
	5/100	17.7015	17.7455	17.2730	16.8701
5	0	21.0940	21.2797	20.8797	20.8196
	1/100	20.1306	19.9624	19.7383	19.1851
	2/100	19.0532	18.9610	18.6522	18.1583
	5/100	16.6308	16.6721	16.2282	15.8497

Table 8 Nondimensional fundamental frequencies of nonuniform E-FGM beam with non-local parameters and Clamped-Free (C-F) boundary conditions. Integration points N=55

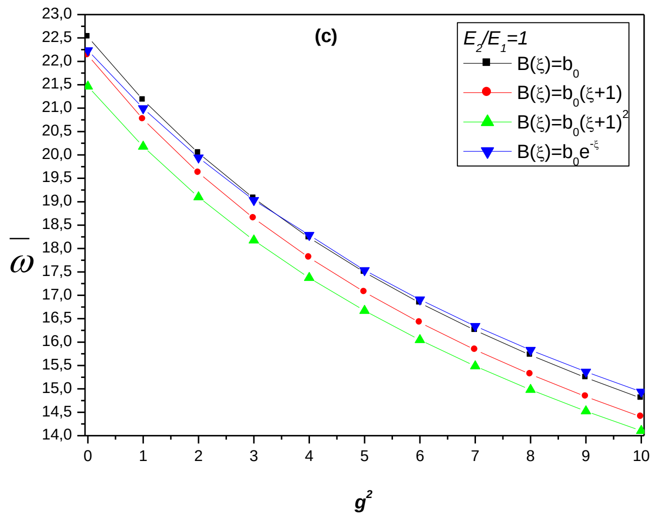
E_2/E_1	Nonlocal parameter g^2	$B(\xi)=b_0$	$B(\xi)=b_0.e^{-\xi}$	$B(\xi)=b_0.(\xi+1)$	$B(\xi)=b_0.(\xi+1)^2$
0.2	0	3.3169	4.4668	2.6679	2.1452
	1/100	3.3023	4.4347	2.6597	2.1407
	2/100	3.2871	4.4014	2.6512	2.1360
	5/100	3.2379	4.2960	2.6233	2.1206
1	0	3.5160	4.7349	2.8397	2.2739
	1/100	3.5006	4.7009	2.8310	2.2692
	2/100	3.4844	4.6656	2.8218	2.2642
	5/100	3.4322	4.5538	2.7922	2.2479
5	0	3.3169	4.4668	2.6679	2.1452
	1/100	3.3023	4.4347	2.6597	2.1407
	2/100	3.2871	4.4014	2.6512	2.1360
	5/100	3.2379	4.2960	2.6233	2.1206



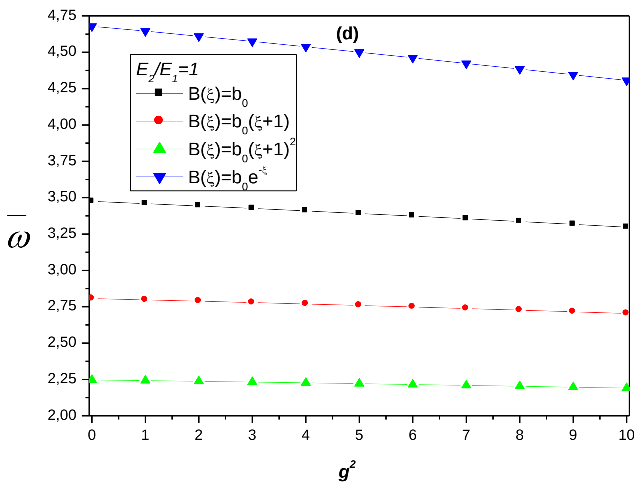
(a) Simply- Simply(S-S)



(b) Simply supported- Clamped (S-C)

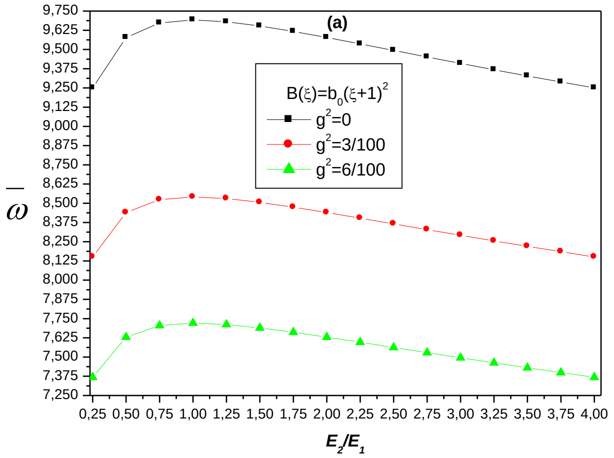


(c) Clamped -Clamped (C-C)

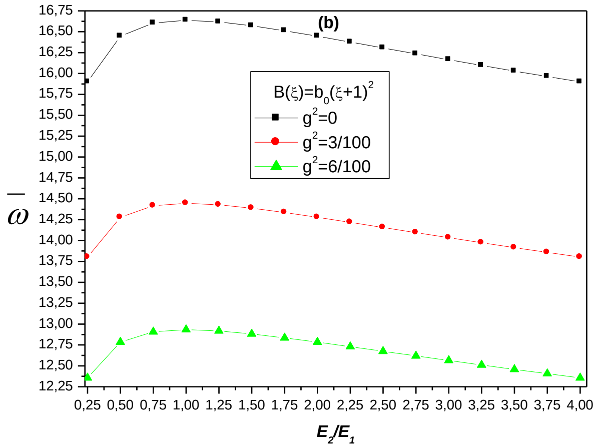


(d) Clamped - Free (C-F)

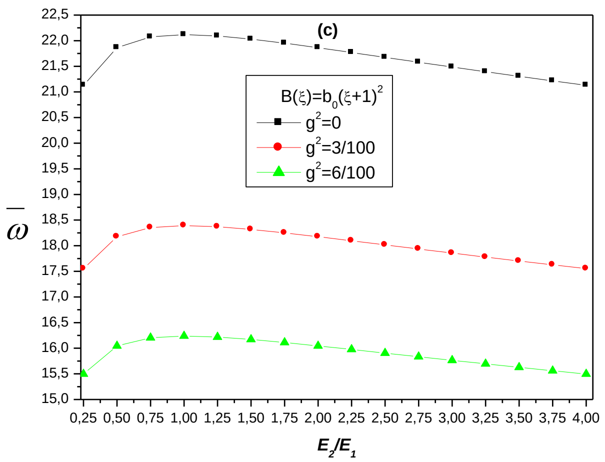
Fig. 4 Variation of nondimensional fundamental frequency with nonlocal parameters and nonuniform width for different boundary conditions:



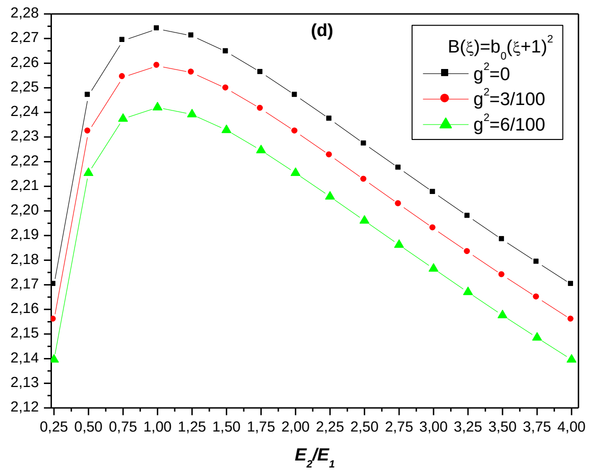
(a) Simply- Simply(S-S)



(b) Simply supported- Clamped (S-C)



(c) Clamped -Clamped (C-C)



(d) Clamped - Free (C-F)

Fig. 5 Variation of nondimensional fundamental frequency with nonlocal parameters and nonuniform width for different boundary conditions:

5. Conclusions

This paper is an attempt to study the application of the differential quadrature method to the longitudinal vibrations of functionally graded material (E-FGM) nonuniform beams, based on the nonlocal elasticity theory. Several beams with nonuniform widths, and for different boundary conditions, were studied while taking into account various parameters such as the material characteristics of the beam, geometric variation of the width, scale effect and nonlocal effect. For various non-uniform widths of the beams, and under different boundary conditions, the increase of the nonlocal parameter causes the nondimensional vibrational frequency to decrease. The non-dimensional frequency varies proportionally with the ratio modulus up to the reference value for isotropic material. Beyond this value of the ratio this frequency evolves in an inversely proportional manner.

The present study demonstrated the effectiveness of the differential quadrature method in solving the differential equation with variable coefficients for E-FGM beams with nonuniform widths and subjected to free vibrations, while applying the nonlocal theory. The effectiveness of this method is proved by the rapid convergence of the results and their concordance with the analytical solutions reported in previous research works.

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