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Buckling of laminated composite plates with elastically restrained boundary conditions

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Abstract. A unified solution is presented for the buckling analysis of rectangular laminated composite plates with elastically restrained edges. The plate is subjected to biaxial in-plane compression, and the boundary conditions are simulated by employing uniform distribution of linear and rotational springs at all edges. The critical values of buckling loads and corresponding modes are calculated based on classical lamination theory and using the Ritz method. The deflection function is defined based on simple polynomials without any auxiliary function. The verifications of the current study are carried out with available combinations of classic boundary conditions in the literature. Through parametric study with a wide range of spring factors with some classical as well as some not classical boundary conditions, competency of the present model of boundary conditions is proved.

Keywords: laminates; arbitrary boundary conditions; buckling; linear and rotational springs; elastic constraints

1. Introduction

The laminated composite plate appears to be a promising candidate for different engineering applications in various industries. The investigation about buckling characteristics of structures made of laminated composites has gained increasing attention over the past few decades. In the stability of structures, the reactions of the edges are one of the most important factors, which have a great effect on the critical load of structures (Belkacem *et al*. (2018)). Usually, the boundary conditions of the structural elements are simplified to classical models such as clamped, simply-supported, sliding, and free, while in practical situations, we deal with elastic (or general) boundary conditions. The classic forms of boundary conditions are extensively examined in various problems including buckling analysis. However, there is a need to present a unified solution of the buckling problems which covers all classic and elastically restrained boundary conditions.

Compared to the number of studies on the buckling of simply-supported (Bohlooly and Mirzavand 2016, 2017, Dietrich *et al*. 1978, Matsunaga 2005, Mirzavand and Bohlooly 2015) or at least two simply-supported opposite edges (Ghasemabadian and Saidi 2017, Hosseini-Hashemi *et al*. 2015, Liew *et al*. 1996, Lopatin and Morozov 2009, Thai and Kim 2011, Xiang *et al*. 1996, Yu and Wang 2008), there exist limited investigations on the critical load of plates with general boundary conditions. Ungbhakorn and Singhatanadgid 2006, Singhatanadgid and Jommalai 2016, and Shufrin *et al*. 2008a,b presented the stability of laminated composite plates with symmetric lay-up configuration using extended Kantorovich method. This method determines the critical load of plates with an iterative procedure in which the function of lateral deflection changes in each iteration. Raju *et al*. (2012) studied buckling analysis of variable angle tow plates by numerical methodology based on differential quadrature method (DQM). In this context, Golmakani and Far (2017) reported the buckling behavior of double-layered graphene plates according to DQM. The DQM methodology is an efficient numerical method for the solution of differential equations by simplifying to algebraic equations at discrete points, although the drawback of this method is its complexity and time consumption. Besides, some researchers have solved this challenge by Ritz methods using various admissible functions. One of these functions is sinusoidal, which leads to quick calculations for limited cases of boundary conditions like simply-supported, clamped, and a combination of them. In this field, Swaminathan and Naveenkumar (2014) studied refined higher-order models for the stability analysis of plates. Iyengar and Karasimhan (1965) solved the buckling problem of an isotropic plate with simply-supported and clamped edges. Gunda (2013) presented thermal buckling and postbuckling of isotropic square plate. Baucke and Mittelstedt (2015) determined the critical load of symmetrically laminated composite plates. Panda and Ramachandra (2010) studied the buckling behavior of symmetric cross-ply plates for in-plane parabolic loads using higher-order shear deformation theory. Mijušković *et al*. (2014, 2015) presented the critical load of isotropic plate subjected to different nonlinear in-plane loads. Also, Latifi *et al*. (2013) calculated the critical load of functionally graded plates using various sinusoidal functions for lateral displacement of non-boundary points and boundary points. The advantage of this difference in deflection (for non-

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boundary and boundary points) is that the edges of plate can be combined of clamped, simply-supported, and free. However, it increases the amount of computation. Some researchers have implemented Chebyshev polynomials into the Ritz method to derive the critical load of plates with different classic boundary conditions such as simply-supported, free, sliding, and clamped. Chebyshev polynomials should be used with special auxiliary functions in each case to fulfill essential boundary conditions. In this subject, Shukla *et al*. (2005) studied the critical load of generally laminated composite plate based on the first-order shear deformation theory. Li and Pan Iu (2010) worked on three-dimensional buckling of isotropic plate using triplicate forms of Chebyshev polynomials for all displacement coefficients. Tang and Wang (2011) investigated the influences of in-plane parabolic loads on the critical load of laminated composite plates with symmetric lay-up configuration. Tamijani and Kapania (2012) developed the Ritz-Chebyshev procedure for the critical load of isotropic plate with eccentrically and/or concentrically curvilinear stiffeners. Uymaz and Aydogdu (2013a, b) presented the critical load of functionally graded plates under in-plane normal and shear loads, respectively. Also, Aydogdu and Aksencer (2018) extended the work on the cross-ply lay-up sequence under in-plane loads with linear variations based on different shear deformation theories. Mirzaei and Kiani (2016) and Kiani (2017) obtained the critical load of functionally graded plates reinforced with carbon nanotubes under thermal and mechanical loads. In Refs. (Aydogdu and Aksencer (2018); Kiani (2017); Li and Pan Iu (2010); Mirzaei and Kiani (2016); Shukla *et al*. (2005); Tamijani and Kapania (2012); Tang and Wang (2011); Uymaz and Aydogdu (2013a, b)), the suggestion of an auxiliary function has critical importance because the accuracy of the predictions depends on how the auxiliary functions can fulfill the essential boundary conditions. On the other hand, the investigators should be aware that 256 different auxiliary functions can be constituted for a plate with four edges and considering four classic boundary conditions like simply-supported, free, sliding, and clamped. So, the finding of suitable auxiliary functions has grueling operations.

The boundary conditions of plates in real applications are different from well-known classic forms. The real boundaries are elastically restrained on complicated shapes of foundations, which can be modeled by simple springs. With considering of this vital topic, Khov *et al*. (2009); Li (2000, 2004); Zhang and Li (2009); Song *et al*. (2015); Jin *et al*. (2014); Qu *et al*. (2013); Song *et al*. (2020); Wang *et al*. (2017) presented the vibrations of beams, plates, and cylindrical shells with linear and rotational springs on edges. Some researchers modeled the interaction of foundations by using linear and nonlinear springs. In this regard, Civalek and Acar (2007) presented the bending analysis of Mindlin plates on two-parameter elastic foundations. Akgoz and Civalek (2011) worked on the vibrational analysis of laminated composite plates with considerations of elastic foundations. Bohlooly and Malekzadeh Fard (2019); Bohlooly and Mirzavand (2018); Bohlooly *et al*. (2018); Mirzavand and Bohlooly (2019) presented the effects of elastic foundations on the buckling and postbuckling of plates and panels. However, the foregoing literature review indicates that no investigations have been performed on the buckling analysis using elastic springs as boundary conditions.

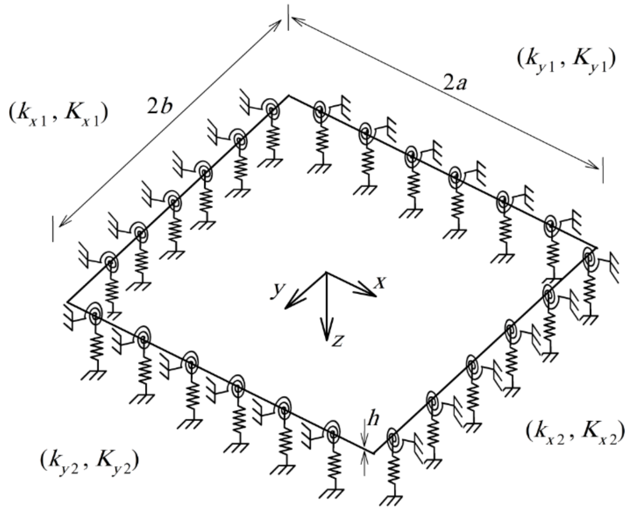


Fig. 1 The geometry of plate with elastic constraints

In the current research, the buckling behavior of a laminated composite plate with symmetric lay-up configuration is investigated. The formulations are derived by the classical theory of plates and shells. The solutions are based on Ritz method. The boundary conditions of the plate are simulated by eight uniform springs (four linear and four rotational) which can be used for all classic and elastically restrained edges. A unified lateral displacement is introduced by simple polynomials without any auxiliary functions. Results for various combinations of classic boundary conditions are verified with the available data in the literature. A range of parameters like different spring factors, lay-up configurations, and geometry are investigated on the critical loads and mode shapes.

2. Fundamental equations

Figure 1 presents a schematic of the rectangular plate. The dimensions of the plate are $2a \times 2b \times h$. The plate is composed of laminated composite with a symmetric lay-up configuration. The laminated composite consists of N plies, whereas all orthotropic plies have equal thickness. A ply is reinforced by parallel unidirectional fibers. These fibers are embedded in a polymer base matrix. A fixed Cartesian coordinate system (x,y,z) is used with the origin on the center of plate. In order to generate arbitrary boundary conditions, two linear and rotational springs (k_i,K_i) are distributed along each edge. The boundary conditions of the plate are controlled by eight springs which can make different combinations of elastic and/or classic boundary conditions such as free, sliding, simply-supported, and clamped.

The laminated composite plate is subjected to in-plane biaxial compression of N_x and N_y where $N_y = RN_x$.

In order to solve the buckling problem and determine the critical value of N_x, the Ritz method is applied in this paper. According to Ritz method (Brush *et al*. (1975)), the total potential energy consists of the strain energy of the structure, the potential energy of external loads, and the energy of line springs along each edge. The integral form of strain energy of a rectangular plate is presented as follow (Brush *et al*. (1975))

$$U_s = \frac{1}{2}\iiint \{\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy}\}\, dxdydz \tag{1}$$

According to classical lamination theory for thin plates (Bohlooly and Mirzavand (2015)) and in the case of symmetrically laminated composites, the strain energy can be expressed in terms of deflection function w as (Ungbhakorn and Singhatanadgid (2006))

$$\begin{aligned} U_s = \frac{1}{2}\int_{-a}^{+a}\int_{-b}^{+b} \{ & D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 \\ & +2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 4D_{66}\left(\frac{\partial^2 w}{\partial y\,\partial x}\right)^2 \\ & +4\left(D_{16}\frac{\partial^2 w}{\partial x^2} + D_{26}\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial^2 w}{\partial y\,\partial x}\}\mathrm{d}y\,\mathrm{d}x \end{aligned} \tag{2}$$

in which the bending stiffness D_{ij} (i,j = 1,2,6) are (Cetkovic and Vuksanovic (2011); Reddy (2004))

$$\begin{aligned} & D_{ij} = \sum_{k=1}^{N}\left(\bar{Q}_{ij}\right)_k \frac{1}{3}(z_k{}^3 - z_{k-1}{}^3) \\ & -\frac{h}{2} < z_k < +\frac{h}{2} \end{aligned} \tag{3}$$

where

$$\begin{Bmatrix} \bar{Q}_{11} \\ \bar{Q}_{12} \\ \bar{Q}_{22} \\ \bar{Q}_{16} \\ \bar{Q}_{26} \\ \bar{Q}_{66} \end{Bmatrix}_k = \begin{Bmatrix} c^4 & 2c^2s^2 & s^4 & 4c^2s^2 \\ c^2s^2 & c^4+s^4 & c^2s^2 & -4c^2s^2 \\ s^4 & 2c^2s^2 & c^4 & 4c^2s^2 \\ c^3s & cs^3-c^3s & -cs^3 & -2cs(c^2-s^2) \\ cs^3 & c^3s-cs^3 & -c^3s & 2cs(c^2-s^2) \\ c^2s^2 & -2c^2s^2 & c^2s^2 & (c^2-s^2)^2 \end{Bmatrix}_k \times \begin{Bmatrix} Q_{11} \\ Q_{12} \\ Q_{22} \\ Q_{66} \end{Bmatrix} \tag{4}$$

$$Q_{11} = \frac{E_{11}}{(1-\nu_{12}\nu_{21})},\ Q_{22} = \frac{E_{22}}{(1-\nu_{12}\nu_{21})}$$

$$Q_{12} = \frac{\nu_{21}E_{11}}{(1-\nu_{12}\nu_{21})},\ Q_{66} = G_{12},\ \nu_{21} = \nu_{12}\frac{E_{22}}{E_{11}}$$

$$c = \cos\theta,\ s = \sin\theta$$

The potential energy of external loads may be calculated as (Ungbhakorn and Singhatanadgid (2006))

$$\begin{aligned} W = \frac{1}{2}\int_{-a}^{+a}\int_{-b}^{+b}\{ & N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 \\ & + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\}\mathrm{d}y\,\mathrm{d}x \end{aligned} \tag{5}$$

in which N_x and N_y are uniform in-plane forces in x and y directions. Also, the shear force in xy plane (N_{xy}) is zero.

The energy of linear springs along each edge is

at $x = -a$,

$$U_1 = \frac{1}{2}\int_{-b}^{+b} k_{x1}\left(w\,|_{x=-a}\right)^2 \mathrm{d}y \tag{6}$$

at $x = +a$,

$$U_2 = \frac{1}{2}\int_{-b}^{+b} k_{x2}\left(w\,|_{x=+a}\right)^2 \mathrm{d}y \tag{7}$$

at $x = -b$,

$$U_3 = \frac{1}{2}\int_{-a}^{+a} k_{y1}\left(w\,|_{y=-b}\right)^2 \mathrm{d}x \tag{8}$$

at $x = +b$,

$$U_4 = \frac{1}{2}\int_{-a}^{+a} k_{y2}\left(w\,|_{y=+b}\right)^2 \mathrm{d}x \tag{9}$$

In addition, the energy of rotational springs along each edge is

at $x = -a$,

$$U_5 = \frac{1}{2}\int_{-b}^{+b} K_{x1}\left(\left.\frac{\partial w}{\partial x}\right|_{x=-a}\right)^2 \mathrm{d}y \tag{10}$$

at $x = +a$,

$$U_6 = \frac{1}{2}\int_{-b}^{+b} K_{x2}\left(\left.\frac{\partial w}{\partial x}\right|_{x=+a}\right)^2 \mathrm{d}y \tag{11}$$

at $x = -b$,

$$U_7 = \frac{1}{2}\int_{-a}^{+a} K_{y1}\left(\left.\frac{\partial w}{\partial y}\right|_{y=-b}\right)^2 \mathrm{d}x \tag{12}$$

at $x = +b$,

$$U_8 = \frac{1}{2}\int_{-a}^{+a} K_{y2}\left(\left.\frac{\partial w}{\partial y}\right|_{y=+b}\right)^2 \mathrm{d}x \tag{13}$$

The total potential energy of the laminated composite plate is the sum of strain energy, potential energy due to external loads, and energy of linear and rotational springs

$$\Pi = U_s - W + \sum_{i=1}^{8} U_i \tag{14}$$

According to Ritz method in previous studies (Aydogdu and Aksencer (2018); Baucke and Mittelstedt (2015); Gunda (2013); Iyengar and Karasimhan (1965); Kiani (2017); Latifi *et al.* (2013); Li and Pan Iu (2010); Mijušković *et al.* (2014, 2015); Mirzaei and Kiani (2016); Panda and Ramachandra (2010); Shukla *et al.* (2005); Tamijani and Kapania (2012); Tang and Wang (2011); Uymaz and Aydogdu (2013a, b)), an approximation of deflection function $w(x,y)$ is suggested in terms of unknown constant coefficients C_{ij}. In such studies, a function like Chebyshev basis polynomials should be used with auxiliary functions to satisfy essential boundary conditions. A serious problem of these studies is that the deflection function depends on boundary conditions. In other words, a new deflection function is required, and all calculations should be repeated in each case. In current study, we suppose that all four edges are free, and boundary conditions appear in the analysis with spring factors. Therefore, there is not any constraint to be satisfied, and a unified deflection function covers all possible cases.

In the current study, a set of separable orthogonal polynomials is considered as an approximation of deflection function

$$w(x,y) = \sum_{m=0}^{M}\sum_{n=0}^{N} C_{mn} \times P_m(\zeta) \times P_n(\eta) \tag{15}$$

where dimensionless parameters ζ and η are x/a and y/b, respectively. Also, the parameters C_{mn} are the undetermined coefficients. According to previous studies, P_m and/or P_n are polynomials which are used from familiar functions like Chebyshev (Mirzaei and Kiani (2016); Uymaz and Aydogdu (2013a)), Legendre (Feng and Xu (2016)), Hermite (Nosier *et al.* (1994)), etc. For instance, the six polynomials of Chebyshev (Mirzaei and Kiani (2016)) are

$$\begin{aligned} P_0(\zeta) &= 1 \\ P_1(\zeta) &= \zeta \\ P_2(\zeta) &= 2\zeta^2 - 1 \\ P_3(\zeta) &= 4\zeta^3 - 3\zeta \\ P_4(\zeta) &= 8\zeta^4 - 8\zeta^2 + 1 \\ P_5(\zeta) &= 16\zeta^5 - 20\zeta^3 + 5\zeta \\ P_6(\zeta) &= 32\zeta^6 - 48\zeta^4 + 18\zeta^2 - 1 \end{aligned} \tag{16}$$

In Eq. (16), the fifth and sixth polynomials have terms (*e.g.* ζ^4 and ζ^3) which are repeated in previous polynomials. These terms add useless complexity to the problem. In the current study, we use a simplified form of polynomials as below

$$\begin{aligned} P_0(\zeta) &= 1 \\ P_1(\zeta) &= \zeta \\ P_2(\zeta) &= \zeta^2 \\ &\vdots \\ P_m(\zeta) &= \zeta^m \end{aligned} \tag{17}$$

Substitution of Eq. (17) into Eq. (15) gives the deflection function as follow

$$\begin{aligned} w &= C_{00} + C_{10}\zeta + C_{20}\zeta^2 + C_{30}\zeta^3 + \cdots + C_{01}\eta \\ &+ C_{11}\zeta\eta + C_{21}\zeta^2\eta + C_{31}\zeta^3\eta + \cdots \\ &+ C_{02}\eta^2 + C_{12}\zeta\eta^2 + C_{22}\zeta^2\eta^2 + C_{32}\zeta^3\eta^2 \\ &+ \cdots + C_{MN}\zeta^M\eta^N \end{aligned} \tag{18}$$

Then, by substitution of deflection function in Eqs. (2, 5-13), integration, inserting the results into Π, and minimizing concerning the coefficients C_{mn} one obtains $(M+1)\cdot(N+1)$ algebraic equations. The matrix form of these equations can be written as

$$\mathbf{K} \cdot \mathbf{P} = 0 \tag{19}$$

where the matrix **K** has $(M+1)\cdot(N+1)$ rows and columns and $\mathbf{P} = [C_{00}, C_{01}, C_{02}, ..., C_{0N}, ..., C_{1N}, ..., C_{MN}]^T$. The lowest eigenvalue of the matrix **K** in Eq. (19) corresponds to the critical buckling load N_{xcr}. The eigenvector corresponding to the lowest eigenvalue indicates the buckling mode shape.

3. Numerical results

In this section, a parametric study is performed to clarify the influences of elastic constraints on the critical buckling load of symmetrically laminated composite. In this regard, a different number of polynomials in the series expansion can be considered (see in Eq. (15)). For example, in the case of $M = N = 9$, the matrix **K** in Eq. (19) has $(9+1)\cdot(9+1) = 100$ rows and columns.

3.1 Comparison examples

Unlikely, there are not reliable references on the buckling of different structures with elastically restrained boundary conditions. Therefore, the results of the current study are validated by classic boundary conditions such as simply-supported (S), clamped (C), sliding (X), and free (F). For instance, the rectangular plate with FXSC boundary conditions can be modeled according to Table 1 for corresponding springs.

Table 1 The stiffness of springs for a plate with FXSC boundary conditions

edge	linear spring [N/m^2]	rotational spring [N/rad·m]	B.C. of edge
$x = -a$	$k_{x1} = 0$	$K_{x1} = 0$	F
$y = -b$	$k_{y1} = 0$	$K_{y1} = 10^{10}$	X
$x = +a$	$k_{x2} = 10^{10}$	$K_{x2} = 0$	S
$y = +b$	$k_{y2} = 10^{10}$	$K_{y2} = 10^{10}$	C

Table 2 Convergence study and comparisons of buckling load factor δ_1 for an isotropic plate under uniaxial compression

$\delta_1 = N_x (2b)^2 / \pi^2 D$	SSSS	CSCS	CCCC
Mijušković *et al.* (2015)	4.000	6.7432	10.0759
FEM (ANSYS), Mijušković *et al.* (2015)	3.966	6.7388	10.0729
Present (M=N=6)	3.9997	6.7471	10.1065
Present (M=N=8)	3.9996	6.7430	10.0737
Present (M=N=10)	3.9995	6.7430	10.0733
Present (M=N=12)	3.9994	6.7430	10.0733
Present (M=N=14)	3.9993	6.7430	10.0733

In the first example, the values of dimensionless critical load $\delta_1 = N_x(2b)^2/(\pi^2 D)$ of an isotropic plate (E=200 GPa, ν=0.3) under uniaxial compression ($R = 0$) are calculated. In the case of a/b=1 the results are verified by Ritz method and FEM presentation of Mijušković *et al.* (2015) in Table 2. According to this table, the convergence study shows the required numbers of polynomials in the series expansion for predicting accurate calculations. Also, it is shown that good agreements can be seen for simply-supported, clamped, and combined conditions.

In the second example, the results of present study are compared with the Levy-type solution based on first-order shear deformation theory (FSDT) and classical lamination theory (CLT) of Thai and Kim (2011) for a square orthotropic plate under both uniaxial and biaxial compression (R=0,1). The results are shown in Table 3. The mechanical properties of ply are

$$E_{11} / E_{22} = 10, G_{12} / E_{22} = 0.5, \nu_{12} = 0.25 \tag{20}$$

The dimensionless critical load $\delta_2 = N_x (2b)^2 / (h^3 E_{22})$ agrees well with those of Thai and Kim (2011). In some cases, lower values of critical load corresponding to different mode shapes are calculated in the current study, which seems to be missed by Thai and Kim (2011).

In the third example, the comparisons of the critical load of cross-ply laminated composite plate $[0/90]_{2S}$ under uniaxial compression (R=0) is presented with the results of Ungbhakorn and Singhatanadgid (2006) in Table 4. The mechanical properties of composite plies are similar to the second example, and the critical loads are normalized by $\delta_3 = N_x (2b)^2 / (\pi^2 D_{22})$. In the study of Ungbhakorn and Singhatanadgid (2006), the calculations are based on extended Kantorovich method. The results of current investigations are well justified with those of Ungbhakorn and Singhatanadgid (2006).

In the last example, the critical load of a square symmetric angle-ply laminated composite plate [30/-30/30] with different boundary conditions are verified by the results of Liu *et al.* (2002); Shufrin *et al.* (2008b); Singhatanadgid and Jommalai (2016) in Table 5. The plate is subjected to uniaxial compression ($R = 0$), and the mechanical properties of plies are

$$E_{11} / E_{22} = 2.45, G_{12} / E_{22} = 0.48, \nu_{12} = 0.23 \tag{21}$$

The dimensionless form of critical load is $\delta_4 = 12(1-\nu_{12}\nu_{21})N_x (2b)^2 / (h^3 E_{11})$. According to Table 5, a good agreement is achieved.

3.2 Parametric study

The effects of different parameters on the stability of laminated composite plates with elastically restrained boundary conditions are studied. The mechanical properties for each ply of Graphite/epoxy laminated composite (Fard and Bohlooly (2017)) are

$$\begin{aligned} E_{11} &= 150 \text{ GPa}, E_{22} = 9 \text{ GPa} \\ G_{12} &= 7.1 \text{ GPa}, \nu_{12} = 0.3. \end{aligned} \tag{22}$$

The critical load is normalized according to the third example of verifications ($\delta = N_x (2b)^2/(\pi^2 D_{22})$). In Figs. 2-5, the critical loads of square symmetric cross-ply laminated composite plate [0/90/0] with length to thickness ratio of $2a/h$=100 under uniaxial compression (R = 0) are presented. Four linear springs have constant values (either zero or a very large value), and the rotational springs vary from null to large values in different conditions. In Fig. 6, the critical loads of square plate with the previous condition (lay-up configuration and uniaxial compression) are studied for different length to thickness ratios. In Fig. 7, the critical loads of square plate ($2a / h = 100$) with various lay-up configurations are examined for biaxial compression ($R = 1$). In Figs. 6 and 7, the plate has variable linear springs in $x = \pm a$, and other linear and rotational springs

Table 3 Comparisons of buckling load factor δ_2 for a square orthotropic plate under uniaxial and biaxial compression

$\delta_2 = N_x(2b)^2 / h^3E_{22}$	R	CSCS	SSCS	SSSS	FSCS	FSSS	FSFS
*Thai and Kim (2011)	0	35.5455	19.5929	11.1415	4.9710	4.0475	2.3122
**Thai and Kim (2011)	0	35.7845	19.6635	11.1628	4.9753	4.0498	2.3131
present	0	35.7841	19.6627	11.1622	4.9747	4.0492	2.3126
*Thai and Kim (2011)	1	20.1558	10.7273	5.5707	1.7733	1.0165	1.2745
**Thai and Kim (2011)	1	20.2904	10.7658	5.5814	1.7747	1.0169	1.2750
present	1	20.2877	10.7648	5.5810	1.7745	1.0168	1.2747
***present	1	12.9426	10.6990	-	-	-	0.8228

*FSDT, **CLT, ***Critical values

Table 4 Comparisons of buckling load factor δ_3 for a square symmetric cross-ply laminated composite plate $[0/90]_{2S}$ under uniaxial compression

$\delta_3 = N_x(2b)^2 / \pi^2D_{22}$	SSSF	SCSF	SCSC	CCCF	CSSC	CSCS
Ungbhakorn and Singhatanadgid (2006)	2.0396	2.2294	7.8342	7.8494	6.5576	8.9399
present	2.0395	2.2293	7.8325	7.8478	6.5562	8.9397

Table 5 Comparisons of buckling load factor δ_4 for a symmetric laminated composite plate [30°/-30°/30°] under uniaxial compression

$\delta_4 = 12(1-\nu_{12}\nu_{21})N_x(2b)^2 / h^3E_{11}$	SSSS	CCCC	SCSC	SCCS
Liu *et al.* (2002)	25.36	62.77	47.77	39.08
Shufrin *et al.* (2008b)	25.25	62.05	47.04	38.54
Singhatanadgid and Jommalai (2016)	25.31	62.02	47.12	38.59
present	25.2481	62.0459	47.0418	38.5429

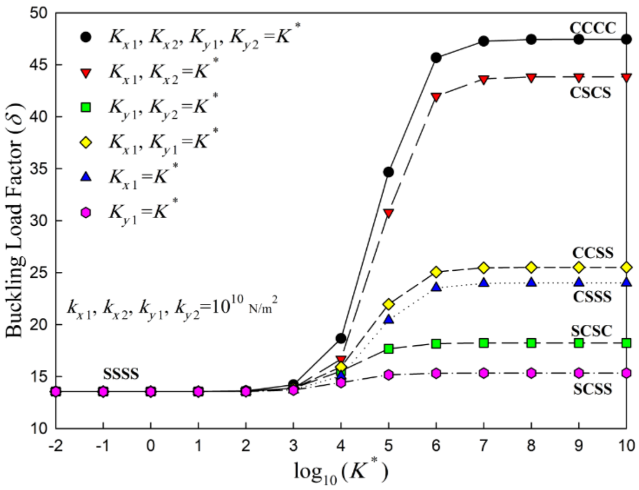


Fig. 2 Effects of rotational springs on the buckling loads for the plate with stiff linear springs in all edges, R=0

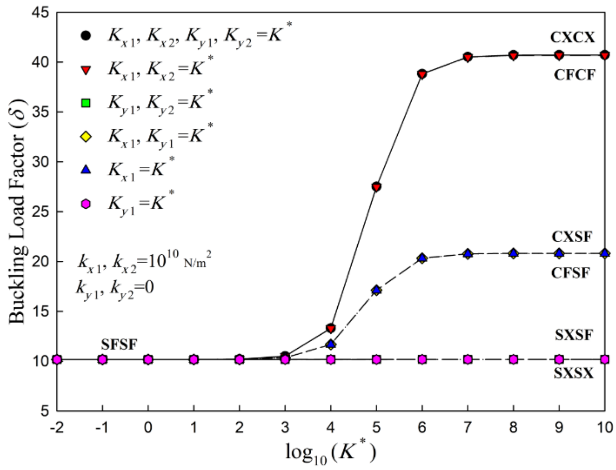


Fig. 3 Effects of rotational springs on the buckling loads for the plate with stiff linear springs in $x=\pm a$, R=0

are considered to have constant values. In order to ensure the correctness of performance of elastic constraints in buckling mode shapes, some selected mode shape associated with specific spring factors are provided in Figs. 8-13. These figures are associated with some results in Figs. 2-5, and 7.

Figs. 2-5 show that the rotational springs have substantial influences on the critical loads. While the rotational springs increase from 10^{-2} to 10^{10}, firstly, a slight increment can be seen for the critical loads. Then, a rapid enhancement happens within the range of 10^3 to 10^7, and finally remains unaltered. In the first and final stages, the edges act as classic boundary conditions. Furthermore, the influence of the rotational springs in $x=\pm a$ is higher than those in $y=\pm b$. This result can be interpreted by the condition of uniaxial compression, in which the rotational springs in $x=\pm a$ increase the moments of resistance against bending. It should be noticed that the effects of the rotational springs in $y=\pm b$ are insignificant when the linear springs of these edges are zero (Fig. 3).

It is observed from Fig. 6 that the effect of variations of linear springs in $x=\pm a$ from 10^{-2} to 10^{10} on the critical

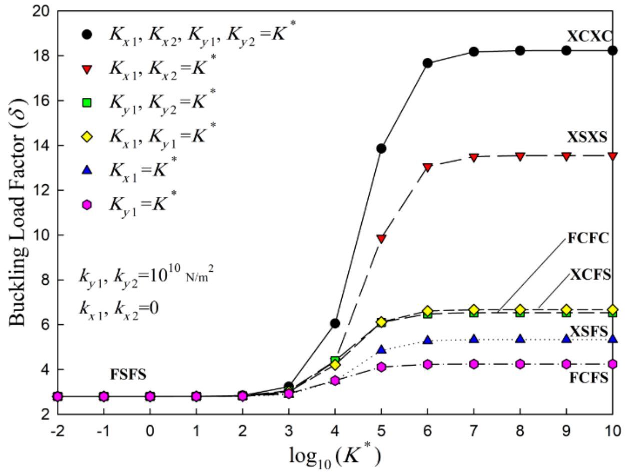


Fig. 4. Effects of rotational springs on the buckling loads for the plate with stiff linear springs in $y=\pm b$, R=0

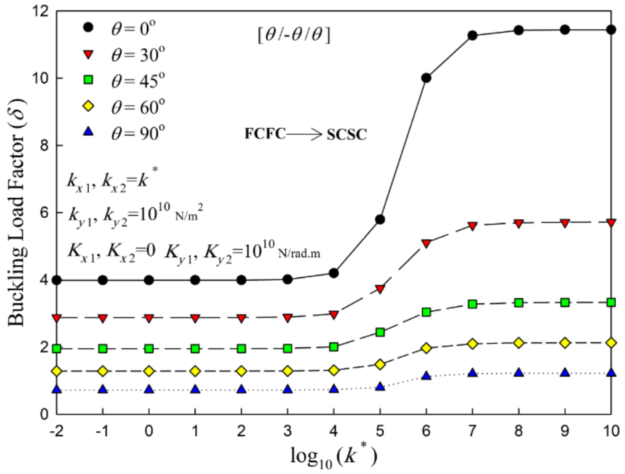


Fig. 7 Effects of linear springs in $x=\pm a$ on the buckling loads for the plates with different lay-up configurations, R=1

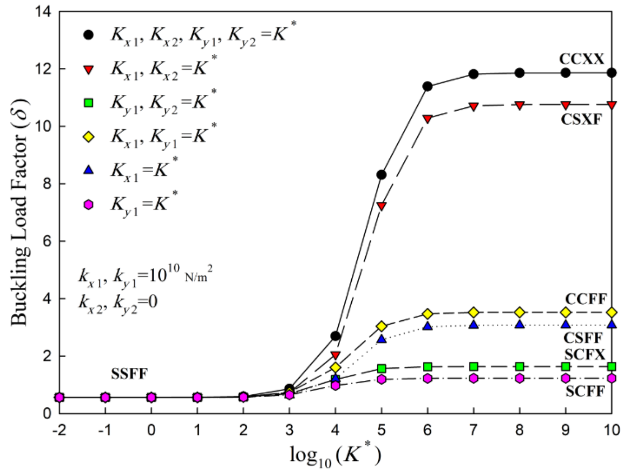


Fig. 5 Effects of rotational springs on the buckling loads for the plate with stiff linear springs in $x=-a$ and $y=-b$, R=0

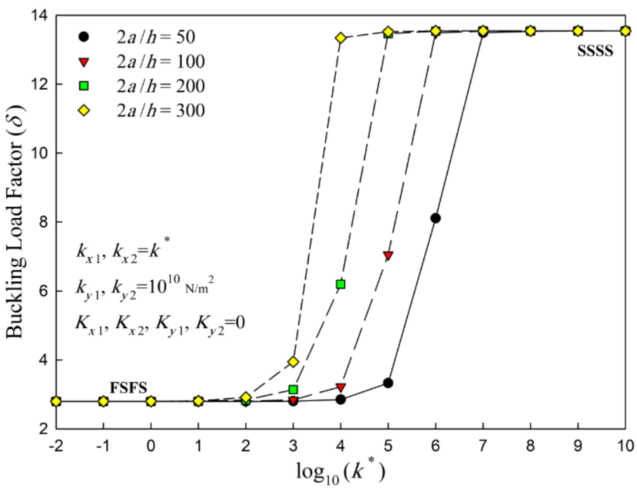
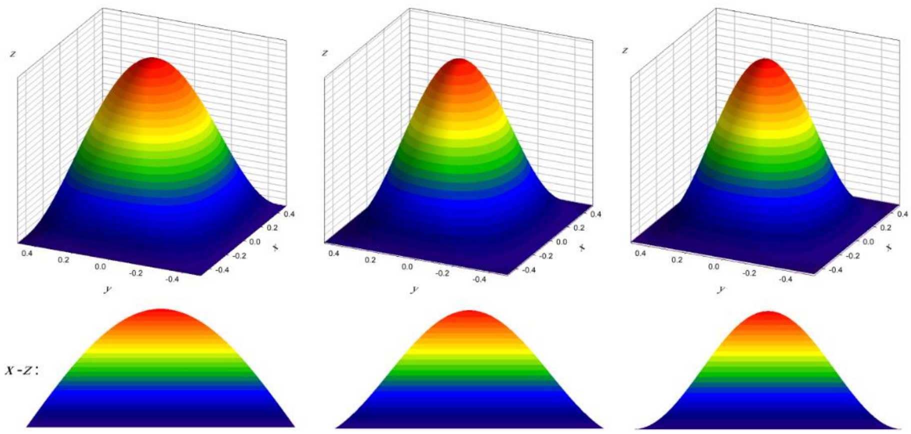


Fig. 6 Effects of linear springs in x=±a on the buckling loads for the plates with different length to thickness ratios, R=0

load are similar to the influence of changes in rotational springs (Figs. 2-5) and the critical loads grow from first to final values. These growths start in lower spring stiffnesses for thinner plates. Obviously, this is related to the bending matrix D_{ij} of laminated composite plates. Hence, a high value of D_{ij} makes plate to has reaction in high elastic restraints.

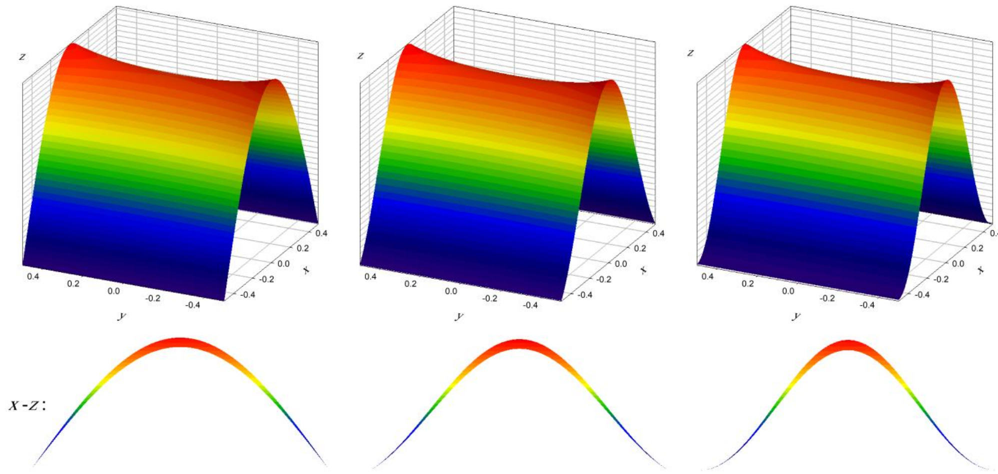
It can be found in Fig. 7 that the influence of the angle of fibers on the critical load is significant. The critical load of the plate $[0]_3$ is the highest. When the linear springs in x=±a are sufficiently large, this influence is more pronounced. Note that for $[90]_3$ configuration, different values of linear springs in x=±a have no considerable change on the critical load. The reason for this insignificant effect may be addressed to the low value of D_{11}, where highly stiffened boundary conditions can't improve the stability of the weak structure.

The corresponding buckling modes for the plate in first case of Fig. 2 (black circles) are presented in Fig. 8. As can be seen, the essential boundary conditions of clamped edges are satisfied by increasing the stiffness of rotational springs from 10^2 to 10^8. By comparing the buckling modes of SSSS and CCCC, it can be shown that the lateral deflection of plate with SSSS edges increases smoothly from edges to center. In contrast, a sudden jump of lateral deflection is occurred for CCCC plate. The buckling modes for the plate in the second case of Fig. 3 (red triangles) are provided in Fig. 9. In this case, the edges x=±a change fromsimply-supported to clamped and the edges y=±b remain free. At the first glance, the buckling modes of plates with SFSF and CFCF edges are very similar. However, the critical buckling load factor of CFCF plate is approximately four times higher than corresponding factor of SFSF plate. The reason for this event is high potential of clamped edges against buckling. Figure 10 depicts the buckling modes of third case of Fig. 4 (green squares), where the boundary conditions of the plate change from FSFS to FCFC. As th e two edges change from simply-supported to clamped conditions, the critical buckling load factor grows twice.



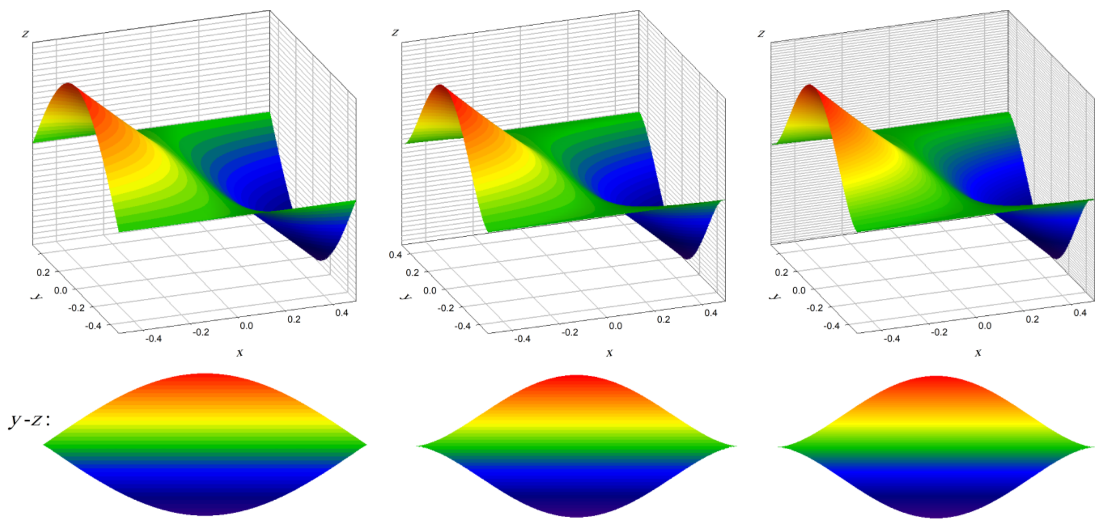
$\log K^*=2$ $\log K^*=5$ $\log K^*=8$

Fig. 8 The variations of mode shapes versus spring factors corresponding to first case in Fig. 2



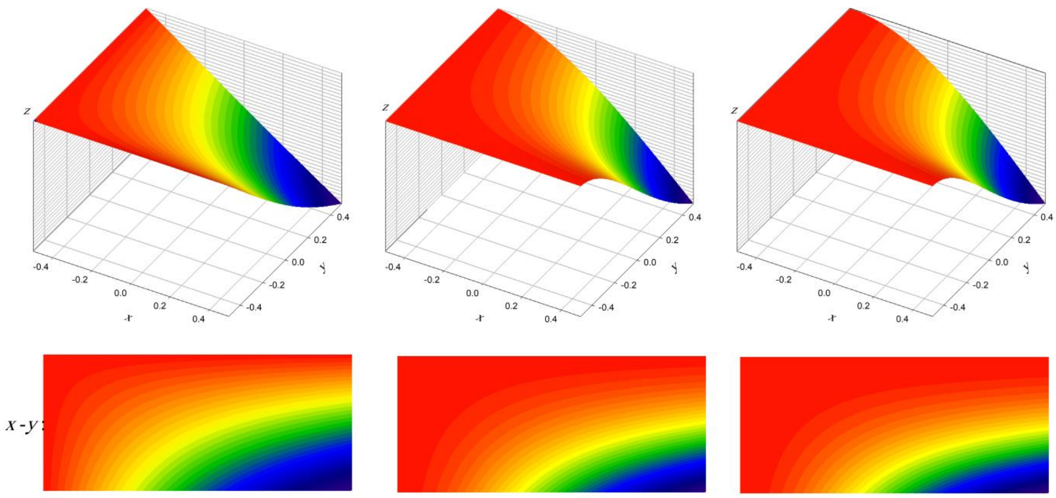
$\log K^*=2$ $\log K^*=5$ $\log K^*=8$

Fig. 9 The variations of mode shapes versus spring factors corresponding to second case in Fig. 3



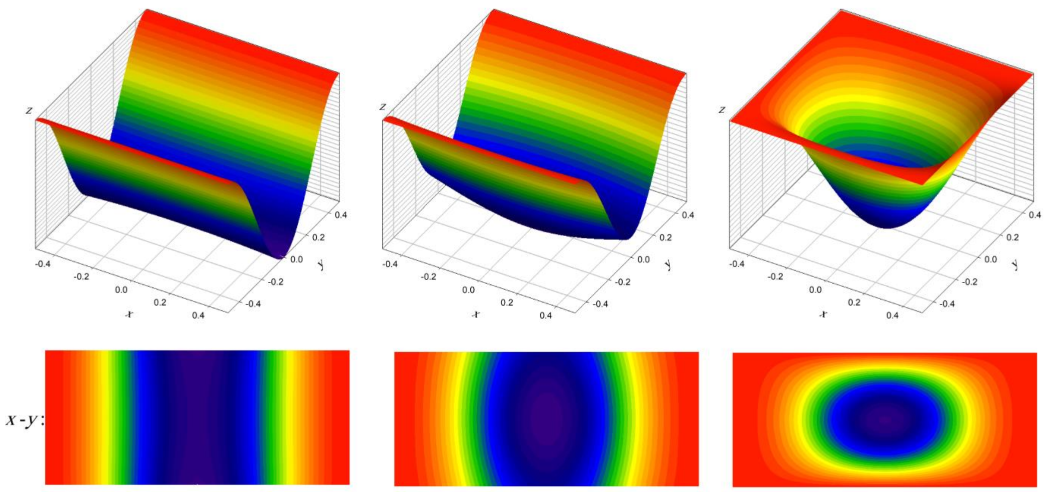
$\log K^*=2$ $\log K^*=5$ $\log K^*=8$

Fig. 10 The variations of mode shapes versus spring factors corresponding to third case in Fig. 4



$\log K^*=2$ $\log K^*=5$ $\log K^*=8$

Fig. 11 The variations of mode shapes versus spring factors corresponding to forth case in Fig. 5



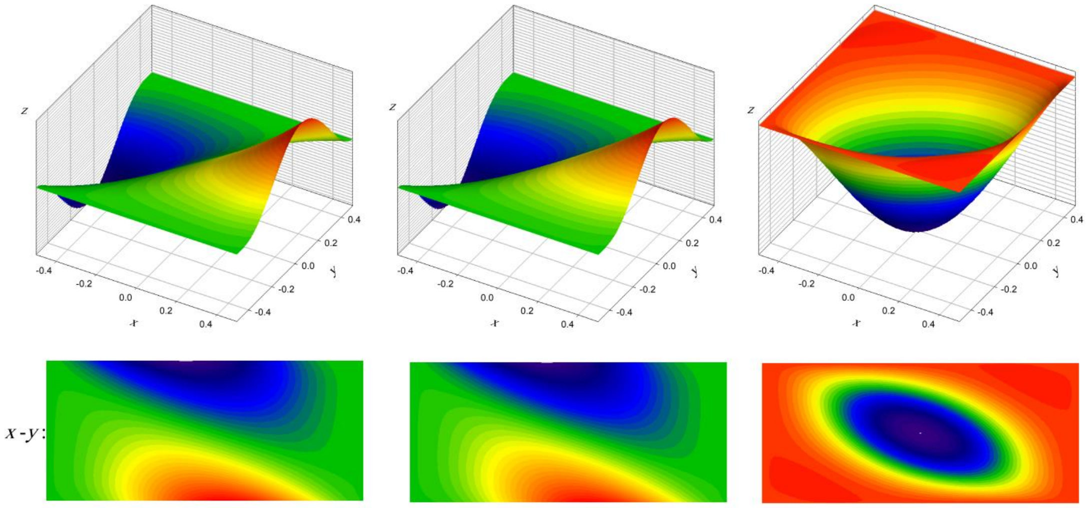
$\log k^*=0$ $\log k^*=5$ $\log k^*=10$

Fig. 12 The variations of mode shapes versus spring factors corresponding to first case in Fig. 7

Here, the compression is on $x=\pm a$ edges, and the clamped edges of $y=\pm b$ have no significant resistance versus bending of plate. The results of forth case of Fig. 5 (yellow rhombus) are demonstrated in Fig. 11 which the edges $x=+a$ and $y=+b$ remain free and opposite edges change from simply-supported to clamped conditions. It can be seen that a large area of plate with CCFF edges have no considerable deflection. In contrast, the plate with SSFF edges has alinear and continuous increment of deflection from $x=-a$ $x=-a$, $y=-b$ to $x=+a$, $y=+b$. Finally, the buckling modes of plate in first and third cases of Fig. 7 with lay-up configuration of $[0]_3$ and [45/-45/45] are presented in Figs. 12 and 13, respectively. In both figures, the deflections of plate in $x=\pm a$ edges decrease to zero by increasing of the stiffness of linear springs. Also Fig. 13 reveals that the mode shapes significantly change with lay-up configuration. In other words, the deflection contours show that the wave lines have non-zero angles with respect to the edges of plate.

4. Conclusions

This article focused on the buckling behavior of rectangular symmetric composite plates with new types of boundary conditions (elastically restrained edges modeled by a set of linear and rotational springs) by utilizing Ritz method and simple polynomials. Compared with the methods in the literature, the present analysis may be considered as a unified solution to deal with buckling analysis of plates under different combinations of classic and elastically restrained boundary conditions with appropriate accuracy. It was illustrated that both springs (*i.e.* linear and rotational) have substantial influence on the critical load, especially the springs of loaded edges. The influence of springs on the critical load changes during the whole range of variations of spring factors. Usually, the critical buckling loads grow between two values, which these values are associated to classic boundary conditions. It is found that the growths start in lower spring stiffnesses



$\log k^*=0$ $\log k^*=5$ $\log k^*=10$

Fig. 13 The variations of mode shapes versus spring factors corresponding to third case in Fig. 7

for thinner plates. Besides, it is concluded that the lay-up configuration is more significant on the buckling load, when the linear springs are sufficiently large.

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