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Variational approximate for high order bending analysis of laminated composite plates

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Abstract. This study presents a 4 node, 11 DOF/node plate element based on higher order shear deformation theory for lamina composite plates. The theory accounts for parabolic distribution of the transverse shear strain through the thickness of the plate. Differential field equations of composite plates are obtained from energy methods using virtual work principle. Differential field equations of composite plates are obtained from energy methods using virtual work principle. These equations were transformed into the operator form and then transformed into functions with geometric and dynamic boundary conditions with the help of the Gâteaux differential method, after determining that they provide the potential condition. Boundary conditions were determined by performing variational operations. By using the mixed finite element method, plate element named HOPLT44 was developed. After coding in FORTRAN computer program, finite element matrices were transformed into system matrices and various analyzes were performed. The current results are verified with those results obtained in the previous work and the new results are presented in tables and graphs.

Keywords: composite plate; high order shear deformation theory; finite element method; static analysis; energy principle

1. Introduction

Laminated composite structures are widely used in many engineering applications such as civil, mechanical and aircraft due to the excellent performance in strength, stiffness and lightness (Mahi and Tounsi 2015, Draiche *et al*. 2016, Bellifa *et al*. 2017, Chikh *et al*. 2017, Bakhadda *et al*. 2018, Kaci *et al*. 2018, Zine *et al*. 2018, Draoui *et al*. 2019). Therefore, knowledge of behavior of laminated composite plates are of great practical importance and significance for design in engineering applications (Zuo *et al*. 2015).

In the literature, different theories were proposed in order to study isotropic and composite plates such as Kirchhoff plate theory, first order and higher-order shear deformation theories which are known equivalent single layer theories (Beldjelili *et al*. 2016, Boukhari *et al*. 2016, Bousahla *et al*. 2016, Fahsi *et al*. 2017, Tu *et al*. 2017, Belkacem *et al*. 2018, Yousfi *et al*. 2018, Bourada *et al*. 2019, Meksi *et al*. 2019).

The Kirchhoff plate theory (KHOPLT), well known as the classical plate theory, which neglects the interlaminar shear deformation and rotary inertia, therefore it is acceptable for thin plates. By neglecting transverse shear effects, the KHOPLT leads to inaccurate results for composites.

The first order shear deformation plate theory (FOPLT) (Whitney 1969) is proposed for in the middle thick plates based on Reissner (1945) and Mindlin (1951) assumptions which include the transverse shear effects with a linear variation of transverse shear strain through the plate thickness. This theory is needed a shear correction to correct variation of transverse shear stress and shear strain through thickness. However, FOPLT does not satisfy the condition of zero transverse shear stress at the top and bottom surfaces of plates and consider a linear transverse shear distribution through the plate thickness (Yaghoubshahi and Alinia 2015). Also, the shear correction factor dependents on layer orientation, loading conditions, geometric parameters and boundary conditions of plates. To eliminate the deficiencies of KHOPLT and FOPLT, the higher-order shear deformation plate theories (HOPLT) are the most favorable due to including non-linear distributions of shear stress through thickness and satisfies the zero transverse shear stress condition on the top and bottom surfaces of plates. Therefore no shear correction factors are used. Regarding ensure to non-linear distributions of shear stress in HOPLT, various models with different shear strain shape functions have been purposed by authors (Daouadji and Adim, 2017, Attia *et al*. 2018, Javed *et al*. 2018, Houari *et al*. 2018). There are cubic shape functions (Ambartsumian 1958, Reissner 1975, Reddy 1984, Zhen and Wanji 2008), trigonometric shape functions (Stein 1986, Touratier 1991, Soldatos 1992, Abualnour *et al*. 2018, Benchohra *et al*. 2018), exponential shape functions (Karama *et al*. 2003, Aydogdu 2009), hyperbolic shape functions (Grover *et al*. 2013, Grover *et al*. 2014, Belabed *et al*. 2018, Taleb *et al*. 2018). The buckling analysis of hybrid FGM plates using a novel four variable refined plate theory is presented by Bourada *et al*. (2018). In this theory the distribution of transverse shear deformation is parabolic

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across the thickness of the plate by satisfying the surface conditions. Bouadi *et al*. (2018) are developed A new nonlocal HOPLT for buckling properties of single graphene sheet. The proposed nonlocal HOPLT contains a new displacement field which incorporates undetermined integral terms and contains only two variables. An analysis of the propagation of waves of functionally graduated plates is presented by Fourn *et al*. (2018) with using a HOPLT. This theory has only four variables, which is less than the FOPLT. Bellifa *et al*. (2017) are proposed an efficient and simple refined theory for buckling analysis of functionally graded plates by using a new displacement field which includes undetermined integral variables. This theory contains only four unknowns, with is even less than the FOPLT. Abdelaziz *et al*. (2017) are developed a simple hyperbolic HOPLT and applied for the bending, vibration and buckling of powerly graded material sandwich plate with various boundary conditions. The displacement field of the present model is selected based on a hyperbolic variation in the in-plane displacements across the plate's thickness. Menasria *et al*. (2017) used a new displacement field that includes undetermined integral terms for analyzing thermal buckling response of FGM sandwich plates. The proposed kinematic uses only four variables, which is even less than the FOPLT and the conventional HOPLT. El-Haina *et al*. (2017) presented a simple analytical approach to investigate the thermal buckling behavior of thick FGM sandwich plates by employing both the sinusoidal HOPLT.

Various numerical methods, such as finite element method (FEM), for analysis of complex structures have been considered. The FEM is a powerful numerical computation method in engineering areas, because of its versatility in handing complex geometries and boundary conditions (Zhang and Yang 2009). Further, developments of FEM to take into account shear deformation theories, and HOPLT in particular, have been performed for composite structures which have exhibited very good performance in analysis (Aagaah *et al*. 2003, Desai *et al*. 2003, Mantari *et al*. 2012, Ramu and Mohanty 2012). A FEM can only be considered in relation with a variational principle and a functional space. Changing the variational principle and the space in which it is posed leads to a different FEM approximation. In the last decade mixed finite elements for the solution of partial differential equations have known substantial developments. The mixed-FEM is far more efficient due to variables can be chosen independently, the forces and moments can be calculated with less number of elements but more sensitive. In the past years, mixed-FEM based on higher-order shear deformation theories has been introduced. The elements have shown quite good performances for beam, plate and shell problems (Zienkiewicz *et al*. 1981, Capsoni and Corradi 1997, Bischoff and Bletzinger 2004, Cervera *et al*. 2010).

In the mixed-FEM, having field equations one needs a method to obtain the functional. Hu-Washizu and Hellinger-Reissner principles or weak formulation are popular approaches to establish a functional, which provide functionals that are essential for finite element formulation (Reddy 1993). The Gâteaux differential (GD) method is a useful alternative to Hu-Washizu and Hellinger-Reissner principles. Recently, Aköz and his co-workers successfully have employed GD to construct the functionals for various problems (Aköz *et al*. 1991, Aköz and Uzcan 1992, Aköz and Kadioğlu 1996, Aköz and Kadioğlu 1999, Akoz and Eratli 2000, Aköz and Özütok 2000, Özütok and Madenci 2013, Ozutok *et al*. 2014). Although, the same functionals can be obtained by the Hellinger-Reissner and Hu-Washizu principles and GD for a relatively simple problem, the GD has some advantages and nice properties. It provides the consistency of the field equations that all field equations are enforced to the functional; the boundary conditions are can be constructed easily; method doesn't use any artificial numerically adjusted factors; potential test provides accuracy of field equations (Eratlı and Aköz 1997, Eratli and Akoz 2002).

The main purpose of this paper is to employ mixed-FEM to investigate the static problems of laminated composite plates with high order shear deformation plate theories. The virtual displacement principle was applied to obtain the partial differential field equations for laminated composite plates. These field equations were written in operator form then by using the Gâteaux differential approach, the functionals which including the dynamic and geometric boundary conditions is obtained after provide potential conditions. Based on HOPLT, the present mixed-FEM models represent non-linear variation of transverse shear stresses across the plate thickness. The transverse shear stresses are vanishes at the plate surfaces. Therefore, the shear correction factor is neglected. The mixed-finite element HOPLT44 is developed based on this function which has four nodes with eleven degrees of freedom per node. The displacements, rotations, internal forces, normal moments and higher-order moments in particular, of composite plates are calculated independently. Several numerical examples are conducted considering side to thickness ratio, modulus ratio, number of layers and layer orientation.

2. Theoretical formulation

Consider a rectangular laminated plate of length "L", width "b" and total thickness "h" with orthotropic layers as shown in Fig. 1. The plate is located in Cartesian coordinate system (x-y-z). Upon the effects of transverse shear deformation and rotary inertia for different plate theories, general displacement fields of laminated composite plates can be defined as Eq (1)

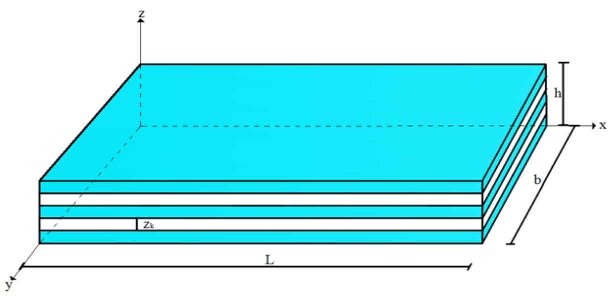


Fig. 1 A n-layered laminated composite plate

$$\begin{aligned} u(x,y,z) &= -z\,w_{0,x} + f(z)\gamma_1 \\ v(x,y,z) &= -z\,w_{0,y} + f(z)\gamma_2 \\ w(x,y,z) &= w_0 \end{aligned} \tag{1}$$

where the in-plane displacement and transverse displacement components are denoted as " u ", " v " and " w ", respectively. The " γ_1 " and " γ_2 " are the transverse shear strains of any point. The term $f(z)$ represents the higher order shape function determining the parabolic distribution of transverse shear effect through thickness of plate in this study . And also " $(...)_{,x}$ " and " $(...)_{,y}$ " are the partial derivatives with respect to x and y axis, respectively.

The strains associated with the displacement field are written as Eq. (2).

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = -z\,w_{,xx} + f(z)\gamma_{1,x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} = -z\,w_{,yy} + f(z)\gamma_{1,y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = f(z)\gamma_{1,y} - 2zw_{,xy} + f(z)\gamma_{2,x} \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = f(z)_{,z}\gamma_1 \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = f(z)_{,z}\gamma_2 \end{aligned} \tag{2}$$

By performing the transformation rule of stresses/strain between the lamina and the laminated coordinate system, the stress-strain relations in the Cartesian x-y-z coordinate system can be obtained as Eq.(3)

$$\{\bar{\sigma}\}^{(k)} = \left[\bar{Q}\right]^{(k)} \{\bar{\varepsilon}\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{3}$$

where " $\{\bar{\sigma}\}^{(k)}$ ", " $\{\bar{\varepsilon}\}$ " and " $\left[\bar{Q}\right]^{(k)}$ " are the stress vector, the strain vector and the transformed rigidity matrix of *k*th lamina, respectively (Reddy 2004). The components of the matrix in Eq. (3) in terms of the stiffness coefficients in the direction of principal axes of material orthotropy can be written as Eq.(4)

$$\begin{aligned} \bar{Q}_{11}^{(k)} &= Q_{11}^{(k)}\,\mathrm{c}^4 + 2(Q_{12}^{(k)} + 2Q_{66}^{(k)})\,\mathrm{c}^2\,\mathrm{s}^2 + Q_{22}^{(k)}\,\mathrm{s}^4 \\ \bar{Q}_{12}^{(k)} &= (Q_{11}^{(k)} + Q_{22}^{(k)} - 4Q_{66}^{(k)})\,\mathrm{c}^2\,\mathrm{s}^2 + Q_{12}^{(k)}(\mathrm{c}^4 + \mathrm{s}^4) \\ \bar{Q}_{22}^{(k)} &= Q_{11}^{(k)}\,\mathrm{s}^4 + 2(Q_{12}^{(k)} + 2Q_{66}^{(k)})\,\mathrm{c}^2\,\mathrm{s}^2 + Q_{22}^{(k)}\,\mathrm{c}^4 \\ \bar{Q}_{16}^{(k)} &= (Q_{11}^{(k)} - Q_{12}^{(k)} - 2Q_{66}^{(k)})\,\mathrm{c}^3\,\mathrm{s} + (Q_{11}^{(k)} - Q_{22}^{(k)} + 2Q_{66}^{(k)})\,\mathrm{c}\,\mathrm{s}^3 \\ \bar{Q}_{26}^{(k)} &= (Q_{11}^{(k)} - Q_{12}^{(k)} - 2Q_{66}^{(k)})\,\mathrm{c}\,\mathrm{s}^3 + (Q_{11}^{(k)} - Q_{22}^{(k)} + 2Q_{66}^{(k)})\,\mathrm{c}^3\,\mathrm{s} \\ \bar{Q}_{66}^{(k)} &= (Q_{11}^{(k)} + Q_{22}^{(k)} - 2Q_{12}^{(k)})\,\mathrm{c}^2\,\mathrm{s}^2 + Q_{66}^{(k)}(\mathrm{c}^2 - \mathrm{s}^2)^2 \\ \bar{Q}_{44}^{(k)} &= Q_{44}^{(k)}\,\mathrm{c}^2 + Q_{55}^{(k)}\,\mathrm{s}^2 \\ \bar{Q}_{45}^{(k)} &= (Q_{44}^{(k)} - Q_{55}^{(k)})\,\mathrm{c}\,\mathrm{s} \\ \bar{Q}_{55}^{(k)} &= Q_{44}^{(k)}\,\mathrm{s}^2 + Q_{55}^{(k)}\,\mathrm{c}^2 \end{aligned} \tag{4}$$

where "c" is " $\cos\theta$ " and "s" is " $\sin\theta$ " and " θ " is the angle between material orientation and x-axis. The terms of engineering constants of the *k*th orthotropic lamina are

$$\begin{aligned} &Q_{11}^{(k)} = \frac{E_1^{(k)}}{1-\upsilon_{12}\upsilon_{21}}, \quad Q_{12}^{(k)} = \frac{\upsilon_{12}E_2^{(k)}}{1-\upsilon_{12}\upsilon_{21}}, \quad Q_{22}^{(k)} = \frac{E_2^{(k)}}{1-\upsilon_{12}\upsilon_{21}}, \\ &Q_{44}^{(k)} = G_{13}, \quad Q_{55}^{(k)} = G_{23}, \quad Q_{66}^{(k)} = G_{12} \end{aligned} \tag{5}$$

The equilibrium equations can be derived using the principle of the virtual displacement; the following expressions can be obtained

$$\begin{aligned} &\int_{-h/2}^{h/2} \int_A \left[\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz} \right] dz\,dA \\ &-\int_A q\delta w\,dx\,dy = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} &\int_A \begin{pmatrix} -K\delta w_{,xx} + \bar{K}\delta\gamma_{1,x} - M\delta w_{,yy} + \bar{M}\delta\gamma_{2,y} + \bar{T}\delta\gamma_{1,y} \\ -2T\delta w_{,xy} + \bar{T}\delta\gamma_{2,x} + \bar{S}\delta\gamma_1 + \bar{Q}\delta\gamma_2 \end{pmatrix} dA \\ &-\int_A q\delta w\,dx\,dy = 0 \end{aligned} \tag{7}$$

where " δ " is an operate to show the variation of any parameter, " q " is the load term, and " K ", " M " and " T " are the resultant moments, " $\bar{K}$ ", " $\bar{M}$ " and " $\bar{T}$ " are the higher-order moments, " $\bar{S}$ " and " $\bar{Q}$ " are the higher-order shear forces, which can be defined as following integrations

$$\begin{aligned} \{K; M; T\} &= \int_{-h/2}^{h/2} \{\sigma_x; \sigma_y; \tau_{xy}\}\, z\,dz \\ &= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \{\sigma_x; \sigma_y; \tau_{xy}\}\, z\,dz \end{aligned} \tag{8}$$

$$\begin{aligned} \{\bar{K}; \bar{M}; \bar{T}\} &= \int_{-h/2}^{h/2} \{\sigma_x; \sigma_y; \tau_{xy}\}\, f(z)\,dz \\ &= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \{\sigma_x; \sigma_y; \tau_{xy}\}\, f(z)\,dz \end{aligned} \tag{9}$$

$$\begin{aligned} \{\bar{S}; \bar{Q}\} &= \int_{-h/2}^{h/2} \{\tau_{xz}; \tau_{yz}\}\, f(z)_{,z}\,dz \\ &= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \{\tau_{xz}; \tau_{yz}\}\, f(z)_{,z}\,dz \end{aligned} \tag{10}$$

The Euler-Lagrange equations are derived from Eq. (7) by integrating the displacement gradients by parts and setting the coefficients of "δw", "$\delta\gamma_1$", "$\delta\gamma_2$" to zero separately. The Euler-Lagrange equations are obtained as

$$\begin{aligned} &\delta w: \; -K_{,xx} - M_{,yy} - 2T_{,xy} - q = 0 \\ &\delta\gamma_1: \; -\bar{K}_{,x} - \bar{T}_{,y} + \bar{S} = 0 \\ &\delta\gamma_2: \; -\bar{M}_{y} - \bar{T}_{,x} + \bar{Q} = 0 \end{aligned} \tag{11}$$

By substituting the stress-strain relations into the Eqs. (8-11) the following constitutive equations are obtained as

$$\begin{Bmatrix} K \\ M \\ T \\ \bar{K} \\ \bar{M} \\ \bar{T} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} & F_{11} & F_{12} & F_{16} \\ D_{12} & D_{22} & D_{26} & F_{12} & F_{22} & F_{26} \\ D_{16} & D_{26} & D_{66} & F_{16} & F_{26} & F_{66} \\ F_{11} & F_{12} & F_{16} & H_{11} & H_{12} & H_{16} \\ F_{12} & F_{22} & F_{26} & H_{12} & H_{22} & H_{26} \\ F_{16} & F_{26} & F_{66} & H_{16} & H_{26} & H_{66} \end{bmatrix} \begin{Bmatrix} -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \\ \gamma_{1,x} \\ \gamma_{2,y} \\ \gamma_{1,y} + \gamma_{2,x} \end{Bmatrix} \tag{12}$$

$$\begin{Bmatrix} \bar{S} \\ \bar{Q} \end{Bmatrix} = \begin{bmatrix} \hat{A}_{55} & \hat{A}_{45} \\ \hat{A}_{45} & \hat{A}_{44} \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \end{Bmatrix}$$

where the laminate material stiffnesses are given by

$$\left\{ D_{ij}; F_{ij}; H_{ij}; \hat{A}_{ij} \right\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left[\bar{Q}_{ij} \right]^{(k)} \left\{ z^2; z\,f(z); [f(z)]^2; [f(z)_{,z}]^2 \right\} dz \tag{13}$$

The present study is corned with cross-ply laminated composite plates. Therefore, the following stiffness components are not considered in the above equations

$$D_{16} = F_{16} = H_{16} = D_{26} = F_{26} = H_{26} = \hat{A}_{45} = 0 \tag{14}$$

Based on plate theories, the constitutive equations of the high order laminated composite HOPLT44 plate element can be expressed in Eq. (15).

$$\begin{Bmatrix} K + D_{11}w_{,xx} - \frac{4}{5}\gamma_{1,x} + D_{12}w_{,yy} - \frac{4}{5}\gamma_{2,y} = 0 \\ M + D_{12}w_{,xx} - \frac{4}{5}\gamma_{1,x} + D_{22}w_{,yy} - \frac{4}{5}\gamma_{2,y} = 0 \\ T + 2D_{66}w_{,xy} - \frac{4}{5}[\gamma_{1,y} + \gamma_{2,x}] = 0 \\ -\bar{K} - \frac{4}{5}D_{11}w_{,xx} + \frac{68}{105}D_{11}\gamma_{1,x} - \frac{4}{5}D_{12}w_{,yy} + \frac{68}{105}D_{12}\gamma_{2,y} = 0 \\ -\bar{M} - \frac{4}{5}D_{12}w_{,xx} + \frac{68}{105}D_{12}\gamma_{1,x} - \frac{4}{5}D_{22}w_{,yy} + \frac{68}{105}D_{22}\gamma_{2,y} = 0 \\ -\bar{T} + \frac{68}{105}D_{66}(\gamma_{1,y} + \gamma_{2,x}) - \frac{8}{5}D_{66}w_{,xy} = 0 \\ -\bar{S} + \hat{A}_{55}\gamma_1 = 0 \\ -\bar{Q} + \hat{A}_{44}\gamma_2 = 0 \end{Bmatrix} \tag{15}$$

Gâteaux differential (GD) method is more suitable that are essential for mixed-type FEM. Dynamic and boundary conditions are written in symbolic form as Eq. (18). Dynamic boundary conditions of the form

$$-\mathbf{R} + \hat{\mathbf{R}} = 0 \qquad -\mathbf{M} + \hat{\mathbf{M}} = 0 \tag{16}$$

Geometric boundary conditions of the form

$$\Omega - \hat{\Omega} = 0; \quad \mathbf{u} - \hat{\mathbf{u}} = 0 \tag{17}$$

The quantities in Eqs. (16-17) with hat have known values on the boundary "$\mathbf{R}$, $\mathbf{M}$, $\mathbf{u}$, Ω "are representing the force, moment rotation and deflection vectors, respectively. All field equations and boundary conditions are completely included to the functional by mathematical manipulations in the Gâteaux differential method. The field equations including boundary conditions for laminate plate can be written in operator form as

$$\left[\begin{array}{c|cccc} L_{ij} & & 0 & & \\ \hline & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ & 0 & -1 & 0 & 0 \\ & -1 & 0 & 0 & 0 \end{array} \right] \begin{Bmatrix} y_i \\ \hline \mathbf{u} \\ \Omega \\ \mathbf{M} \\ \mathbf{R} \end{Bmatrix} = \begin{Bmatrix} f_i \\ \hline -\hat{\mathbf{R}} \\ -\hat{\mathbf{M}} \\ \hat{\Omega} \\ \hat{\mathbf{u}} \end{Bmatrix} \tag{18}$$

where L_{ij}, y_i and f_i are given in Eq. (19)

$$\begin{gathered} \{y_1 = w\} \quad \{y_2 = \gamma_1\} \quad \{y_3 = \gamma_2\} \\ \left\{ \begin{aligned} y_4 = {} & \frac{85D_{22}K}{-D_{22}D_{11} + (D_{12})^2} + \frac{105D_{12}\bar{M}}{-D_{22}D_{11} + (D_{12})^2} \\ & - \frac{85D_{12}M}{-D_{22}D_{11} + (D_{12})^2} - \frac{105D_{22}\bar{K}}{-D_{22}D_{11} + (D_{12})^2} \end{aligned} \right\} \\ \left\{ \begin{aligned} y_5 = {} & -\frac{105D_{11}\bar{M}}{(-D_{22}D_{11} + (D_{12})^2)} + \frac{85D_{11}M}{(-D_{22}D_{11} + (D_{12})^2)} \\ & + \frac{105D_{12}\bar{K}}{(-D_{22}D_{11} + (D_{12})^2)} - \frac{85D_{12}K}{(-D_{22}D_{11} + (D_{12})^2)} \end{aligned} \right\} \\ \left\{ y_6 = \frac{105\bar{T}}{D_{66}} - \frac{85T}{D_{66}} \right\} \\ \left\{ \begin{aligned} y_7 = {} & -\frac{525D_{22}\bar{K}}{4(-D_{22}D_{11} + D_{12}^2)} + \frac{105D_{22}K}{(-D_{22}D_{11} + D_{12}^2)} \\ & + \frac{525D_{12}\bar{M}}{4(-D_{22}D_{11} + (D_{12})^2)} - \frac{105D_{12}M}{(-D_{22}D_{11} + (D_{12})^2)} \end{aligned} \right\} \\ \left\{ \begin{aligned} y_8 = {} & -\frac{525D_{11}\bar{M}}{4(-D_{22}D_{11} + (D_{12})^2)} + \frac{105D_{11}M}{(-D_{22}D_{11} + (D_{12})^2)} \\ & + \frac{525D_{12}\bar{K}}{4(-D_{22}D_{11} + (D_{12})^2)} - \frac{105D_{12}K}{(-D_{22}D_{11} + (D_{12})^2)} \end{aligned} \right\} \\ \left\{ y_9 = \frac{525\bar{T}}{4D_{66}} - \frac{105T}{D_{66}} \right\} \quad \left\{ y_{10} = \frac{\bar{S}}{A_{55}} \right\} \quad \left\{ y_{11} = \frac{\bar{Q}}{A_{44}} \right\} \end{gathered} \tag{19}$$

$$L_{1,4} = D_{11}(.)_{,xx} + D_{12}(.)_{,yy}; L_{1,5} = D_{12}(.)_{,xx} + D_{22}(.)_{,yy};$$

$$L_{1,6} = 2D_{66}(.)_{,xy}; L_{1,7} = -\frac{4}{5}D_{11}(.)_{,xx} - \frac{4}{5}D_{12}(.)_{,yy};$$

$$L_{1,8}=-\frac{4}{5}D_{12}(.)_{,xx}-\frac{4}{5}D_{22}(.)_{,yy};L_{1,9}=-\frac{8}{5}D_{66}(.)_{xy}$$

$$L_{2,4}=\frac{68}{85}D_{11}(.)_{,x};L_{2,5}=\frac{68}{85}D_{12}(.)_{,x};L_{2,6}=\frac{68}{85}D_{66}(.)_{,y};$$

$$L_{2,7}=-\frac{68}{105}D_{11}(.)_{,x};L_{2,8}=-\frac{68}{105}D_{12}(.)_{,x};$$

$$L_{2,9}=\frac{68}{105}D_{66}(.)_{,y};L_{2,10}=A_{55}(.);L_{3,4}=\frac{68}{85}D_{12}(.)_{,x};$$

$$L_{3,5}=\frac{68}{85}D_{22}(.)_{,x};L_{3,6}=\frac{68}{85}D_{66}(.)_{,x};L_{3,7}=-\frac{68}{105}D_{12}(.)_{,x};$$

$$L_{3,8}=-\frac{68}{105}D_{22}(.)_{,x};L_{3,9}=\frac{68}{105}D_{66}(.)_{,x};L_{3,10}=A_{44}(.);$$

$$L_{4,1}=D_{11}(.)_{,xx}+D_{12}(.)_{,yy};L_{4,2}=-\frac{4}{5}D_{11}(.)_{,x};$$

$$L_{4,3}=-\frac{4}{5}D_{12}(.)_{,y};L_{4,4}=-D_{11}(.);L_{4,5}=-D_{12}(.);$$

$$L_{4,7}=\frac{4}{5}D_{11}(.);L_{4,8}=\frac{4}{5}D_{12}(.);L_{5,1}=D_{12}(.)_{,xx}+D_{22}(.)_{,yy};$$

$$L_{5,2}=-\frac{4}{5}D_{12}(.)_{,x};L_{5,3}=-\frac{4}{5}D_{22}(.)_{,y};L_{5,4}=-D_{12}(.);$$

$$L_{5,5}=-D_{22}(.);L_{5,7}=\frac{4}{5}D_{12}(.);L_{5,8}=\frac{4}{5}D_{22}(.) \tag{19}$$

$$L_{6,1}=2D_{66}(.)_{,xy};L_{6,2}=-\frac{4}{5}D_{66}(.)_{,y};L_{6,3}=-\frac{4}{5}D_{66}(.)_{,x};$$

$$L_{6,2}=-D_{66}(.);L_{6,9}=\frac{4}{5}D_{66}(.);L_{7,1}=-\frac{4}{5}D_{11}(.)_{,xx}-\frac{4}{5}D_{12}(.)_{,yy};$$

$$L_{7,2}=\frac{68}{105}D_{11}(.)_{,x};L_{7,3}=\frac{68}{105}D_{12}(.)_{,y};L_{7,4}=\frac{4}{5}D_{11}(.);$$

$$L_{7,5}=\frac{4}{5}D_{12}(.);L_{7,7}=-\frac{68}{105}D_{11}(.);L_{7,8}=-\frac{68}{105}D_{12}(.);$$

$$L_{8,1}=-\frac{4}{5}D_{12}(.)_{,xx}-\frac{4}{5}D_{22}(.)_{,yy};L_{8,2}=\frac{68}{105}D_{12}(.)_{,x};$$

$$L_{8,3}=\frac{68}{105}D_{22}(.)_{,y};L_{8,4}=\frac{4}{5}D_{12}(.);L_{8,5}=\frac{4}{5}D_{22}(.);$$

$$L_{8,7}=-\frac{68}{105}D_{12}(.);L_{8,8}=-\frac{68}{105}D_{22}(.);L_{9,1}=-\frac{8}{5}D_{66}(.)_{,xy};$$

$$L_{9,2}=\frac{68}{105}D_{66}(.)_{,y};L_{9,3}=\frac{68}{105}D_{66}(.)_{,x};L_{9,6}=\frac{4}{5}D_{66}(.);$$

$$L_{9,9}=-\frac{68}{105}D_{66}(.);L_{10,2}=A_{55}(.);L_{10,10}=-A_{55}(.);$$

$$L_{11,3}=A_{44}(.);L_{11,11}=-A_{44}(.)$$

The Gâteaux derivative of an operator is defined as;

$$\mathrm{d}\mathbf{P}(\mathbf{y},\bar{\mathbf{y}})=\left.\frac{\partial\mathbf{P}(\mathbf{y}+\tau\bar{\mathbf{y}})}{\partial\tau}\right|_{\tau=0} \tag{20}$$

where "τ" is a scalar quantity. A required and sufficient condition for "**P**" to be a potential is

$$<\mathbf{dP}(\mathbf{y},\bar{\mathbf{y}}),\mathbf{y}^{*}>=<\mathbf{dP}(\mathbf{y},\mathbf{y}^{*}),\bar{\mathbf{y}}> \tag{21}$$

where parentheses indicate the inner products. If the operator "**P**" is a potential, then the functional corresponding to the field equations will be given as for HOPLT;

$$\begin{aligned}I_{[y]}&=\left[K_{,x},w_{,x}\right]+\left[M_{,y},w_{,y}\right]+\left[w_{,x},T_{,y}\right]+\left[w_{,y},T_{,x}\right]\\&+\left[\bar{K},\gamma_{1,x}\right]+\left[(\gamma_{1,y}+\gamma_{2,x}),\bar{T}\right]+\left[\bar{S},\gamma_{1}\right]+\left[\bar{Q},\gamma_{2}\right]\\&+\left[\bar{M},\gamma_{2,y}\right]+\frac{\alpha}{2}[K,K]+\Delta\left\{\left[K,\bar{M}\right]+\left[M,\bar{K}\right]\right\}\\&-\Upsilon[K,M]-\Theta\left[\bar{K},K\right]-\Pi\left[M,\bar{M}\right]+\frac{\beta}{2}[M,M]\\&+\frac{105}{D_{66}}\left[T,\bar{T}\right]-\frac{85}{2D_{66}}[T,T]+\frac{\bar{\alpha}}{2}\left[\bar{K},\bar{K}\right]-\Lambda\left[\bar{K},\bar{M}\right]\\&+\frac{\bar{\beta}}{2}\left[\bar{M},\bar{M}\right]-\frac{525}{8D_{66}}\left[\bar{T},\bar{T}\right]-\frac{1}{2A_{55}}\left[\bar{S},\bar{S}\right]-\frac{1}{2A_{44}}\left[\bar{Q},\bar{Q}\right]\\&-[q,w]+boundary\ conditions\end{aligned} \tag{22}$$

where the boundary conditions and constants are given Eqs. (23-25)

$$\begin{aligned}[\mathbf{R},\mathbf{u}]&=-\left[K_{,x},w\right]_{0}-\left[M_{,y},w\right]_{0}-\left[\bar{K},\gamma_{1}\right]_{0}-\left[\bar{M},\gamma_{2}\right]_{0}\\&-\left[T_{,x},w\right]_{0}-\left[T_{,y},w\right]_{0}-\left[\bar{T},\gamma_{1}\right]_{0}-\left[\bar{T},\gamma_{2}\right]_{0}\end{aligned} \tag{23}$$

$$\begin{aligned}[\mathbf{M},\Omega]&=-\left[w_{,x},K\right]_{0}-\left[w_{,y},M\right]_{0}-\left[\gamma_{1},\bar{K}\right]_{0}-\left[\gamma_{2},\bar{M}\right]_{0}\\&-\left[w_{,x},T\right]_{0}-\left[w_{,y},T\right]_{0}-\left[\gamma_{1},\bar{T}\right]_{0}-\left[\gamma_{2},\bar{T}\right]_{0}\end{aligned} \tag{24}$$

$$\begin{aligned}&\alpha=\frac{85D_{22}}{(-D_{22}D_{11}+D_{12}^{2})};\quad\beta=\frac{85D_{11}}{(-D_{22}D_{11}+D_{12}^{2})};\\&\bar{\alpha}=\frac{525D_{22}}{4(-D_{22}D_{11}+D_{12}^{2})};\quad\Theta=\frac{105D_{22}}{(-D_{22}D_{11}+D_{12}^{2})};\\&\bar{\beta}=\frac{525D_{11}}{4(-D_{22}D_{11}+D_{12}^{2})};\quad\Delta=\frac{105D_{12}}{(-D_{22}D_{11}+D_{12}^{2})};\\&\Upsilon=\frac{85D_{12}}{(-D_{22}D_{11}+D_{12}^{2})};\quad\Pi=\frac{105D_{11}}{(-D_{22}D_{11}+D_{12}^{2})};\\&\Lambda=\frac{525D_{12}}{4(-D_{22}D_{11}+D_{12}^{2})}\end{aligned} \tag{25}$$

The development of the finite element matrix is presented for laminated plates. A rectangular serendipity element is used as in Fig. 2.

The parent shape functions is calculated as

$$\begin{aligned}&\psi_{i}(\xi,\eta)=\frac{1}{4}(1+\xi\xi_{i})(1+\eta\eta_{i})\\&\xi_{i}=\pm1,\eta_{i}=\pm1\,(i=1,\ldots,4)\end{aligned} \tag{26}$$

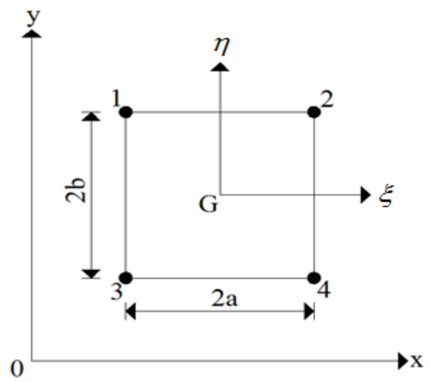


Fig. 2 A four noded rectangular a plate element with the local and the global coordinate system

where the subscripted non-dimensional parent coordinates take the values

$$\begin{aligned} \xi_1 = \xi_2 = \eta_1 = \eta_3 = -1 \\ \xi_3 = \xi_4 = \eta_2 = \eta_4 = 1 \end{aligned} \tag{27}$$

The sub-matrices are defined below for the each theory for composite plates given in Eq. (28).

$$\begin{aligned} [k_1] &= \int_A \psi_i \psi_j \, dA; \quad [k_2] = \int_A \psi_{i,x} \psi_j \, dA; \quad [k_3] = \int_A \psi_{i,y} \psi_j \, dA \\ [k_4] &= \int_A \psi_{i,x} \psi_{j,x} dA; \quad [k_5] = \int_A \psi_{i,y} \psi_{j,y} dA; \quad [k_6] = \int_A \psi_{i,x} \psi_{j,y} dA \end{aligned} \tag{28}$$

The explicit expressions of submatrices are given in the Eq. (29).

$$\int_A \psi_i \psi_j \, dA = \begin{bmatrix} 4ab/9 & 4ab/18 & 4ab/18 & 4ab/36 \\ 4ab/18 & 4ab/9 & 4ab/36 & 4ab/18 \\ 4ab/18 & 4ab/36 & 4ab/9 & 4ab/18 \\ 4ab/36 & 4ab/18 & 4ab/18 & 4ab/9 \end{bmatrix} \tag{29a}$$

$$\int_A \psi_{i,x} \psi_{j,x} \, dA = \begin{bmatrix} b/3a & b/6a & -b/3a & -b/6a \\ b/6a & b/3a & -b/6a & -b/3a \\ -b/3a & -b/6a & b/3a & b/6a \\ -b/6a & -b/3a & b/6a & b/3a \end{bmatrix} \tag{29b}$$

$$\int_A \psi_{i,y} \psi_{j,y} \, dA = \begin{bmatrix} a/3b & -a/3b & a/6b & -a/6b \\ -a/3b & a/3b & -a/6b & a/6b \\ a/6b & -a/6b & a/3b & -a/3b \\ -a/6b & a/6b & -a/3b & a/3b \end{bmatrix} \tag{29c}$$

$$\int_A \psi_{i,y} \psi_{j,x} \, dA = \begin{bmatrix} 1/4 & 1/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \end{bmatrix} \tag{29d}$$

$$\int_A \psi_{i,x} \psi_j \, dA = \begin{bmatrix} -b/3 & -b/6 & -b/3 & -b/6 \\ -b/6 & -b/3 & -b/6 & -b/3 \\ b/3 & b/6 & b/3 & b/6 \\ b/6 & b/3 & b/6 & b/3 \end{bmatrix} \tag{29e}$$

Table 1 Material properties

E_1/E_2	υ_{12}	G_{12}	G_{13}	G_{23}
25	0.25	$0.5E_2$	$0.5E_2$	$0.2E_2$

Table 2 Non-dimensional maximum displacements (w*) in three-layer (0°/90°/0°) SSSS laminated

Study	$2a/h$		
	10	20	100
HOPLT44	1.0970	0.7787	0.6709
Reddy (2004)	1.0900	0.7760	0.6705
Sheikh and Chakrabarti (2003)	1.0910	0.7763	0.6708
Sahoo and Singh (2013)	1.1237	---	0.6709
Xiao et. al. (2008)	1.1055	0.7694	---
Pagano and Hatfield (1972)	1.1533	---	0.6712

$$\int_A \psi_{i,y} \psi_j \, dA = \begin{bmatrix} -a/3 & -a/3 & -a/6 & -a/6 \\ a/3 & a/3 & a/6 & a/6 \\ -a/6 & -a/6 & -a/3 & -a/3 \\ a/6 & a/6 & a/3 & a/3 \end{bmatrix} \tag{29f}$$

The finite element matrix of HOPLT44 plate element is obtained as given in Eq. (30).

3. Numerical examples

In this section, the present mixed-type finite element approach is assessed through different static tests. Bending and stress analysis of cross-ply laminated composite plates are considered. To show the efficacy of the present models several numerical examples were compared with those of other shear deformation theories. For analysis of laminated composite plate, a program is made on FORTRAN. The following normalized quantities (Eq. 31) are defined for deflection, stresses and moments

$$\begin{aligned} & w^* = w\left(\frac{E_2 h^3}{(2b)^4 q_0}\right).100 \\ & \sigma^*_{xx}, \sigma^*_{yy}, \tau^*_{xy} = \left[\sigma_{xx}, \sigma_{yy}, \tau_{xy}\right] h^2 \big/ (2b)^2 \\ & \tau^*_{xz}, \tau^*_{yz} = \left[\tau_{xz}, \tau_{yz}\right] h \big/ (2b) \\ & K^* = K/q(2a)^2,\ M^* = M/q(2a)^2,\ T^* = T/q(2a)^2 \\ & S^* = S/q(2a),\ Q^* = Q/q(2a) \end{aligned} \tag{31}$$

The mechanical properties of each layer are given in Table 1. All layers have the same thickness. Simply supported plate subjected.

3.1 Three layer symmetric cross-ply composite plate

In this example, a three-layered simply-supported (SSSS) symmetric (0°/90°/0°) cross-ply square laminated

$$
\left[k_{el}\right]=
\begin{array}{c}
\begin{array}{ccccccccccc}
[w] & [\gamma_1] & [\gamma_2] & [K] & [M] & [T] & [\bar{K}] & [\bar{M}] & [\bar{T}] & [S] & [Q]
\end{array}\\
\begin{bmatrix}
0 & 0 & 0 & [k_4] & [k_5] & [k_6] & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & [k_2]^T & 0 & [k_3]^T & [k_1] & 0 \\
 & & 0 & 0 & 0 & 0 & 0 & [k_3]^T & [k_2]^T & 0 & [k_1] \\
 & & & \alpha[k_1] & -\Upsilon[k_1] & 0 & -\Theta[k_1] & \Delta[k_1] & 0 & 0 & 0 \\
 & & & & \beta[k_1] & 0 & \Delta[k_1] & -\Pi[k_1] & 0 & 0 & 0 \\
 & & & & & \frac{-85}{D_{66}}[k_1] & 0 & 0 & \frac{105}{D_{66}}[k_1] & 0 & 0 \\
 & & & & & & \bar{\alpha}[k_1] & -\Lambda[k_1] & 0 & 0 & 0 \\
 & & & & & & & \bar{\beta}[k_1] & 0 & 0 & 0 \\
 & & sym. & & & & & & \frac{-525}{4D_{66}}[k_1] & 0 & 0 \\
 & & & & & & & & & \frac{-1}{A_{55}}[k_1] & \\
 & & & & & & & & & & \frac{-1}{A_{44}}[k_1]
\end{bmatrix}
\end{array}
\qquad (30)
$$

Table 3 Non-dimensional stresses in three-layer (0°/90°/0°) SSSS laminated plate

$2a/h$	Study	σ^*_{xx}	σ^*_{yy}	τ^*_{xy}	τ^*_{xz}	τ^*_{yz}
100	HOPLT44	0.80702	0.1917	0.0423	0	0.2466
	Reddy (2004)	0.8072	0.1925	0.0426	0.7744	0.2842
20	HOPLT44	0.8186	0.2299	0.0428	0	0.2536
	Reddy (2004)	0.7983	0.2227	0.0453	0.7697	0.2902
	Xiao et. al. (2008)	0.8125	0.2300	0.0458	1.070	0.3570
10	HOPLT44	0.8434	0.3393	0.0436	0	0.2772
	Reddy (2004)	0.7719	0.3072	0.0514	0.7548	0.3107
	Xiao et. al. (2008)	0.7660	0.2900	0.0484	0.660	0.285

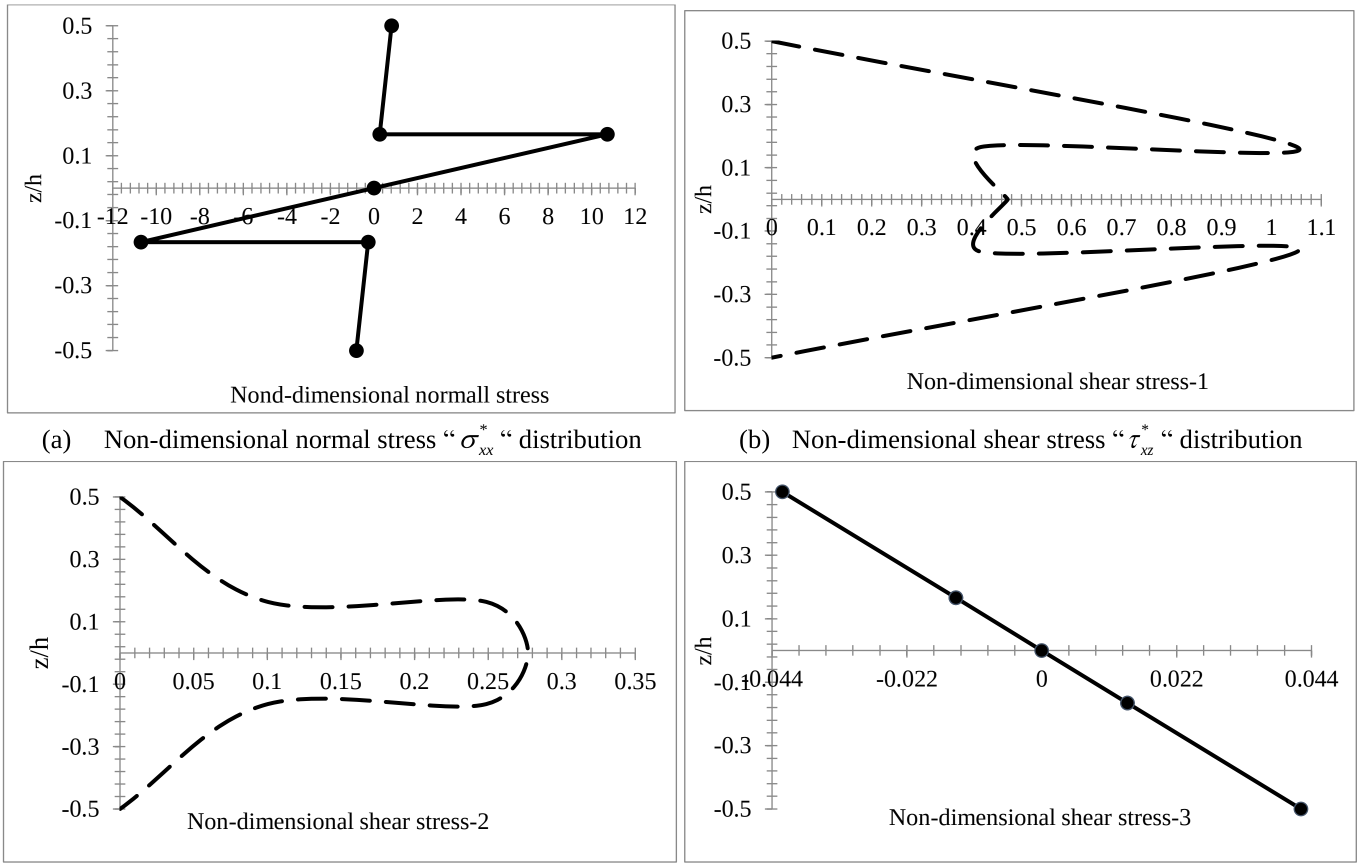
plate under uniform load is studied. The non-dimensional deflection and stresses are computed using present method. By varying the side to thickness ratio "$2a/h$" is 10 to 100 (thick to thin), the static analysis is performed and the results are tabulated in Table 2 and Table 3.

The membrane stress was evaluated at the this locations"$\sigma^*_{xx}(a,b,h/2)$","$\sigma^*_{yy}(a,b,h/4)$"and"$\tau^*_{xy}(2a,2b,h/2)$". The transverse shear stresses are calculated using the constitutive equations. The "τ^*_{xz}" is evaluated at in layers 1 and 3, and "τ^*_{yz}" is computed at $(2a,0)$, $(0,2b)$ in layers 2. The results of the present method are in excellent agreement with other solutions. For further comparisons, the variation of distribution of normal stress "σ^*_{xx}" and the transverse shear stress "τ^*_{xz}" "τ^*_{xy}" and "τ^*_{yz}" of the present models are plotted through the thickness in Figure 3. The HOPLT44 plate element is satisfy the condition of zero transverse shear stress at the top and bottom surfaces of plates and considers a parabolic transverse shear stress.

In order to investigate the normal and higher-order moments values which are effective in stresses of plate calculated with shear deformation theories. As the displacement of components of composite plate element based upon high order theory was obtained with the aim of virtual displacement principle, moment values expressions caused by higher-order terms are included in field equation like the one shown at Eq. 20. In Tables 4-9., the variation between the moment values $\bar{K}$, $\bar{M}$ and $\bar{T}$ caused by high order terms and normal moments K, M and T is shown.

Table 4 Non-dimensional moments $K^*(a,b)$, $M^*(a,\underline{b})$, $T^*(a,b)$ and higher-order moments $\bar{K}^*(a,b)$, $\bar{M}^*(a,b)$, $\bar{T}^*(2a,2b)$ values for SSSS (0°/90°/0°) laminated plate (E_1/E_2=10)

$2a/h=10$					
K^*	M^*	T^*	$\bar{K}^*$	$\bar{M}^*$	$\bar{T}^*$
0.1085	0.0182	0.0151	0.0866	0.0146	0.0120
$2a/h=20$					
K^*	M^*	T^*	$\bar{K}^*$	$\bar{M}^*$	$\bar{T}^*$
0.1118	0.0160	0.0144	0.0120	0.0894	0.0128
$2a/h=100$					
K^*	M^*	T^*	$\bar{K}^*$	$\bar{M}^*$	$\bar{T}^*$
0.1128	0.0153	0.0141	0.0902	0.0122	0.0011



(a) Non-dimensional normal stress “ σ_{xx}^{*} “ distribution (b) Non-dimensional shear stress “ τ_{xz}^{*} “ distribution

(c) Non-dimensional shear stress “ τ_{yz}^{*} “distribution (d) Non-dimensional shear stress “ τ_{xy} “ distribution

Fig. 3 Non-dimensional stress distributions of HOPLT44 (0°/90°/0°) composite plate element

Table 5 Non-dimensional higher-order shear forces $\bar{S}^{*}(2a,0)$ and $\bar{Q}^{*}(0,2b)$ values for SSSS (0°/90°/0°) laminated plate (E_1/E_2=10)

$2a/h=10$		$2a/h=20$		$2a/h=100$	
$\bar{S}^{*}$	$\bar{Q}^{*}$	$\bar{S}^{*}$	$\bar{Q}^{*}$	$\bar{S}^{*}$	$\bar{Q}^{*}$
0.1128	0.0153	0.0141	0.0902	0.0122	0.0011

Table 6 Non-dimensional moments $K^{*}(a,b)$, $M^{*}(a,\underline{b})$, $T^{*}(a,b)$ and higher-order moments $\bar{K}^{*}(a,b)$, $\bar{M}^{*}(a,b)$, $\bar{T}^{*}(2a,2b)$ values for SSSS (0°/90°/0°) laminated plate (E_1/E_2=25)

$2a/h=10$					
K^{*}	M^{*}	T^{*}	$\bar{K}^{*}$	$\bar{M}^{*}$	$\bar{T}^{*}$
0.1232	0.0140	0.0085	0.0978	0.0112	0.0068
$2a/h=20$					
K^{*}	M^{*}	T^{*}	$\bar{K}^{*}$	$\bar{M}^{*}$	$\bar{T}^{*}$
0.1285	0.0099	0.0075	0.1026	0.0079	0.0059
$2a/h=100$					
K^{*}	M^{*}	T^{*}	$\bar{K}^{*}$	$\bar{M}^{*}$	$\bar{T}^{*}$
0.1304	0.0086	0.0072	0.1041	0.0068	0.0056

Table 7 Non-dimensional higher-order shear forces $\bar{S}^{*}(2a,0)$ and $\bar{Q}^{*}(0,2b)$ values for SSSS (0°/90°/0°) laminated plate (E_1/E_2=25)

$2a/h=10$		$2a/h=20$		$2a/h=100$	
$\bar{S}^{*}$	$\bar{Q}^{*}$	$\bar{S}^{*}$	$\bar{Q}^{*}$	$\bar{S}^{*}$	$\bar{Q}^{*}$
0.3845	0.1243	0.4035	0.1138	0.4105	0.1106

Table 8 Non-dimensional moments $K^{*}(a,b)$, $M^{*}(a,\underline{b})$, $T^{*}(a,b)$ and higher-order moments $\bar{K}^{*}(a,b)$, $\bar{M}^{*}(a,b)$, $\bar{T}^{*}(2a,2b)$ values for SSSS (0°/90°/0°) laminated plate laminated plate (E_1/E_2=40)

$2a/h=10$					
K^{*}	M^{*}	T^{*}	$\bar{K}^{*}$	$\bar{M}^{*}$	$\bar{T}^{*}$
0.1265	0.0139	0.0063	0.1000	0.0111	0.0050
$2a/h=20$					
K^{*}	M^{*}	T^{*}	$\bar{K}^{*}$	$\bar{M}^{*}$	$\bar{T}^{*}$
0.1337	0.0082	0.0054	0.1061	0.0068	0.0041
$2a/h=100$					
K^{*}	M^{*}	T^{*}	$\bar{K}^{*}$	$\bar{M}^{*}$	$\bar{T}^{*}$
0.1349	0.0064	0.0047	0.1079	0.0051	0.0038

Table 9 Non-dimensional higher-order shear forces $\bar{S}^*(2a,0)$ and $\bar{Q}^*(0,2b)$ values for SSSS (0°/90°/0°) laminated plate (E_1/E_2=40)

$2a/h=10$		$2a/h=20$		$2a/h=100$	
$\bar{S}^*$	$\bar{Q}^*$	$\bar{S}^*$	$\bar{Q}^*$	$\bar{S}^*$	$\bar{Q}^*$
0.3836	0.1194	0.4092	0.1050	0.4816	0.1004

Table 10 Non-dimensional displacement of SSSS (0°/90°/0°) laminated plate for different " E_1/E_2 " ratio

E_1/E_2	$2a/h$		
	10	20	100
3	3.247	3.0325	2.961
7	2.180	1.9213	1.837
10	1.798	1.5188	1.426
15	1.438	1.138	1.038
20	1.231	0.920	0.8154
40	0.8788	0.5510	0.4382

Table 11 Non-dimensional maximum displacements (w^*) in four-layer (0°/90°/90°/0°) SSSS laminated

Study	$2a/h$		
	10	20	100
HOPLT44	1.120	0.7981	0.6848
Thai et. al. (2012)	1.1184	0.7966	---
Xiao et.al. (2008)	0.9730	0.7506	---
Sahoo and Sing (2013)	1.1143	0.7951	0.6844
Pagano and Hatfield (1972)	1.1393	0.8026	0.6847

3.2 Four layer symmetric cross-ply composite plate

In this example, the same example as in the previous section (Section 3.1.) is considered with four-layer symmetric (0°/90°/90°/0°) laminated composite plate. Other approaches available in open literature are given for further comparison. Table 11. and Table 12 contain the non-dimensional deflection and stresses as defined in example 3.1., respectively. Table 11 and Table 12 show that the present results are in good agreement with 3D-elasticity solution in deflection and stresses.

Table 13 Non-dimensional moments $K^*(a,b)$, $M^*(a,\underline{b})$, $T^*(a,b)$ and higher-order moments $\bar{K}^*(a,b)$, $\bar{M}^*(a,b)$, $\bar{T}^*(2a,2b)$ values for SSSS (0°/90°/90°/0°) laminated plate (E_1/E_2=10)

K^*	M^*	T^*	$\bar{K}^*$	$\bar{M}^*$	$\bar{T}^*$
$2a/h=10$					
0.0981	0.0295	0.0147	0.0782	0.0236	0.0117
$2a/h=20$					
0.1029	0.0259	0.0139	0.0823	0.0207	0.0111
$2a/h=100$					
0.1046	0.0246	0.0136	0.0836	0.0197	0.0109

Table 14 Non-dimensional higher-order shear forces $\bar{S}^*(2a,0)$ and $\bar{Q}^*(0,2b)$ values for SSSS (0°/90°/90°/0°) laminated plate (E_1/E_2=10)

$2a/h=10$		$2a/h=20$		$2a/h=100$	
$\bar{S}^*$	$\bar{Q}^*$	$\bar{S}^*$	$\bar{Q}^*$	$\bar{S}^*$	$\bar{Q}^*$
0.3433	0.1783	0.3579	0.1705	0.3630	0.1680

Table 15 Non-dimensional moments $K^*(a,b)$, $M^*(a,\underline{b})$, $T^*(a,b)$ and higher-order moments $\bar{K}^*(a,b)$, $\bar{M}^*(a,b)$, $\bar{T}^*(2a,2b)$ values for SSSS (0°/90°/90°/0°) laminated plate (E_1/E_2=25)

K^*	M^*	T^*	$\bar{K}^*$	$\bar{M}^*$	$\bar{T}^*$
$2a/h=10$					
0.1080	0.0312	0.0079	0.0856	0.0250	0.0062
$2a/h=20$					
0.1174	0.0233	0.0069	0.0937	0.0187	0.0055
$2a/h=100$					
0.1211	0.0201	0.0065	0.0968	0.0161	0.0052

Table 12 Non-dimensional stresses in four-layer (0°/90°/90°/0°) SSSS laminated plate

$2a/h$	Study	σ^*_{xx}	σ^*_{yy}	τ^*_{xy}	τ^*_{xz}	τ^*_{yz}
100	HOPLT44	0.8246	0.3565	0.0394	0	0.3071
	Belinha and Dinis (2006)	0.8230	0.3548	0.0391	0.8430	0.3330
20	HOPLT44	0.82247	0.4168	0.0397	0	0.3202
	Reddy (2004)	0.7980	0.4819	0.0399	0.6856	0.3108
	Xiao et. al. (2008)	0.7975	0.4725	0.0385	0.600	0.2845
10	HOPLT44	0.8230	0.5654	0.0392	0	0.3588
	Reddy (2004)	0.7561	0.4966	0.0425	0.713	0.3320
	Xiao et. al. (2008)	0.7577	0.5006	0.0470	0.7986	0.3499

Table 16 Non-dimensional higher-order shear forces $\bar{S}^*(2a,0)$ and $\bar{Q}^*(0,2b)$ values for SSSS (0°/90°/90°/0°) laminated plate (E_1/E_2=25)

$2a/h=10$		$2a/h=20$		$2a/h=100$	
$\bar{S}^*$	$\bar{Q}^*$	$\bar{S}^*$	$\bar{Q}^*$	$\bar{S}^*$	$\bar{Q}^*$
0.3478	0.1676	0.3784	0.1495	0.3901	0.1434

Table 17 Non-dimensional moments $K^*(a,b)$, $M^*(a,b)$, $T^*(a,b)$ and higher-order moments $\bar{K}^*(a,b)$, $\bar{M}^*(a,b)$, $\bar{T}^*(2a,2b)$ values for SSSS (0°/90°/90°/0°) laminated plate (E_1/E_2=40)

$2a/h=10$					
K^*	M^*	T^*	$\bar{K}^*$	$\bar{M}^*$	$\bar{T}^*$
0.1079	0.0349	0.005671	0.0852	0.0279	0.004487
$2a/h=20$					
K^*	M^*	T^*	$\bar{K}^*$	$\bar{M}^*$	$\bar{T}^*$
0.1207	0.0237	0.004777	0.0962	0.0189	0.003807
$2a/h=100$					
K^*	M^*	T^*	$\bar{K}^*$	$\bar{M}^*$	$\bar{T}^*$
0.1262	0.01887	0.004340	0.1010	0.01510	0.003471

Table 18 Non-dimensional higher-order shear forces $\bar{S}^*(2a,0)$ and $\bar{Q}^*(0,2b)$ values for SSSS (0°/90°/90°/0°) laminated plate (E_1/E_2=40)

$2a/h=10$		$2a/h=20$		$2a/h=100$	
$\bar{S}^*$	$\bar{Q}^*$	$\bar{S}^*$	$\bar{Q}^*$	$\bar{S}^*$	$\bar{Q}^*$
0.3376	0.1712	0.3806	0.1455	0.3986	0.1358

Table 19 Non-dimensional displacement of SSSS (0°/90°/90°/0°) laminated plate for different " E_1/E_2 " ratio

E_1/E_2	$2a/h$		
	10	20	100
3	3.284	3.047	2.967
7	2.220	1.941	1.848
10	1.838	1.540	1.440
15	1.472	1.160	1.052
20	1.260	0.939	0.829
40	0.886	0.567	0.4508

5. Conclusions

The conclusions that quit from this study can be summarized as follows:

- A generalized high order functional and formulation are developed for shear deformation theories that includes shear strain function. Gâteaux differential method has been used.
- Using the virtual work principle the governing differential equations are derived.
- Boundary conditions terms are constructed and introduced to the functional in a systematic way.
- The closed form of the element equation HOPLT44 is obtained which eliminate the time-consuming numerical inversion of the element matrix. It has has four nodes with eleven degrees of freedom per node and total fourty-four freedoms.
- The variables chosen independently, the displacements, rotations, shear forces, moments and higher-order moments calculated clearly with less number of elements but more sensitive.
- A computer program is developed in FORTRAN language to carry out the analyses.
- The comparison studies concerning the influences of length-to-thickness ratios, layer numbers and Young's modulus ratios demonstrate deflections, stress, moments, shear forces and higher-order moments of cross-ply laminated plates.
- Numerical results are in excellent agreement with those of available results in literature.

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