Structural Engineering and Mechanics, *Vol. 73, No. 3 (2020) 331-337*
DOI: https://doi.org/10.12989/sem.2020.73.3.331

Structural health monitoring through nonlinear frequency-based approaches for conservative vibratory systems

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(Received March 21, 2019, Revised September 16, 2019, Accepted October 25, 2019)

Abstract. This paper proposes a new approximate analytical solution for highly nonlinear vibration of mechanical systems called Hamiltonian Approach (HA) that can be widely use for structural health monitoring systems. The complete procedure of the HA approach is studied, and the precise application of the presented approach is surveyed by two familiar nonlinear partial differential problems. The nonlinear frequency of the considered systems is obtained. The results of the HA are verified with the numerical solution using Runge-Kutta's [RK] algorithm. It is established the only one iteration of the HA leads us to the high accurateness of the solution.

Keywords: Hamiltonian Approach (HA); nonlinear vibration; analytical solution; Runge-Kutta's algorithm

1. Introduction

Structural health monitoring (SHM) refers to the process of implementing a damage detection and characterization strategy for engineering structures such as bridges, buildings, plates and mechanical systems and etc. An important application of Structural Health Monitoring is to improve the reliability of existing infrastructure. In structural health monitoring, the main challenging is how to evaluate the exact behavior of the structures.

One of the characteristics of the mechanical systems or general physical phenomena is natural frequency. As long as the frequency of the structures and systems are showing their internal situation, there is a need to extract the exact frequency of the systems. Depending on the oscillation experiences of any forces or not, the frequency of the system can be called forced frequency and natural frequency. Natural frequency is the vibrational response of the oscillation when it is not subjecting to any external force. Resonance occurs when the forced vibration frequency is equal to the and natural frequency of an oscillation, which increases the amplitude of the system. It has become essential to obtain the natural frequency of the system in the design procedure.

In a mechanical system, there are different elements used and combined to form it. These elements are behaving linear o nonlinear based on their force-displacement relationship. Developing a finite element modeling is a challenging task that could be able to predict the response of the structures accurately. The mathematical models and governing equations that represent the response of the system are usually presented as nonlinear partial differential equations.

Finding an exact solution for nonlinear partial differential equations is still a challenge in science. Generally, it is not an easy task to prepare an analytical solution for nonlinear problems. Some scientists are studied different approximate analytical solutions to solve these kinds of equations in recent years. Based on the recently developed approaches, fuzzy Hermite-Hadamard inequality proposed and applied by Avazpour *et al.* (2016). Their proposed approach is effectively able to strong problems such (s,m)-Godunova-Levin functions via fractional integrals. This can be applied to find new inequalities for exclusive means such as geometric, arithmetic and logarithmic means which will be used to show the existence of ordinary differential equations and partial differential equations. They also extended their work by investigating some problems in fractional cases that have been considered widely (Avazpour,2018). They made inquiries about fractional Ostrowski inequality for the functions whose derivatives are preinvex, prequasiinvex and logarithmic preinvex.

Olvera *et al.* (2015) proposed a new approach called enhanced multistage homotopy perturbation method (EMHPM). The method applied to find the approximate solution of differential equations with strong nonlinearities.

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http://www.techno-press.com/journals/sem&subpage=7
ISSN: 1225-4568 (Print), 1598-6217 (Online)

Hieu (2018) developed the governing equation of the nonlinear free vibration of microbeams. He considered the post-buckling of the pinned–pinned and clamped–clamped end conditions. An approximate solution using weighted averaging value was applied, and the results were discussed. Zhang *et al.* (2018) studied the analytical model of the vibration of rotors. They considered the interaction between the transition force, axial vibration, contact stiffness, and speed fluctuation is investigated. Joubari (2015), worked on the nonlinear vibration of tapered beams by Modified Iteration Perturbation Method (MIPM). Kalita *et al.* (2016), focused on the vibration of a thick rectangular plate and they tried to obtain the natural frequency of it and discussing the effects of rotary inertia. They made a comparison between the numerical, analytical and experimental data. Mohammadian *et al.* (2017) applied global residue harmonic balance method (GRHBM) for strongly nonlinear vibration problems.

Omran *et al.* (2013) proposed the variational approach method as a powerful method to obtain the nonlinear frequencies of the nonlinear vibration problem with only one iteration.

Panigrahi *et al.* (2014) studied the harmonic balance for large-amplitude vibrations in snap-through structures. The problem that they worked on was a nonlinear Duffing oscillator for its snap-through behavior at large-amplitude vibrations.

In the last recent years, many authors have been working and developing analytical, semi-analytical approaches and also advanced nonlinear modelling of structures (Avazpour 2018, Avazpour *et al.* 2016, Avci *et al.* 2018, Bayat *et al.* 2012, Bennamia,2016, Dehghani *et al.* 2019, Ellis *et al.* 2018, Ganji *et al.* 2009, Golafshani *et al.* 2013, Hashemiparast *et al.* 2008, He,2008,2010, Hieu *et al.* 2018, Hosseinzadeh *et al.* 2018, Jalili *et al.* 2018, Jamshidi *et al.* 2012, Javed *et al.* 2018, Jiang *et al.* 2016, Johnson *et al.* 2017, Joubari *et al.* 2014, Kahya *et al.* 2018, Kalita *et al.* 2015,2016, Kia *et al.* 2016, Kutanaei *et al.* 2019, Martínez *et al.* 2019, Nayfeh,2011, Olvera *et al.* 2014, Sarokolayi *et al.* 2016, Tan *et al.* 2017, Tee *et al.* 2018,Wu *et al.* 2019, Zangooee *et al.* 2019, Zhao *et al.* 2019, Akgoz *et al.* 2014, Sun *et al.* 2016, Lu *et al.* 2019, Mackie *et al.* 2019, Martínez *et al.* 2019, Kroworz *et al.* 2019, Deng *et al.* 2018, Feng *et al.* 2019, Zhuang *et al.* 2019, Suresh *et al.* 2019, Bayat *et al.* 2017a,b, 2015).

In this paper, a new approach, called Hamiltonian Approach (HA) is studied and applied to well-known nonlinear partial differential equations. The HA does not need any small parameters and any linearizations. Therefore, firstly the basic idea of the HA is presented, and the procedure of the Runge-Kutta's algorithm is described in detail. Applications of the HA are examined on two problems and the results are compared with numerical solutions. It has been demonstrated that the HA can be a powerful approach to obtain the natural frequency of the nonlinear conservative systems.

2. The basic idea of Hamiltonian Approach (HA)

Recently, He (2010) has proposed the Hamiltonian approach to overcome the shortcomings of the energy balance method. This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, consider the following general oscillator;

$$\ddot{\theta} + f\ (\theta, \dot{\theta}, \ddot{\theta}) = 0 \tag{1}$$

With initial conditions:

$$\theta(0) = A, \qquad \dot{\theta}(0) = 0. \tag{2}$$

Oscillatory systems contain two critical physical parameters, i.e. the frequency ω and the amplitude of oscillation A. It is easy to establish a variational principle for Eq. (1), which reads;

$$J(\theta) = \int_0^{T/4} \left\{ -\frac{1}{2}\dot{\theta}^2 + F(\theta) \right\} dt \tag{3}$$

Where T is the period of the nonlinear oscillator, $\partial F / \partial \theta = f$.

In the Eq (3), $\frac{1}{2}\dot{\theta}^2$ is kinetic energy and $F(\theta)$ potential energy, so the Eq (3) is the least Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads:

$$H(\theta) = \frac{1}{2}\dot{\theta}^2 + F(\theta) = \text{constant} \tag{4}$$

From Eq. (4), we have:

$$\frac{\partial H}{\partial A} = 0 \tag{5}$$

Introducing a new function, $\bar{H}(\theta)$, defined as:

$$\bar{H}(u) = \int_0^{T/4} \left\{ \frac{1}{2}\dot{\theta}^2 + F(\theta) \right\} dt = \frac{1}{4}TH \tag{6}$$

Eq. (5) is, then, equivalent to the following one

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial T} \right) = 0 \tag{7}$$

or

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial (1/\omega)} \right) = 0 \tag{8}$$

From Eq. (8) we can obtain approximate frequency–amplitude relationship of a nonlinear oscillator.

3. The basic idea of Runge-Kutta's Method

For the numerical approach to verify the analytic solution, the fourth RK (Runge-Kutta) method has been used. This iterative algorithm is written in the form of the following formulation:

$$\dot{\theta} = f\ (t, \theta), \quad \theta(t_0) = \theta_0 \tag{9}$$

θ is an unknown function of time t which we would like to approximate. Then the RK4 method is given for this problem as below:

$$\theta_{n+1} = \theta_n + \tfrac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$
$$t_{n+1} = t_n + h. \tag{10}$$

For n = 0, 1, 2, 3, . . . , using

$$\begin{aligned} k_1 &= f\left(t_n, \theta_n\right), \\ k_2 &= f\left(t_n + \tfrac{1}{2}h,\ \theta_n + \tfrac{1}{2}hk_1\right), \\ k_3 &= f\left(t_n + \tfrac{1}{2}h,\ \theta_n + \tfrac{1}{2}hk_3\right), \\ k_4 &= f\left(t_n + h,\ \theta_n + hk_3\right). \end{aligned} \tag{11}$$

Where θ_{n+1} is the RK4 approximation of $\theta(t_{n+1})$. The fourth-order Runge-Kutta method requires four evaluations of the right-hand side per step *h*.

4. Applications

In this section, we will present three examples to illustrate the applicability, accuracy, and effectiveness of the proposed approach.

4.1 Example 1

In this example, we consider the following nonlinear oscillator (Nayfeh 2011):

$$\left(\frac{1}{12}l^2 + r^2\theta^2\right)\ddot{\theta} + r^2\theta\dot{\theta}^2 + r\,g\,\theta cos\left(\theta\right) = 0 \tag{12}$$

With the boundary conditions of:

$$\theta(0) = A, \qquad \dot{\theta}(0) = 0 \tag{13}$$

In order to apply the Hamiltonian approach method to solve the above problem, the Taylor's series expansion for $\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$ and by some manipulation in Eq. (19) we can re-write Eq. (12) in the following form.

$$\left(\frac{1}{12}l^2 + r^2\theta^2\right)\ddot{\theta} + r^2\theta\dot{\theta}^2 + r\,g\,\theta\left(1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4\right) = 0 \tag{14}$$

The Hamiltonian of Eq. (14) is constructed as

$$H = -\tfrac{1}{24}l^2\dot{\theta}^2 - \tfrac{1}{2}\dot{\theta}^2\theta^2 r^2 + \tfrac{1}{2}g\,r\,\theta^2 - \tfrac{1}{8}g\,r\,\theta^4 + \tfrac{1}{144}g\,r\,\theta^6 \tag{15}$$

Integrating Eq. (15) concerning t from 0 to $T/4$, we have;

$$\bar{H} = \int_0^{T/4}\left(\begin{array}{l} -\tfrac{1}{24}l^2\dot{\theta}^2 - \tfrac{1}{2}\dot{\theta}^2\theta^2 r^2 + \tfrac{1}{2}g\,r\,\theta^2 \\ -\tfrac{1}{8}g\,r\,\theta^4 + \tfrac{1}{144}g\,r\,\theta^6 \end{array}\right)dt \tag{16}$$

Assume that the solution can be expressed as

$$\theta(t) = A\cos(\omega t) \tag{17}$$

Substituting Eq. (17) into Eq. (16), we obtain

$$\begin{aligned} \bar{H} &= \int_0^{T/4}\left(\begin{array}{l} -\frac{1}{24}\omega^2 l^2 A^2\sin^2(\omega t) - \frac{1}{2}\omega^2 r^2 A^4\sin^2(\omega t)\cos^2(\omega t) \\ +\frac{1}{2}g\,r\,A^2\cos^2(\omega t) - \frac{1}{8}g\,r\,A^4\cos^4(\omega t) \\ +\frac{1}{144}g\,r\,A^6\cos^6(\omega t) \end{array}\right)dt \\ &= \int_0^{\pi/2}\left(\begin{array}{l} -\frac{1}{24}\omega^2 l^2 A^2\sin^2 t - \frac{1}{2}\omega^2 r^2 A^4\sin^2 t\cos^2 t \\ +\frac{1}{2}g\,r\,A^2\cos^2 t - \frac{1}{8}g\,r\,A^4\cos^4 t + \frac{1}{144}g\,r\,A^6\cos^6 t \end{array}\right)dt \\ &= \frac{5}{4608}\frac{A^6\pi g r}{\omega} - \frac{1}{32}\pi\omega r^2 A^4 + \frac{1}{8}\frac{\pi g\,r\,A^2}{\omega} \\ &\quad - \frac{3}{128}\frac{\pi}{\omega}g\,r\,A^4 - \frac{1}{96}\pi\omega l^2 A^2 \end{aligned} \tag{18}$$

Setting

$$\begin{aligned} \frac{\partial}{\partial A}\left(\frac{\partial\bar{H}}{\partial(1/\omega)}\right) &= -\frac{1}{4}\omega^2\alpha\pi A - \frac{1}{4}\beta\pi A + \frac{1}{8}\beta\pi A^3 - \frac{5}{256}\beta\pi A^5 \\ &\quad + \frac{35}{36864}\beta\pi A^7 + \frac{1}{4}\lambda\pi A - \frac{1}{32}\lambda\pi A^3 \end{aligned} \tag{19}$$

Solving the above equation, an approximate frequency as a function of amplitude equal to

$$\omega_{HA} = \frac{1}{4}\sqrt{\frac{r\,g\,\left(192 - 72A^2 + 5A^4\right)}{6A^2r^2 + l^2}} \tag{20}$$

Hence, the approximate solution can be readily obtained:

$$\theta(t) = A\cos\left(\frac{1}{4}\sqrt{\frac{r\,g\,\left(192 - 72A^2 + 5A^4\right)}{6A^2r^2 + l^2}}\;t\right) \tag{21}$$

The numerical solution (with the Runge-Kutta method of order 4) for the nonlinear equation is:

$$\begin{array}{ll} \dot{\theta} = \mathrm{y} & \theta(0) = \mathrm{A} \\ \dot{y} = -\dfrac{r^2\theta^2 + r\,g\,\theta\cos(\theta)}{\frac{1}{12}l^2 + r^2\theta^2} & y(0) = 0 \end{array} \tag{22}$$

4.2 Example 2

We consider mathematical pendulum. The differential equation governing for the free oscillation of the mathematical pendulum is given by (Nayfeh 2011):

$$\ddot{\theta} - \Delta^2 cos\left(\theta\right)sin\left(\theta\right) + \frac{g}{r}sin\left(\theta\right) = 0 \tag{23}$$

With the boundary conditions of:

$$\theta(0) = A, \qquad \dot{\theta}(0) = 0 \tag{24}$$

In order to apply the Hamiltonian approach method to solve the above problem, the approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$ and $\sin\theta \approx \theta - \frac{1}{6}\theta^3$ is used.

The Hamiltonian of Eq. (23) is constructed as

$$H=-\frac{1}{2}\dot{\theta}^2-\frac{1}{2}\Delta^2\theta^2+\frac{1}{6}\Delta^2\theta^4-\frac{1}{48}\Delta^2\theta^6+\frac{1}{1152}\Delta^2\theta^8+\frac{1}{2}\frac{g}{r}\theta^2-\frac{1}{24}\frac{g}{r}\theta^4 \tag{25}$$

Integrating Eq. (25) with respect to t from 0 to $T/4$, we have;

$$\bar{H}=\int_0^{T/4}\left(\begin{array}{l}-\frac{1}{2}\dot{\theta}^2-\frac{1}{2}\Delta^2\theta^2+\frac{1}{6}\Delta^2\theta^4-\frac{1}{48}\Delta^2\theta^6\\+\frac{1}{1152}\Delta^2\theta^8+\frac{1}{2}\frac{g}{r}\theta^2-\frac{1}{24}\frac{g}{r}\theta^4\end{array}\right)dt \tag{26}$$

Assume that the solution can be expressed as

$$\theta(t)=A\cos(\omega t) \tag{27}$$

Substituting Eq. (27) into Eq. (26), we obtain

$$\begin{aligned}\bar{H}&=\int_0^{T/4}\left(\begin{array}{l}-\frac{1}{2}\omega^2A^2\sin^2(\omega t)-\frac{1}{2}\Delta^2A^2\cos^2(\omega t)\\+\frac{1}{6}\Delta^2A^4\cos^4(\omega t)-\frac{1}{48}\Delta^2A^6\cos^6(\omega t)\\+\frac{1}{1152}\Delta^2A^8\cos^8(\omega t)+\frac{1}{2}\frac{g}{r}A^2\cos^2(\omega t)\\-\frac{1}{24}\frac{g}{r}A^4\cos^4(\omega t)\end{array}\right)dt\\&=\int_0^{\pi/2}\left(\begin{array}{l}-\frac{1}{2}\omega^2A^2\sin^2 t-\frac{1}{2}\Delta^2A^2\cos^2 t+\frac{1}{6}\Delta^2A^4\cos^4 t\\-\frac{1}{48}\Delta^2A^6\cos^6 t+\frac{1}{1152}\Delta^2A^8\cos^8 t+\frac{1}{2}\frac{g}{r}A^2\cos^2 t\\-\frac{1}{24}\frac{g}{r}A^4\cos^4 t\end{array}\right)dt\\&=\frac{1}{32}\frac{\pi}{w}\Delta^2A^4+\frac{1}{8}w\pi A^2-\frac{1}{8}\frac{\pi}{w}\Delta^2A^2-\frac{5}{1536}\frac{\pi}{w}\Delta^2A^6\\&\quad+\frac{35}{294912}\frac{\pi}{w}\Delta^2A^8+\frac{1}{8}\frac{g\pi}{wr}A^2-\frac{1}{128}\frac{g\pi}{wr}A^4\end{aligned} \tag{28}$$

Setting

$$\frac{\partial}{\partial A}\left(\frac{\partial\bar{H}}{\partial(1/\omega)}\right)=\frac{1}{8}\pi\Delta^2A^3+\frac{1}{4}\pi w^2A-\frac{1}{4}\Delta^2\pi-\frac{5}{256}\pi\Delta^2A^5+\frac{35}{3664}\pi\Delta^2A^7+\frac{1}{4}\frac{g\pi}{r}A-\frac{1}{32}\frac{g\pi}{r}A^3 \tag{29}$$

Solving the above equation, an approximate frequency as a function of amplitude equal to

$$\omega_{HA}=\sqrt{-\Delta^2+\frac{1}{2}\Delta^2A^2-\frac{5}{64}\Delta^2A^4+\frac{35}{9216}\Delta^2A^6+\frac{g}{r}-\frac{1}{8}\frac{g}{r}A^2} \tag{30}$$

Hence, the approximate solution can be readily obtained:

$$\theta(t)=A\cos\left(\sqrt{\begin{array}{l}-\Delta^2+\frac{1}{2}\Delta^2A^2-\frac{5}{64}\Delta^2A^4\\+\frac{35}{9216}\Delta^2A^6+\frac{g}{r}-\frac{1}{8}\frac{g}{r}A^2\end{array}}\;t\right) \tag{31}$$

The numerical solution (with the Runge-Kutta method of order 4) for nonlinear equation is:

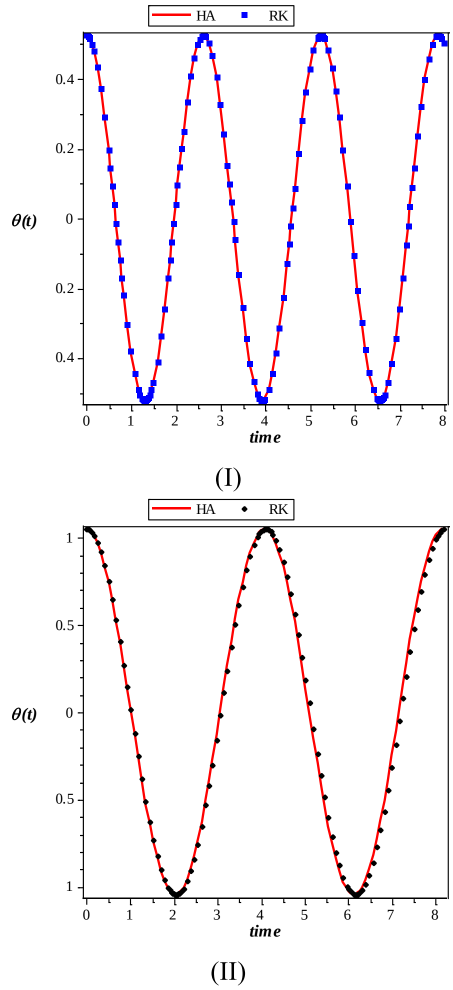


Fig. 1 Comparison of time response of the HA solution with RK solution (I): $A=\pi/6,\ g=10,\ l=3,\ r=0.5$ (II): $A=\pi/3,\ g=10,\ l=5,\ r=1$

$$\begin{array}{ll}\dot{\theta}=y & \theta(0)=A\\ \dot{y}=\Delta^2\cos(\theta)\sin(\theta)-\frac{g}{r}\sin(\theta) & y(0)=0\end{array} \tag{32}$$

5. Results and discussions

In the previous sections, the basic idea of the Hamiltonian approach has been presented, and the detail application and formulation of it on two well-known problems were applied.

In this part the, results of the HA are comparing with numerical solutions. Figures 1 and 5 are showing the motion of the problems are periodic, and the graphs are presenting the displacement versus time for the first period of the motion. The results have an excellent agreement with numerical solutions.

For each figure 1 and 5, the results are presented for different parameters to consider the accuracy of the proposed approach. Figure 2 represents the variation of the frequency of respect to r for the fixed values of A, g, and l. It can be observed from this figure that by increasing the r,

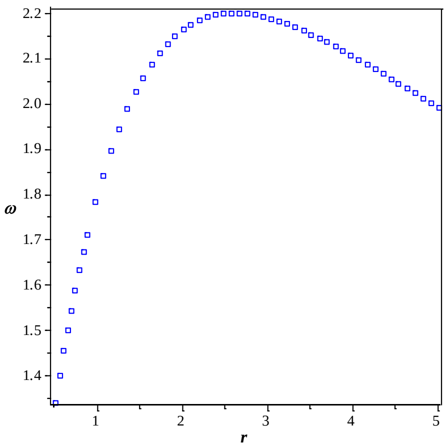


Fig. 2. Comparison of the frequency corresponding to various parameters of r $A=\pi/4,\ g=10,\ l=5$

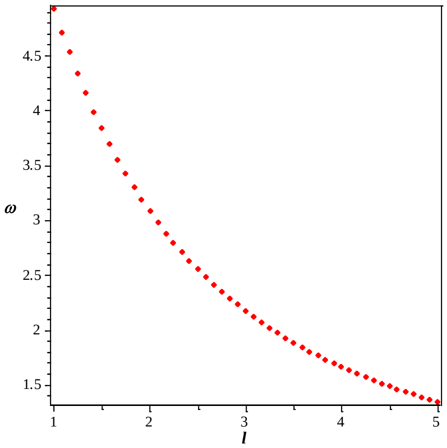


Fig. 3. Comparison of the frequency corresponding to various parameters of l $A=\pi/4,\ g=10,\ r=0.5$

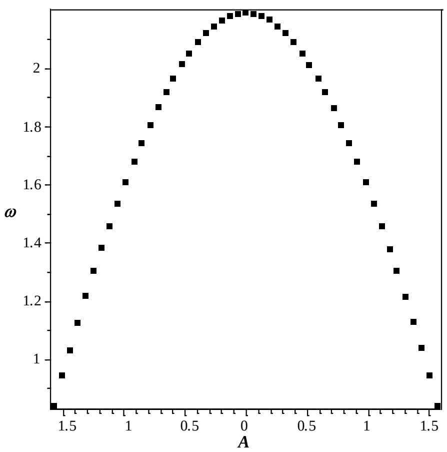
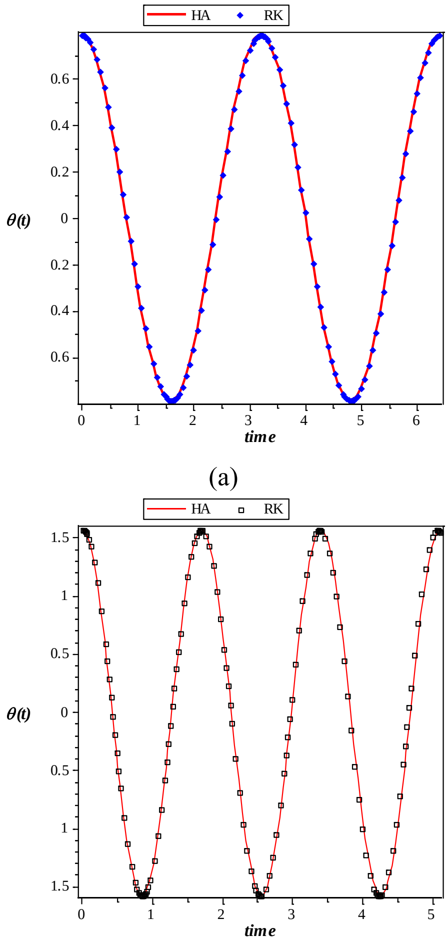


Fig. 4. Comparison of the frequency corresponding to various parameters of A $g=10,\ l=5,\ r=1$

the frequencies are increased until its maximum value which is 2.2 and decreasing by increasing r. Figures 3 and 4 are the effects of l and A on the frequency of the system, the increases on l decreases the frequency, and the frequency response has symmetric behavior by increasing A, the maximum value is 2.2.



(a)

(b)

Fig. 5 Comparison of time response of the HA solution with RK solution (a): $A=\pi/4,\ g=10,\ \Delta=\pi/3,\ r=2$ (b): $A=\pi/2,\ g=10,\ \Delta=\pi/3,\ r=0.5$

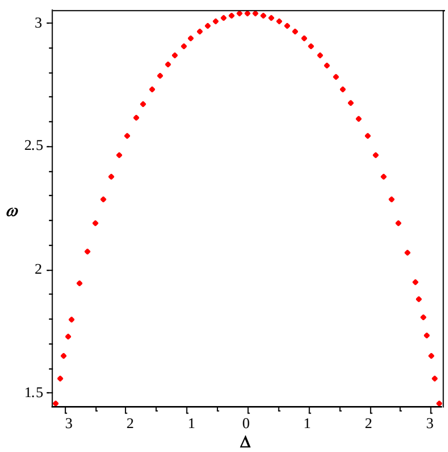


Fig. 6. Comparison of the frequency corresponding to various parameters of Δ $A=\pi/4,\ g=10,\ r=1$

In examples 2, the variation of the frequency respect to Δ is the same trend as we had for the amplitude of in example 1. The maximum values of the frequency in this example is 3, and it has symmetric behavior respect to Δ equal to zero. The variation of the frequency respect to r is

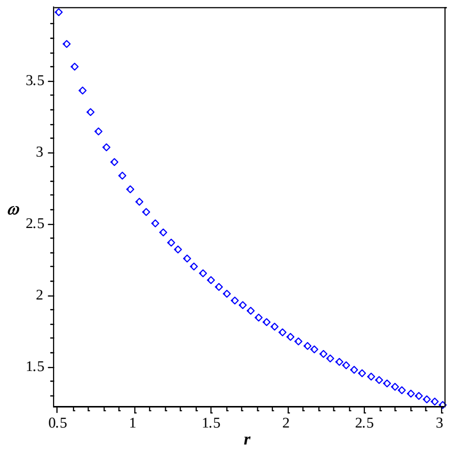


Fig. 7. Comparison of the frequency corresponding to various parameters of r $A=\pi/3,\ g=10,\ \Delta=\pi/2$

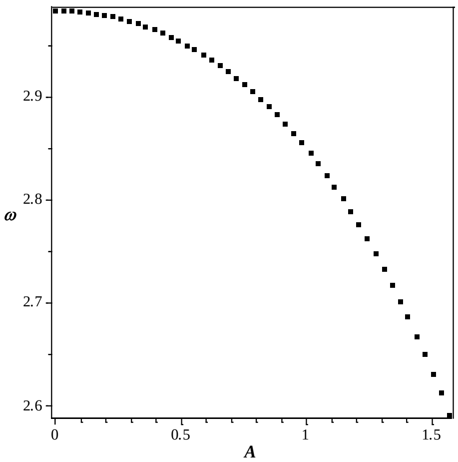


Fig. 8. Comparison of the frequency corresponding to various parameters of amplitude $g=10,\ \Delta=\pi/3,\ r=1$

decreasing by increasing r, with positive curvature and the same trend is related to A, but the main difference is, in this case it has a negative curvature. It can be seen that by applying accurately a mathematical tool, the effects of the essential parameters are easily obtained and the variation of them is captured.

The HA is a powerful mathematical tool, that does not need any linearization and small parameters and doesn't have the limitation of the traditional methods. This approach can be easily extended to any kind of conservative oscillation.

6. Conclusion

In this paper, The Hamiltonian approach has been implemented successfully for studying highly nonlinear vibration problems. The detail procedure and application of the proposed approach have been presented and the results were verified with the numerical solution. By the gained results, it has been shown that the results are in agreement with the numerical solution. The effects of the critical parameters were shown on the frequency response of the problems. Only one iteration of the HA leads us to a highly accurate solution to the problems. The proposed method can be extended to other kinds of conservation problems.

Acknowledgements

A.H. Alavi acknowledges the startup fund from the Swanson School of Engineering at the University of Pittsburgh.

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