**Structural Engineering and Mechanics**, *Vol. 73, No. 1 (2020) 83-95*
DOI: https://doi.org/10.12989/sem.2020.73.1.083


# Formulae for the frequency equations of beam-column system carrying a fluid storage tank

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*(Received December 29, 2018, Revised September 4, 2019, Accepted September 12, 2019)*

**Abstract.** In this work, a mathematical model of beam-column system carrying a double eccentric end mass system is investigated, and solved analytically based on the exact solution analysis. The model considers the case in which the double eccentric end mass is a rigid storage tank containing fluid. Both Timoshenko and Bernoulli-Euler beam bending theories are considered. Equation of motion, general solution and boundary conditions for the present system model are developed and presented in dimensional and non-dimensional format. Several important non-dimensional design parameters are introduced. Symbolic and/or explicit formulae of the frequency and mode shape equations are formulated. To the authors knowledge, the present reduced closed form symbolic and explicit frequency equations have not appeared in literature. For different applications, the results are validated using commercial finite element package, namely ANSYS. The beam-column system investigated in this paper is significant for many engineering applications, especially, in mechanical and structural systems.

**Keywords:** thick beam-column, free vibration, exact solution, tank with fluid, eccentric mass

## 1. Introduction

Axially loaded beams are important structural members and have many applications in structural, mechanical and aerospace systems such as water tanks, wind turbines, and construction members. The loading conditions for these beams may be quasi-static or dynamic, therefore close attention should be paid to their dynamic and vibrational characteristics.

Timoshenko (1922) was the first to introduce the effect of transverse shear deformation on the vibration of beams. Huang (1961) introduced the solution of Timoshenko beam without axial load under different boundary conditions. The shear coefficients in the Timoshenko's beam theory was driven by Cowper (1966). The effect of axial load on the transverse vibration of Timoshenko beams was introduced by (Kounadis 1980, Sato 1991). Farghaly and Shebl (1995) investigated the vibration problem of axially loaded single span Timoshenko beams with system of complex end conditions, in which extensive comparisons between Sato and Kounadis contributions were presented. Several investigators discussed the axially loaded vibrating beams with simple end conditions, with and without end mass. From these contributors,(Takahashi 1980, Grossi and Laura 1982, Abbas 1984, Kanaka Raju and Venkateswara Rao 1984, Bokaian 1988, Stephen 1989, Abramovich 1992, Farghaly 1992, Demirdag and Yesilce 2011, El-Sayed and Farghaly 2017)

Recently, researchers have shown an increased interest in investigating beam systems with complex end conditions, see for example, (El-Sayed and Farghaly 2016, Farghaly and El-Sayed 2016, Malaeke and Moeenfard 2016, El-Sayed and Farghaly 2018, Rezaiee Pajand, Aftabi Sani *et al*. 2018). El-Sayed and Farghaly (2016) investigated the natural frequencies, mode shapes and the critical buckling load coefficients of a single span axially loaded Timoshenko beam system with complex end conditions, including mass spring sub-system. Malaeke and Moeenfard (2016) investigated the large amplitude flexural–extensional free vibration of tapered cantilever beams carrying both transversely and axial eccentric end mass. A few number of literature concerning the exact free vibration and stability of partially filled tank are available, from these (Feodosyev 1983, Ibrahim 2005, Badran and Gaith *et al*. 2012).

In the present work the transverse vibration of an axially loaded beam-column system with double eccentric end-mass is investigated. The end mass is represented as a partially filled rigid storage tank. The symbolic and explicit formulae of the modal frequency equation and mode shape for such a complex model are developed. The study includes both of Timoshenko's beam theory (TBT), as well as Bernoulli-Euler beam theory (EBT). Both elevated and suspended tank situations are considered. The sloshing inside the tank is neglected. The generalized formula for frequency equations according to both TBT and EBT are derived. In addition, two categories of application examples with clamped base models are investigated numerically using the present analytical solutions. For the validation of the current work, selected examples are modeled using ANSYS finite element software and the obtained results are compared with the present frequency equations results.

## 2. Mathematical model

Figure 1 shows an axially loaded beam-column system with root and end flexibilities. An eccentric mass is rigidly

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http://www.techno-press.com/journals/sem&subpage=7

ISSN: 1225-4568 (Print), 1598-6217 (Online)

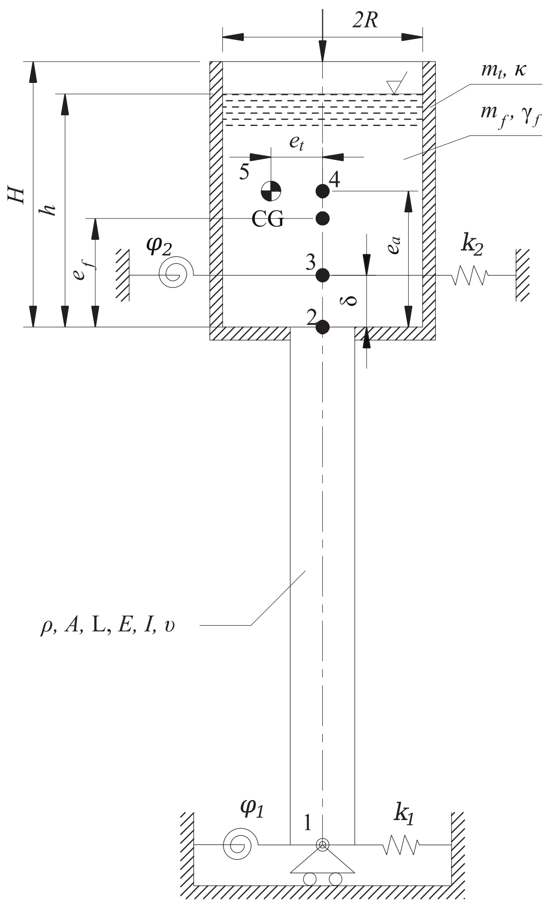


Fig. 1 Schematic for complex beam-column model

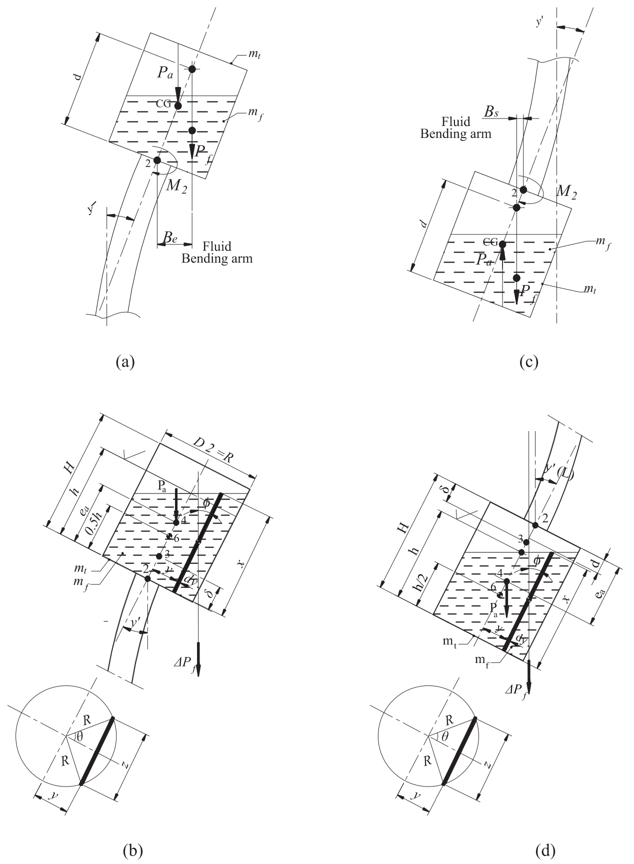

Fig. 2 Elevated (a, b), and suspended (c, d), cylindrical tank situations

connected to the end of the column. This is mass cylindrical partially filled tank. A set of design variables, such as the modulus of elasticity $E$, moment to of inertia $I$, beam area $A$, beam material density$\rho$, shear modulus of elasticity $G$, shear geometrical factor $k'$ and span length $L$, are considered. End translational and rotational spring stiffness's $(k_1, \phi_1)$, $(k_2, \phi_2)$ are acting at the base point 1 and at a distance $\delta$ from the end point of attachment 2, respectively. The mass of the tank is $m_t$ and its radius of gyration is $\kappa$. The center of gravity of the tank mass is located at distance $e_a, e_t$ from the point 2 in the axial and transverse directions respectively. The centroid of the fluid mass $m_f$ is located at distance $e_f$ from point 2. The system is also subjected to a constant axial load $P_a$ applied at point 4 and fluid load $P_f$. Two situations of the storage tank are considered which are the elevated and suspended situations as shown in Fig. 2(a) and (c) respectively.

## 3. Equation of motion and boundary conditions

In the current paper, the transverse free vibration is investigated using both Timoshenko and Bernoulli-Euler beam theories. The boundary conditions include an eccentric rigid partially filled storage tank connected at point 2. The detailed derivations of the bending moment and its arm acting on the end point of attachment 2 are given in section 3.2.1.

### *3.1 Equation of motion*

#### *3.1.1 According to Timoshenko beam theory (TBT)*

The equation of motion for uniform classical Timoshenko beam theory are presented using two coupled Eqs. (1) and (2). The two coupled equations are function in lateral deflection $y(x,t)$ and slope due to bending $\psi(x,t)$, see for example, (Sato 1991).

$$(k'GA)(y''-\psi')(x,t)-Py''(x,t)-\rho A\ddot{y}(x,t)=0; \quad (1)$$

$$EI\psi''(x,t)+(k'GA)(y'-\psi)(x,t)-\rho I\ddot{\psi}(x,t)=0 \quad (2)$$

The prime $(\ )'$ indicates $\frac{\partial()}{\partial x}$ and the dot $(\dot{\ })$indicates $\frac{\partial()}{\partial t}$

Let

$$y(x,t)=\mathrm{Y}(\zeta)e^{j\omega t} \quad (3)$$

$$\psi(x,t)=\Psi(\zeta)e^{j\omega t} \quad (4)$$

$$\zeta=\frac{x}{\mathrm{L}} \quad (5)$$

where $Y(\zeta)$ is the normal function of y,$\Psi(\zeta)$ is the normal function of $\psi, \zeta$ is the non-dimensional length of beam span, and $j=\sqrt{-1}$.

Substituting by Eqs. (3)-(5) into Eqs. (1) and (2) results in the following non-dimensional equations

$$p_s^2 Y''(\zeta) + \lambda^4 s^2 Y(\zeta) - L\Psi'(\zeta) = 0; \quad (6)$$

$$s^2 L\Psi''(\zeta) + (\lambda^4 s^2 r^2 - 1)L\Psi(\zeta) + Y'(\zeta) = 0 \quad (7)$$

Here, the prime $(.)'$ means d ( )/d$\zeta$ and

$$r^2 = I/AL^2; \quad \lambda^4 = \rho A\omega^2 L^4/EI; \; s^2 = r^2 E/Gk'; \quad p_s^2 = 1 - s^2 p^2, p^2 = PL^2/EI \quad (8a\text{-}e)$$

Substituting by $Y'(\zeta)$ in Eq. (7) into the differentiation of Eq. (6) results in the decoupled fourth order differential Eq. (9). Also, substituting by $L\Psi'(\zeta)$ in Eq. (6) into the differentiation of Eq. (7) results in Eq. (10).

$$Y''''(\zeta) + \bar{\alpha}\, Y''(\zeta) + \bar{\beta}^2\, Y(\zeta) = 0, \quad (9)$$

$$L\Psi''''(\zeta) + \bar{\alpha}\, L\Psi''(\zeta) + \bar{\beta}^2 L\Psi(\zeta) = 0 \quad (10)$$

The complementary solutions, for Eqs. (9) and (10) are as follows:

$$Y(\zeta) = A_1 \sin(a\,\zeta) + A_2 \cos(a\,\zeta) + A_3 \sinh(b\,\zeta) + A_4 \cosh(b\,\zeta); \quad (11)$$

$$L\,\Psi(\zeta) = -\left(\frac{\delta_1}{a}\right) A_1 \cos(a\,\zeta) + \left(\frac{\delta_1}{a}\right) A_2 \sin(a\,\zeta) + \left(\frac{\delta_2}{b}\right) A_3 \cosh(b\,\zeta) + \left(\frac{\delta_2}{b}\right) A_4 \sinh(b\,\zeta) \quad (12)$$

For the present model,

$$\delta_1 = \lambda^4 s^2 - a^2 p_s^2 \; and \; \delta_2 = \lambda^4 s^2 + b^2 p_s^2 \quad (13\text{ a-b})$$

where

$$a^2 = (\bar{\alpha}/2) + \left[(\bar{\alpha}/2)^2 - \bar{\beta}^2\right]^{1/2},$$
$$b^2 = -(\bar{\alpha}/2) + \left[(\bar{\alpha}/2)^2 - \bar{\beta}^2\right]^{1/2},$$
$$\bar{\alpha} = [\lambda^4(r^2 + s^2) - p^2(\lambda^4 r^2 s^2 - 1)]/p_s^2;$$
$$\text{and } \bar{\beta}^2 = \lambda^4(\lambda^4 r^2 s^2 - 1)/p_s^2; \quad (14\text{ e-h})$$

### 3.1.2 According to Euler-Bernoulli theory (EBT)

The equation of motion for the axially loaded beam according to Bernoulli-Euler beam bending theory can be written in the form

$$EIy''''(x,t) + Py''(x,t) + \rho A\ddot{y}(x,t) = 0 \quad (15)$$

Let $y(x,t) = Y(\zeta)e^{j\omega t}$ and omitting the factor $e^{j\omega t}$, Eq. (15) becomes

$$Y''''(\zeta) + p^2 Y''(\zeta) - \lambda^4 Y(\zeta) = 0. \qquad \zeta = x/L \quad (16)$$

The complementary solution for the differential Eq. (16) is as follows:

$$Y(\zeta) = A_1 \sin(a\,\zeta) + A_2 \cos(a\,\zeta) + A_3 \sinh(b\,\zeta) + A_4 \cosh(b\,\zeta); \quad (17)$$

where,

$$a^2 = (p^2/2) + [(p^2/2)^2 + \lambda^4]^{1/2} \text{ and}$$
$$b^2 = -(p^2/2) + [(p^2/2)^2 + \lambda^4]^{1/2}. \quad (18a,b)$$

## 3.2 Boundary conditions

### 3.2.1 Derivation of the bending moment due to fluid inside the tank

The storage tank consists of two masses $m_t$ for the rigid tank and $m_f$ for the fluid mass inside the tank. When the storage tank is fully filled or empty, the end mass can be considered as a rigid mass $M = (m_t + m_f)$. The rigid tank is connected inflexibly to the point of attachment 2. In this section, $\phi$ and $\theta$ are trigonometrical angles used to define the fluid element as shown in Fig. 2. The derivation of the fluid bending moment for elevated and suspended tank situations are obtained as follows

#### 3.2.1.1 Elevated partially filled tank, Fig. 2 (a, b)

Referring to Fig. 2 (b), the volume of the fluid slice is $x\,z\,dy$ . This slice applies vertical load $dp = \gamma\,x\,z\,dy$. The following analysis is based on the assumption of small oscillations; therefore, the following approximations can be used. $\sin\phi = \phi$ , $\tan\phi = \phi$, and $\cos\phi = 1$

The bending moment $dM_{fe}$ due to $dp$ becomes

$$dM_{fe} = dp(y\cos\phi + (x/2)\sin\phi) = dp\,[\,y + \phi(h + y\phi/2)] \quad (19)$$

where $x = h + y\phi$

By expanding Eq. (19) and eliminating the higher order, of $\phi$ , Eq. (19) can be reduced to:

$$dM_{fe} = z\,\gamma\left(y^2\,\phi + hy + h^2\frac{\phi}{2}\right) dy \quad (20)$$

From the geometric relation of the fluid element Fig. 2b, the following equations can be obtained

$$z = 2\,R\sin\theta \text{ and} \quad (21)$$

$$y = R\cos\theta \quad (22)$$

Differentiating both sides of Eq. (22)

$$dy = -R\sin\theta \quad d\,\theta \quad (23)$$

Substituting by Eq. (23) and Eq. (21) into Eq. (20) results in the following equation

$$dM_{fe} = -2\,\gamma R^2\left(R^2\phi\,\cos^2\theta\,\sin^2\theta + hR\cos\theta\,\sin^2\theta + \frac{h^2\phi}{2}\sin^2\theta\right)d\theta \quad (24)$$

Substituting by the trigonometric relations in Appendix A, Eqs. (76-82) into Eq. (24) results in the following equation

$$dM_{fe} = -2\,\gamma R^2\left[R^2\phi\,\frac{\sin^2 2\theta}{4} + \frac{hR}{4}(\cos\theta - \cos 3\theta) + \frac{h^2\phi}{2}\sin^2\theta\right]d\theta \quad (25)$$

by integrating over $\theta$ from $\pi$ to $0$;

$$M_{fe}=\int_{\pi}^{0}dM_{fe}=\left\{-2\,\gamma R^2\left[\frac{R^2\phi}{4}\left(\frac{\theta}{2}-\frac{sin4\theta}{8}\right)+\frac{hR}{4}\left(sin\theta-\frac{\sin3\theta}{3}\right)+\frac{h^2\phi}{2}\left(\frac{\theta}{2}-\frac{sin2\theta}{4}\right)\right]\right\}_{\pi}^{0} \quad (26)$$

$$M_{fe}=\gamma\frac{\pi R^2}{4}(2h^2+R^2)\,\phi=\gamma A_f h\left(\frac{h}{2}+\frac{R^2}{4h}\right)y'(L)=P_f\frac{h}{4}\left(2+\left(\frac{R}{h}\right)^2\right)y'(L)$$

The expression of the fluid bending arm $B_e$, for the case of elevated tank situation, becomes

$$B_e=\frac{h}{4}\left[2+\left(\frac{R}{h}\right)^2\right]y'(L) \quad (27)$$

### 3.2.1.2 Suspended partially filled tank, Fig. 2 (c, d)

The second case of *Fig.* 2 (d) is concerned with the derivation of the expressions governing the bending moment and its associated bending arm for partially filled tank suspended from the point of attachment 2. In this case the axial load due to fluid inside the tank causes a tensile load in the beam. The volume of the used fluid element is $x\,z\,dy$ and the vertical load due to this fluid element $dp=\gamma\,x\,z\,dy$.

The bending moment $dM_{fs}$ due to $dp$ becomes

$$dM_{fs}=dp\left[y\cos\phi-\sin\phi\left(H-\frac{y}{2}\right)\right]=dp\left[\left(1-\frac{\phi^2}{2}\right)y-\phi\left(H-\frac{y}{2}\right)\right]=\gamma\,x\,z\left[\left(1-\frac{\phi^2}{2}\right)y-\phi\left(H-\frac{y}{2}\right)\right]dy \quad (28)$$

Substituting by $x=h+y\tan\phi=h+y\phi$ into Eq. (28) and expanding with elimination of higher order of $\phi$. The fluid element bending moment can be written as follow

$$dM_{fs}=z\gamma\left[y+\left(\frac{h}{2}-H\right)\phi\right](h+y\phi)\,dy=z\gamma\left[y^2\phi+hy+h\left(\frac{h}{2}-H\right)\phi\right]dy \quad (29)$$

Using the relations in Eq. (21-23) leads to

$$\therefore dM_{fs}=-2\,\gamma\,R^2\left[y^2\phi+hy+h\left(\frac{h}{2}-H\right)\phi\right]\sin^2\theta\,d\theta \quad (30)$$

$$=-2\,\gamma\,R^2\left[\phi\,R^2\sin^2\theta\cos^2\theta+h\,R\,\sin^2\theta\cos\theta+\phi\,h\left(\frac{h}{2}-H\right)\sin^2\theta\right]d\theta$$

$$=-2\,\gamma\,R^2\left[\phi\,R^2\frac{\sin^2 2\theta}{4}+\frac{h\,R}{2}(\cos\theta-\cos\,3\theta)+h\left(\frac{h}{2}-H\right)\phi\,\sin^2\theta\right]d\theta$$

By integration over $\theta$ from $\pi$ to 0;

$$M_{fs}=\int_{\pi}^{0}dM_{fs}=\left\{-2\,\gamma R^2\left[\frac{\phi\,R^2}{4}\left(\frac{\theta}{2}-\frac{sin4\theta}{8}\right)+\frac{hR}{2}\left(\sin\theta-\frac{\sin3\theta}{3}\right)+h\left(\frac{h}{2}-H\right)\phi\left(\frac{\theta}{2}-\frac{sin2\theta}{4}\right)\right]\right\}_{\pi}^{0} \quad (31)$$

$$=\gamma\frac{\pi R^2}{4}(R^2+2h^2-4hH)\,\phi$$

$$M_{fs}=\gamma\frac{\pi R^2}{4}(R^2+2h^2-4hH)y'(L)=\gamma\,A_f\,h\frac{h}{4}\left(2+\left(\frac{R}{h}\right)^2-4\frac{H}{h}\right)y'(L) \quad (32)$$

$$=P_f\cdot\frac{h}{4}\left(2+\left(\frac{R}{h}\right)^2-4\frac{H}{h}\right)y'(L)$$

where,

$$P_f=\gamma_f A_f h \quad \text{and} \quad A_f=\pi R^2 \quad (33a\text{-}b)$$

Therefore, the expression of the fluid bending arm $B_s$, for the suspended tank situation, can be written in the form

$$B_s=\frac{h}{4}\left[2+\left(\frac{R}{h}\right)^2-4\left(\frac{H}{h}\right)\right]y'(L) \quad (34)$$

### 3.2.2 Boundary conditions according to TBT

The boundary conditions for the system shown in Fig. 1 may be developed in the dimensional expressions at $x=0$ and at $x=L$ as follows

at $x=0$;

$$EI\psi'(0)=\phi_1\psi(0); \quad (35)$$

$$(k'GA)\big(y'(0)-\psi(0)\big)=\big(P_a+P_f\big)y'(0)+k_1\,y(0) \quad (36)$$

at $x=L$;

$$EI\psi'(L)=-\left[-\omega^2 m_t(e_a^2+e_t^2+\kappa^2)-\omega^2 m_f e_f^2+(\phi_2+k_2\delta^2)\right]\psi(L)+\big(P_a e_a+P_f B_a\big)y'(L)-\big(-\omega^2 m_t e_a-\omega^2 m_f\,h/2+k_2\delta\big)y(L) \quad (37)$$

$$(k'GA)\left(y'(L)-\psi(L)\right) = -\left(-\omega^2 m_t e_a - \omega^2 m_f e_f + k_2\delta\right)\psi(L) + \left(P_a+P_f\right)y'(L) - \left(-\omega^2 m_t - \omega^2 m_f + k_2\right)y(L) \quad (38)$$

where $B_a = B_e$ in case of elevated tank and $B_a = B_s$ in case of suspended tank. $e_f = h/2$ in case of elevated tank.

Eqs. (35-38) can be rewritten in the non-dimensional form, at $\zeta = 0$ and $\zeta = 1$, as follows:

at $\zeta = 0$;

$$L\Psi'(0) - \Phi_1 L\Psi(0) = 0, \quad (39)$$

$$p_s^2 Y'(0) - Z_1 s^2 Y(0) - L\Psi(0) = 0 \quad (40)$$

and at $\zeta = 1$;

$$-p_t^2 Y'(1) - \beta\, Y(1) + L\Psi'(1) - \alpha_T L\Psi(1) = 0. \quad (41)$$

$$p_s^2 Y'(1) - s^2\,\theta\, Y(1) - \beta_s\, L\Psi(1) = 0 \quad (42)$$

where $\alpha_T, \beta, \beta_s$ and $\theta$ are developed and introduced as follows

$$\alpha_T = \lambda^4\left[\bar{m}_t(\bar{e}_a^2 + \bar{e}_t^2 + \bar{\kappa}^2) + \bar{m}_f \bar{e}_f^2\right] - \left(Z_2\bar{\delta}^2 + \Phi_2\right);$$

$$\theta = \lambda^4(\bar{m}_t + \bar{m}_f) - Z_2; \quad (43\text{ a-d})$$

$$\beta = \lambda^4\left(\bar{m}_t\bar{e}_a + \bar{m}_f\bar{e}_f\right) - Z_2\bar{\delta};$$

$$\beta_s = 1 + s^2\beta$$

where

$\Phi_1 = \phi_1 L/EI$; $\bar{e}_a = e_a/L$; $p_a^2 = P_a L^2/EI$;

$Z_1 = k_1 L^3/EI$; $\bar{e}_t = e_t/L$; $p_f^2 = P_f L^2/EI$;

$\Phi_2 = \phi_2 L/EI$; $\bar{\kappa} = \kappa/L$; $\bar{h} = h/L$; (44 a-r)

$Z_2 = k_2 L^3/EI$; $\bar{m}_t = m_t/\rho AL$; $\bar{R} = R/L$;

$\bar{\delta} = \delta/L$; $\bar{m}_f = m_f/\rho AL$; $\bar{H} = H/L$

$\bar{e}_f = e_f/L$ $\quad \bar{e}_f = \bar{h}/2$ (elevated tank)

### *3.2.3 Boundary conditions according to EBT*

The boundary conditions for the system shown in Fig. 1, may be written in the dimensional format due to the bending moment and the shear force, acting at the end point, respectively, as follows

at $x = 0$;

$$EIy''(0) = \phi_1 y'(0), \quad (45)$$

$$EIy'''(0) = -\left(P_a + P_f\right)y'(0) + k_1 y(0) \quad (46)$$

at $x = L$;

$$EIy''(L) = -\left[-\omega^2 m_t(e_a^2 + e_t^2 + \kappa^2) - \omega^2 m_f\,(h/2)^2 + \left(k_2\delta^2 + \phi_2 + P_a e_a + P_f B_a\right)\right]y'(L) - (-\omega^2 m_t e_a - \omega^2 m_t\, h/2 + k_2\delta)y(L) \quad (47)$$

$$EIy'''(L) = -\left[-\omega^2 m_t e_a + \left(k_2\delta + P_a + P_f\right)\right]y'(L) - \left(-\omega^2 m_t - \omega^2 m_f\, h/2 + k_2\right)y(L) \quad (48)$$

Equations (45-48) can be rewritten in the following non-dimensional format as shown below

*at* $\zeta = 0$;

$$Y''(0) - \Phi_1 Y'(0) = 0; \quad (49)$$

$$Y'''(0) + p^2 Y'(0) - Z_1 Y(0) = 0 \quad (50)$$

*at* $\zeta = 1$

$$Y''(1) - \alpha_E Y'(1) - \beta Y(1) = 0; \quad (51)$$

$$Y'''(1) - \gamma\, Y'(1) - \theta Y(1) = 0 \quad (52)$$

where

$$\alpha_E = \alpha_T - p_t^2; \quad \text{and} \quad \gamma = \beta - p^2 \quad (53\text{ a, b})$$

## 4. Characteristics determinants

### *4.1 Derivation of the characteristics elements*

Substitution by the general solutions into the non-dimensional boundary conditions leads to a system of four homogenous linear algebraic equations in $A_1, A_2, A_3$ and $A_4$. These equations may be presented in a matrix format of order 4×4. The elements according to this matrix of TBT are presented in Eqs. (55 a-p) and in case of EBT are presented in Eqs. (56 a-p), all are based on the suggested equation.

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \\ \epsilon_5 & \epsilon_6 & \epsilon_7 & \epsilon_8 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = 0 \quad (54)$$

where,

For TBT

$a_1 = \Phi_1(\delta_1/a)$; $\epsilon_1 = [-(\beta - \delta_1)\sin a + \eta_1 \cos a]$

$a_2 = \delta_1$; $\quad \epsilon_2 = -[\eta_1 \sin a + (\beta - \delta_1)\cos a]$

$a_3 = -\Phi_1(\delta_2/b)$; $\quad \epsilon_3 = -[(\beta - \delta_2)\sinh b + \eta_2 \cosh b]$ (55 a-p)

$a_4 = \delta_2$; $\quad \epsilon_4 = -[\eta_2 \sinh b + (\beta - \delta_2)\cosh b]$

$a_5 = \delta_3$; $\quad \epsilon_5 = [-s^2\,\theta \sin a + \eta_3 \cos a]$

$a_6 = -Z_1 s^2;\quad \epsilon_6 = -[\eta_3 \sin a + s^2\theta \cos a]$

$a_7 = \delta_4;\quad \epsilon_7 = [-s^2\theta \sinh b + \eta_4 \cosh b]$

$a_8 = -Z_1 s^2;\quad \epsilon_8 = [\eta_4 \sinh b - s^2\theta \cosh b]$

For EBT

$$
\begin{aligned}
&a_1 = -\Phi_1 a; && \epsilon_1 = [(\beta + a^2)\sin a + a\,\alpha_E \cos a] \\
&a_2 = -a^2; && \epsilon_2 = [-a\,\alpha_E \sin a + (\beta + a^2)\cos a] \\
&a_3 = -\Phi_1 b; && \epsilon_3 = [(\beta - b^2)\sinh b + b\,\alpha_E \cosh b] \\
&a_4 = b^2; && \epsilon_4 = [b\,\alpha_E \sinh b + (\beta - b^2)\cosh b] \qquad (56\text{ a-p}) \\
&a_5 = a\,(p^2 - a^2); && \epsilon_5 = [-\theta \sin a + a\,(\gamma + a^2)\cos a] \\
&a_6 = Z_1; && \epsilon_6 = [-a\,(\gamma + a^2)\sin a - \theta \cos a] \\
&a_7 = b\,(p^2 + b^2); && \epsilon_7 = [-\theta \sinh b + b\,(\gamma - b^2)\cosh b] \\
&a_8 = -Z_1; && \epsilon_8 = [b\,(\gamma - b^2)\sinh b - \theta \cosh b]
\end{aligned}
$$

### *4.2 Suggested formulae for expanding of the characteristics determinant*

For non-trivial solution of Eq. (54), the following determinant should be equal to zero.

$$
\det\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \\ \epsilon_5 & \epsilon_6 & \epsilon_7 & \epsilon_8 \end{bmatrix} = 0 \qquad (57)
$$

The 4×4 determinant equations can be expanded as follows

$$
G_1 + G_2 + G_3 + G_4 + G_5 + G_6 = 0 \qquad (58)
$$

where,

$$
\begin{aligned}
G_1 &= (a_1 a_6 - a_5 a_2)_{14}(\epsilon_3\epsilon_8 - \epsilon_7\epsilon_4)_{22}; \\
G_2 &= (a_3 a_8 - a_7 a_4)_{22}(\epsilon_1\epsilon_6 - \epsilon_5\epsilon_2)_{14}; \\
G_3 &= -(a_1 a_7 - a_5 a_3)_{16}(\epsilon_2\epsilon_8 - \epsilon_6\epsilon_4)_{20}; \qquad (59\text{ a-f}) \\
G_4 &= -(a_2 a_8 - a_6 a_4)_{20}(\epsilon_1\epsilon_7 - \epsilon_5\epsilon_3)_{16}; \\
G_5 &= (a_1 a_8 - a_5 a_4)_{18}(\epsilon_2\epsilon_7 - \epsilon_6\epsilon_3)_{18}; \text{ and} \\
G_6 &= (a_2 a_7 - a_6 a_3)_{18}(\epsilon_1\epsilon_8 - \epsilon_5\epsilon_4)_{18}
\end{aligned}
$$

### *4.3 Generalized explicit frequency equation*

#### *4.3.1 Timoshenko beam generalized frequency equation*

The Timoshenko beam frequency equation can be obtained by substituting Eqs. (55 a-p) into Eqs.(58-59). After extensive mathematical manipulations, one can obtain the frequency equation in its explicit form, for system with different classical and/or non-classical, simple and/or complex end conditions. Therefore, the generalized frequency equation according to TBT can be written in the following explicit form:

$$
\begin{aligned}
f(\lambda) = {} & (\eta_5\beta_1 + \eta_6\beta_2) + [-\delta_5(\eta_3\eta_2 + \eta_1\eta_4) - s^2\theta\,\delta_6(\delta_1 - \delta_2) + (\eta_7\beta_3 - \eta_8\beta_4)]\,\hat{s}\,\hat{\text{s}}\text{h} + [(\delta_5\beta_3 - \\
&\delta_6\beta_4) - \eta_7(\eta_3\eta_2 + \eta_1\eta_4) - s^2\theta\eta_8(\delta_1 - \delta_2)]\,\hat{s}\,\hat{\text{c}}\text{h} + \\
&[-(\delta_5\beta_4 + \delta_6\beta_3) - s^2\theta\eta_7(\delta_1 - \delta_2) + \eta_8(\eta_3\eta_2 + \eta_1\eta_4)]\,\hat{c}\,\hat{\text{s}}\text{h} + [-s^2\theta\delta_5(\delta_1 - \delta_2) + \delta_6(\eta_3\eta_2 + \eta_1\eta_4) - \\
&(\eta_7\beta_4 + \eta_8\beta_3)]\,\hat{c}\,\hat{\text{c}}\text{h} = 0
\end{aligned} \qquad (60)
$$

where

$$
\begin{aligned}
\eta_1 &= -p_t^2 a + \alpha_T(\delta_1/a); \\
\eta_2 &= p_t^2 b + \alpha_T(\delta_2/b); \\
\eta_3 &= p_s^2 a + \beta_s(\delta_1/a); \\
\eta_4 &= p_s^2 b - \beta_s(\delta_2/b); \\
\eta_5 &= -[\Phi_1 Z_1 s^2(\delta_1/a) + \delta_1\delta_3]; \\
\eta_6 &= [\Phi_1 Z_1 s^2(\delta_2/b) - \delta_2\delta_4]; \\
\eta_7 &= -[\Phi_1 Z_1 s^2(\delta_1/a) + \delta_2\delta_3]; \\
\eta_8 &= -[\Phi_1 Z_1 s^2(\delta_2/b) - \delta_1\delta_4]; \qquad (61\text{ a-p}) \\
\beta_1 &= [\eta_2 s^2\theta + \eta_4(\beta - \delta_2)]; \\
\beta_2 &= [-\eta_1 s^2\theta + \eta_3(\beta - \delta_1)]; \\
\beta_3 &= [\eta_1 s^2\theta - \eta_3(\beta - \delta_2)]; \\
\beta_4 &= [\eta_2 s^2\theta + \eta_4(\beta - \delta_1)] \text{ and} \\
\delta_3 &= p_s^2 a + (\delta_1/a); \\
\delta_4 &= p_s^2 b - (\delta_2/b); \\
\delta_5 &= -\Phi_1[(\delta_1\delta_4/a) + (\delta_2\delta_3/b)]; \\
\delta_6 &= Z_1 s^2(\delta_1 - \delta_2)
\end{aligned}
$$

where

$$
\begin{aligned}
&p_{te}^2 = p_a^2\bar{e}_a + p_f^2\bar{B}_e; && p_{ts}^2 = p_a^2\bar{e}_a + p_f^2\bar{B}_s; \\
&\bar{B}_e = \bar{h}\left[2 + \left(\bar{R}/\bar{h}\right)^2\right]/4; && \bar{B}_s = \bar{B}_e - (\bar{H}/4)
\end{aligned} \qquad (62\text{ a-d})
$$

The sign of $p_a^2$ and $p_f^2$ in Eqs. (62 a, b) is obtained according to Figure 1 (a), (c). The positive sign for $p_a^2$ represents a compressive axial load, and negative sign for $p_a^2$ represents a tensile load condition. The positive sign for $p_f^2$ is for elevated tank, and the negative sign for $p_f^2$ is for suspended tank situation.

Equations (60-62) are functions in the following dimensionless parameters $\lambda, s^2, r^2, \Phi_1, Z_1, \bar{m}, \Phi_2, Z_2,$

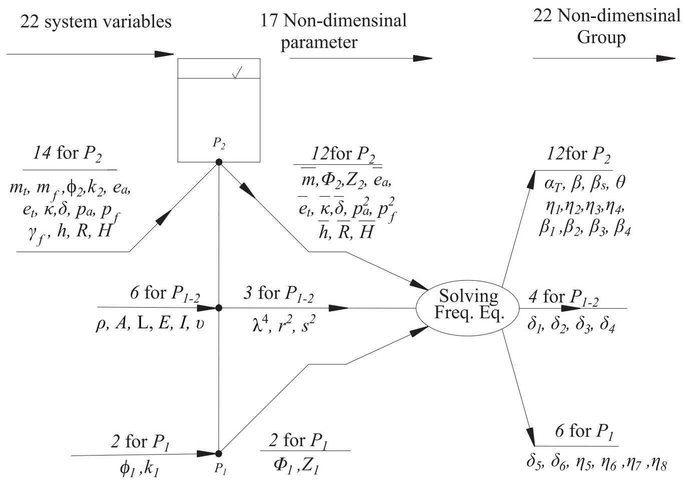


Fig. 3 Timoshenko model design variables, non-dimensional parameters and non-dimensional groups. $P_1$, means point 1, $P_2$ means point of attachment 2, and $P_{1-2}$ means span between point 1 and 2

$\bar{e}_a$, $\bar{e}_t$, $\bar{\kappa}$, $\bar{\delta}$, $p_a^2$, $p_f^2$, $\bar{h}$, $\bar{H}$ and $\bar{R}$ . It is important to note that, during the expansion procedures, several non-dimensional system design groups, $\alpha_T$, $\alpha_E$, $\gamma$, $\beta$, $\beta_s$, $p_{te}^2$ , $p_{ts}^2$ ,$\bar{B}_e$ and $\bar{B}_s$ are developed. Fig. 3 shows a schematic representation for the present model variables, non-dimensional parameters and non-dimensional groups.

#### *4.3.2 Generalized frequency equation for Bernoulli-Euler beam*

The frequency equation of the beam based on Bernoulli-Euler model can be obtained by substituting Eqs. (56 a-p) into Eqs. (58-59). The generalized frequency equation according to EBT can be written in the following explicit form:

$$\begin{gathered} f(\lambda) = (\eta_9 \beta_5 + \eta_{10} \beta_6) + [\delta_7 \alpha_E \, ab \, (a^2 + b^2) - \theta \delta_8 (a^2 + b^2) + (\eta_{11} \beta_7 + \eta_{12} \beta_8)] \hat{s} \, \hat{s}\mathrm{h} + \\ [(\delta_7 \beta_7 - \delta_8 \beta_8) + \eta_{11} \alpha_E \, ab \, (a^2 + b^2) + \theta \eta_{12} (a^2 + b^2)] \, \hat{s} \, \hat{c}\mathrm{h} + [(\delta_7 \beta_8 + \delta_8 \beta_7) + \\ \theta \eta_{11} (a^2 + b^2) - \eta_{12} \alpha_E ab (a^2 + b^2)] \, \hat{c} \, \hat{s}\mathrm{h} + \\ [\theta \delta_7 (a^2 + b^2) - \delta_8 \alpha_E ab (a^2 + b^2) + (\eta_{11} \beta_8 - \eta_{12} \beta_7)] \, \hat{c} \, \hat{c}\mathrm{h} = 0 \end{gathered} \tag{63}$$

$$\begin{aligned} \eta_9 &= a\,[\Phi_1 Z_1 + a^2(p^2 - a^2)]; \\ \eta_{10} &= b\,[\Phi_1 Z_1 - b^2(p^2 + b^2)]; \\ \eta_{11} &= a\,[\Phi_1 Z_1 - b^2(p^2 - a^2)]; \\ \eta_{12} &= -b\,[\Phi_1 Z_1 + a^2(p^2 + b^2)]; \\ \beta_5 &= b[\theta\,\alpha_E - (\beta - b^2)(\gamma - b^2)]; \\ \beta_6 &= a[\theta\,\alpha_E - (\beta + a^2)(\gamma + a^2)] \\ \beta_7 &= -a[\,\alpha_E \theta - (\gamma + a^2)(\beta - b^2)]; \\ \beta_8 &= -b\,[\theta \alpha_E - (\beta + a^2)(\gamma - b^2)] \\ \delta_7 &= \Phi_1 ab(a^2 + b^2) \\ \delta_8 &= -Z_1(a^2 + b^2) \end{aligned} \tag{64 a-i}$$

It is interesting to note that, the frequency Eqs. (60) and (63), in their simplified explicit formulae, have not appeared before in literature.

### *4.4 Reduced generalized explicit frequency equations*

#### *4.4.1 Reduced frequency equations for clamped base models*

In many industrial applications, the base is approximately clamped, i.e. $Z_1$ and $\Phi_1 \to \infty$ . This simplifies the general Eqs. (60) and (63), by dividing their terms by $\Phi_1, Z_1 \to \infty$ . TBT frequency equation for clamped base model in an explicit form becomes

$$\begin{gathered} f(\lambda) = (\delta_2 \beta_2 \, a - \delta_1 \beta_1 b) + (-\delta_1 \beta_3 \, b + \delta_2 \beta_4 \, a) \, \hat{s} \, \hat{s}\mathrm{h} + [s^2 \theta \;\, a\, \delta_2 (\delta_1 - \delta_2) + \delta_1 b (\eta_3 \, \eta_2 + \\ \eta_1 \, \eta_4)] \, \hat{s} \, \hat{c}\mathrm{h} + [s^2 \theta \, \delta_1 b \, (\delta_1 - \delta_2) - \delta_2 a (\eta_3 \, \eta_2 + \eta_1 \, \eta_4)] \, \hat{c} \, \hat{s}\mathrm{h} + (\delta_1 \beta_4 b + \delta_2 \, \beta_3 a) \, \hat{c} \, \hat{c}\mathrm{h} = 0 \end{gathered} \tag{65}$$

similarly, EBT frequency equation for clamped base model in an explicit form becomes

$$\begin{aligned} f(\lambda) = (a\,\beta_5 + b\beta_6) &+ (a\,\beta_7 - b\,\beta_8)\,\hat{s}\,\hat{s}\mathrm{h} \\ &+ \mathrm{b}(a^2 + b^2)(a^2 \alpha_E - \theta)\,\hat{s}\,\hat{c}\mathrm{h} \\ &+ a\,(a^2 + b^2)(b^2 \alpha_E + \theta)\,\hat{c}\,\hat{s}\mathrm{h} \\ &+ (a\,\beta_8 + b\,\beta_7)\,\hat{c}\,\hat{c}\mathrm{h} = 0 \end{aligned} \tag{66}$$

#### *4.4.2 Formulae for simple practical applications*

It is worth noting that Eqs. (65-66) may be reduced to simpler formulae, according the values of the following non-dimensional parameters $\alpha_E$, $\beta$, $\gamma$, $\theta$, $\eta_1$, $\eta_2$, $\beta_5$, $\eta_3$, $\eta_4$ and $\alpha_T$. Five simplified cases with single effect boundary conditions [group I] are selected and presented in Fig. 4. The corresponding inputs parameters for these cases are a listed in Table 1. The formulae are tested through numerical examples as shown in Table 2. In addition, Fig. 4 shows

Table 1 Input parameters for the five models shown in Fig. 4 (I)

| Model | $\alpha_E$ | $\beta$ | $\gamma$ | $\theta$ | $\eta_1$ | $\eta_2$ | $\beta_5$ | $\eta_3$ | $\eta_4$ | $\alpha_T$ |
|---|---|---|---|---|---|---|---|---|---|---|
| I(a) | 0 | 0 | 0 | 0 | 0 | 0 | 1 | $\frac{a^2-\delta_1}{a}$ | $\frac{b^2-\delta_2}{b}$ | 0 |
| I(b) | 0 | 0 | 0 | $-Z_2$ | 0 | 0 | 1 | $\frac{a^2-\delta_1}{a}$ | $\frac{b^2-\delta_2}{b}$ | 0 |
| *I(c)* | $-\Phi_2$ | 0 | 0 | 0 | $\frac{-\Phi_2\delta_1}{a}$ | $\frac{-\Phi_2\delta_1}{b}$ | 1 | $\frac{a^2-\delta_1}{a}$ | $\frac{b^2-\delta_2}{b}$ | $\Phi_2$ |
| I(d) | 0 | 0 | 0 | $\lambda^4\overline{m}$ | 0 | 0 | 1 | $\frac{a^2-\delta_1}{a}$ | $\frac{b^2-\delta_2}{b}$ | 0 |
| I(e) | 0 | 0 | $-p^2$ | 0 | 0 | 0 | 1 | $\frac{p_s^2a^2+\delta_1}{a}$ | $\frac{p_s^2b^2-\delta_2}{b}$ | 0 |

Table 2 Explicit formulae for reduced frequency equations, for the five models shown in Fig. 4 (I)

| Model | TBT | EBT |
|---|---|---|
| I (a) | $2-[(\delta_1^2+\delta_2^2)/\delta_1\delta_2]\,\hat{c}\,\hat{c}\mathrm{h}-[(a^2-b^2)/ab\,]\,\hat{s}\,\hat{s}\mathrm{h}=0\Rightarrow(CFT)$ | $1+\hat{c}\,\hat{c}h=0$ (reference equation) $\Rightarrow(CFE)$ |
| I (b) | $(CFT)+[\,Z_2(a^2+b^2)(\delta_2\,a\,\hat{s}\,\hat{c}\mathrm{h}+\delta_1 b\,\hat{c}\,\hat{s}\mathrm{h})/(\lambda^4\delta_1\delta_2)]=0$ | $\lambda^3(CFE)+Z_2(\hat{s}\,\hat{c}\mathrm{h}-\hat{c}\,\hat{s}\mathrm{h})=0$ |
| I (c) | $(CFT)-\Phi_2(\delta_1-\delta_2)\,(b\,\delta_1\,\hat{s}\,\hat{c}\mathrm{h}-a\,\delta_2\hat{c}\,\hat{s}\mathrm{h})/(ab\,\delta_1\delta_2)=0$ | $\lambda(CFE)+\Phi_2(\hat{s}\,\hat{c}\mathrm{h}+\hat{c}\,\hat{s}\mathrm{h})=0$ |
| I (d) | $(CFT)-[\overline{m}(\delta_1-\delta_2)(\delta_2\,a\,\hat{s}\,\hat{c}\mathrm{h}+\delta_1\,b\,\hat{c}\,\hat{s}\mathrm{h})/\delta_1\delta_2]=0$ | $(CFE)-\lambda\overline{m}(\hat{c}\,\hat{s}\mathrm{h}-\hat{s}\,\hat{c}\mathrm{h})=0$ |
| I (e) | $[(p_s^2a^2+\delta_1)-(p_s^2b^2-\delta_2)]$<br>$+\{[-\delta_1^2(p_s^2b^2-\delta_2)+\delta_2^2(p_s^2a^2+\delta_1)]/\delta_1\delta_2\}\,\hat{c}\,\hat{c}\mathrm{h}$<br>$+\{[b^2(p_s^2a^2+\delta_1)+a^2(p_s^2b^2-\delta_2)]/ab\}\,\hat{s}\,\hat{s}\mathrm{h}=0$ | $[a^2(p^2-a^2)-b^2(p^2+b^2)]$<br>$+[b^2(p^2-a^2)-a^2(p^2+b^2)]\,\hat{c}\,\hat{c}\mathrm{h}$<br>$+ab[(p^2-a^2)+(p^2+b^2)]\,\hat{s}\,\hat{s}\mathrm{h}=0$ |

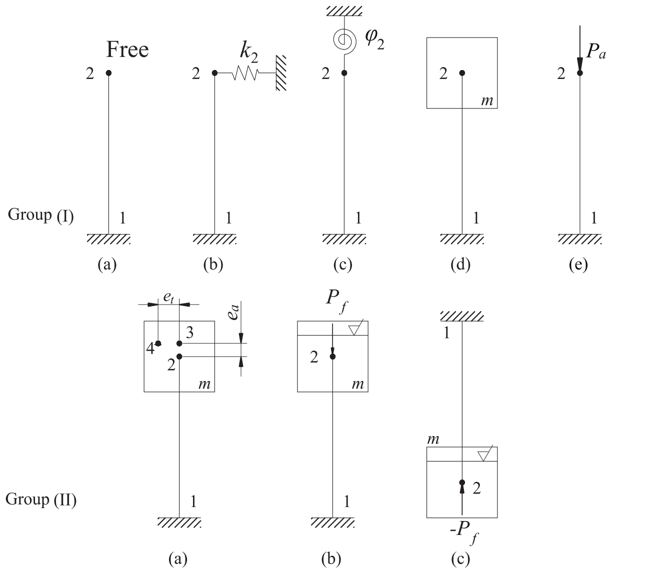


Fig. 4 Practical applications with a clamped base subjected to different design parameters: I, single effect. Π, double effects

three different application examples in which two effects are acting at the point of attachment 2, as shown in the cases from II (a) to II (c). The last three cases are tested through numerical examples of Tables (3-5).

### *4.5 Mode shapes*

The mode shape equation Eq.(11) can be plotted after obtaining the equation constants from the boundary conditions. The expressions for the mode shape constants $A_1.A_2$ and $A_3$ are obtained by solving the first three equations in Eq. (54) and letting $A_4=1$.

$$\begin{bmatrix}A_1\\A_2\\A_3\end{bmatrix}=-\begin{bmatrix}a_1&a_2&a_3\\a_5&a_6&a_7\\\epsilon_1&\epsilon_2&\epsilon_3\end{bmatrix}^{-1}\begin{bmatrix}a_4\\a_8\\\epsilon_4\end{bmatrix}\tag{67}$$

The symbolic constants $A_1.A_2$ and $A_3$, respectively become

$$A_1=[-a_4(a_6\epsilon_3-a_7\epsilon_2)+a_8(a_2\epsilon_3-a_3\epsilon_2)-\epsilon_4(a_2a_7-a_6a_3)]/\Delta;\tag{68}$$

$$A_2=[a_4(a_5\epsilon_3-a_7\epsilon_1)-a_8(a_1\epsilon_3-a_3\epsilon_1)+\epsilon_4(a_1a_7-a_5a_3)]/\Delta;\tag{69}$$

$$A_3=[-a_4(a_5\epsilon_2-a_6\epsilon_1)+a_8(a_1\epsilon_2-a_2\epsilon_1)-\epsilon_4(a_1a_6-a_5a_2)]/\Delta;\tag{70}$$

where,

$$\Delta=a_1(a_6\epsilon_3-a_7\epsilon_2)-a_2(a_5\epsilon_3-a_7\epsilon_1)+a_3(a_5\epsilon_2-a_6\epsilon_1)\tag{71}$$

Substituting by Eqs. (55) into the above symbolic relations Eqs. (68-71) and using the relations in Eqs. (61), the explicit formulae for the mode shape constants can be obtained as follows,

$$\Delta A_1=+\eta_1\eta_6\,\hat{s}+\eta_6\,(\beta-\delta_1)\hat{c}+(\delta_6(\beta-\delta_2)+\eta_2\eta_8)\,\hat{s}h+(\eta_8(\beta-\delta_2)+\delta_6\,\eta_2)\hat{c}h;\tag{72}$$

$$\Delta A_2=-\eta_6(\beta-\delta_1)\hat{s}+\eta_1\,\eta_6\hat{c}+\left(\eta_2\delta_5+\eta_7(\beta-\delta_2)\right)\hat{s}h+(\delta_5(\beta-\delta_2)+\eta_2\eta_7)\hat{c}h;\tag{73}$$

Table 3 The first four frequency parameters $\lambda_i$ and their associated relative frequency $f_r = (\lambda_i/\lambda_r)^2$ using formulae shown in Table 5, for the models shown in Fig. 5 (I)

| Input design parameters | $f_r$ | $\lambda_i$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | TBT ($r^2 = 0.0016$) | | | | EBT | | | |
| | | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
| case I (a), | $\lambda_i$ | 1.8615 | 4.4798 | 7.1286 | 9.4252 | 1.8751 | 4.6941 | 7.8548 | 10.9955 |
| free | $f_r$ | 0.9924 | 0.9543 | 0.9075 | 0.8572 | 1 | 1 | 1 | 1 |
| case I (b), | $\lambda_i$ | 3.5557 | 5.4020 | 7.3766 | 9.5136 | 3.6405 | 5.6159 | 8.0800 | 11.0748 |
| $Z_2 = 100$ | $f_r$ | 1.8963 | 1.1508 | 0.9391 | 0.8652 | 3.7694 | 1.4313 | 1.0581 | 1.0145 |
| case I (c), | $\lambda_i$ | 2.3206 | 5.1689 | 7.7030 | 9.9277 | 2.3564 | 5.4708 | 8.5982 | 11.7256 |
| $\Phi_2 = 100$ | $f_r$ | 1.2376 | 1.1011 | 0.9807 | 0.9029 | 1.5770 | 1.3583 | 1.1982 | 1.1372 |
| case I (d), | $\lambda_i$ | 0.86649 | 3.8156 | 6.5151 | 8.8881 | 0.8700 | 3.9499 | 7.0825 | 10.2199 |
| $\bar{m} = 5$ | $f_r$ | 0.4621 | 0.8129 | 0.8294 | 0.8083 | 0.2153 | 0.7208 | 0.8130 | 0.8639 |
| case I (e), | $\lambda_i$ | 1.2346 | 4.3063 | 7.0366 | 9.3548 | 1.2573 | 4.5282 | 7.7737 | 10.9414 |
| $p^2 = 2$ | $f_r$ | 0.6584 | 0.9174 | 0.8958 | 0.8508 | 0.4496 | 0.9306 | 0.9795 | 0.9902 |

Fig. 5 Finite element mesh for partially elevated tank

$$\Delta A_3 = -(\delta_6(\beta-\delta_1)+\eta_1\eta_7)\hat{s} - (\eta_7(\beta-\delta_1) - \delta_6\eta_1)\,\hat{c} + \eta_2\,\eta_5\hat{s}h + \eta_5(\beta-\delta_2)\hat{c}h; \tag{74}$$

$$\Delta \;\; = -(\eta_8(\beta-\delta_1)+\delta_5\,\eta_1)\hat{s} - (\delta_5(\beta-\delta_1) - \eta_1\eta_8)\hat{c} - \eta_5(\beta-\delta_2)\hat{s}h - \eta_2\eta_5\,\hat{c}h \tag{75}$$

Note that, $\eta_1.\eta_2.\ \eta_5.\eta_6.\eta_7.\eta_8.\delta_1.\delta_2.\delta_5.\delta_6.$ and $\beta$ are already defined in Eqs. (61 a-p), and Eqs. (13 a,b).

### *4.6 Finite element analysis*

For the validation of the current analytical model, selected examples were modeled using ANSYS workbench finite element model. The models are created using ANSYS workbench design modeler module. These models are then meshed using ANSYS workbench mesh module. SOLID186 element is used for the beam and the solid tank, meanwhile, FLUID220 element is chosen to model the fluid inside the tank. Fine mapped mesh is utilized in order to obtain accurate results. Fig.5 shows the mesh used to obtain the results of the model in Table 5. The boundary conditions and the interface between the fluid and solid elements are then identified. ANSYS modal analysis module is used to evaluate the partially filled tank natural frequencies.

## 5. Results and discussion

In this section, the results of the proposed explicit general formulae and simplified formulae are presented. Initially, the results when the end boundary condition includes only the effect of single parameter such as linear or torsional spring. Then the results of the cases when there are multiple boundary conditions at the end of the beam is considered using the general frequency equations presented in Eqs. (60) and (65) in case of TBT and Eqs. (63) and (66) in case of EBT.

### *5.1 Application models are subjected to a single effect*

In this section, group (I) applications shown in Fig.4 and Table 2 are considered. It consists of five simple models as shown in Fig. 4(I), (from I (a) to I (e)). Each case is subjected to a single effect of either, free, $Z_2\,.\,\Phi_2.\bar{m}$ and $p_a^2$ . The values of design parameters for cases (I (b) to I (e)), $Z_2\,.\,\Phi_2.\ \bar{m}$ and $p_a^2$ are taken as, 100, 100, 5 and 2 respectively. In the TBT models $r^2 = 0.0016$ . The reduced simple explicit formulae, shown in Table 2 are used to evaluate the results shown in Table 3. As can be seen from Table 3, that $Z_2.\Phi_2.\bar{m}\ and\ p^2$ has the greatest effect on the fundamental mode of vibration compared with the other four modes. For these applications, the flexibility element $Z_2$ has higher effect on $f_r$ than $\Phi_2$, Meanwhile, its effect on $f_r$ is negligible at the fourth mode. Also, the effect of tensile axial load parameter $p^2$on $f_r$ is very small at higher modes. The end mass $\bar{m}$ has a great effect on $f_r$ in the first mode and its effect decreases for higher modes. It should be noted that the present results are validated using published analysis by the authors (Farghaly and El-Sayed 2016).

### *5.2 Application models are subjected to double effect*

In this section, the second group (II) in Fig. 4 is considered. This group consists of three end mass loaded models as shown in Fig. 4 (II), (from II (a) to II (c)). The results of $\lambda_i\ and\ f_r$ are estimated, considering $\bar{m} = 5$ using either Eq. (60) or Eq. (65). In the first application, II (a), the end mass has double eccentricity parameters

Table 4 The first four natural frequencies $\omega_i$ for system with end mass of $\bar{m} = 5$ and is subjected to double effect of in-plane eccentricity parameters $(\bar{e}_a, \bar{e}_t)$

| | Eccentricity $(\bar{e}_a, \bar{e}_t)$ | Method | $\omega_i$ (Hz) | | | |
|---|---|---|---|---|---|---|
| | | | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ |
| (a) | (0.00, 0.00) | Eq. (65)* | 3.0403 | 62.6518 | 201.3563 | 418.8763 |
| | | FEM | 3.2039 | 62.7040 | 201.2800 | 418.7900 |
| (b) | (0.05, 0.00) | Eq. (65) | 2.8353 | 56.9052 | 183.2147 | 382.4469 |
| (c) | (0.00, 0.05) | Eq. (65) | 3.0322 | 50.2694 | 121.1405 | 260.3442 |
| (d) | (0.05, 0.05) | Eq. (65) | 2.8285 | 47.7747 | 127.0790 | 269.0598 |
| (e) | (0.10, 0.00) | Eq. (65) | 2.6523 | 52.1494 | 169.7502 | 358.8845 |
| (f) | (0.00, 0.10) | Eq. (65) | 3.0082 | 32.6426 | 98.7286 | 251.6450 |
| (g) | (0.10, 0.10) | Eq. (65) | 2.6292 | 33.4830 | 105.0598 | 256.9155 |
| (h) | (0.20, 0.00) | Eq. (65) | 3.2421 | 45.1991 | 153.5284 | 335.3363 |
| (i) | (0.00, 0.20) | Eq. (65) | 2.9152 | 18.0708 | 93.0857 | 249.6480 |
| (j) | (0.20, 0.20) | Eq. (65) | 2.2766 | 21.4252 | 97.2589 | 293.0364 |
| (k) | (0.50, 0.50) | Eq. (65) | 1.5572 | 13.1026 | 94.1444 | 251.5693 |
| | | FEM | 1.6362 | 13.1000 | 93.8240 | 249.9000 |

* Eq. (60), can be used also.

Table 5 the first four frequencies $\omega_i$ for system with end storage tank partially filled and in both elevated and suspended positions

| | Method | $\omega_i$(Hz) | | | |
|---|---|---|---|---|---|
| | | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ |
| (a) Rigid tank without fluid: | | | | | |
| elevated situation | Eq. (65)* | 5.9778 | 57.5347 | 175.876 | 400.4476 |
| | FEM | 5.9834 | 57.4820 | 176.090 | 401.380 |
| suspended situation | Eq. (65) | 5.9778 | 57.5347 | 175.876 | 400.4476 |
| | FEM | 5.9834 | 57.4820 | 176.090 | 401.380 |
| (b) Rigid tank with fluid: | | | | | |
| elevated situation | Eq. (65) | 4.8779 | 57.1996 | 172.0884 | 394.9390 |
| | FEM | 4.8512 | 56.7320 | 171.5700 | 394.2000 |
| suspended situation | Eq. (65) | 4.6470 | 56.0677 | 175.372 | 400.4026 |
| | FEM | 4.7873 | 57.1170 | 175.130 | 398.4200 |

* Eq. (60) can be used also.

$(\bar{e}_a, \bar{e}_t)$. The beam diameter and length are 0.02 m and 1 m respectively. The beam material elastic modulus and density are $2.1 \times 10^{11}$Pa and 7850 $\mathrm{kg/m^3}$ respectively.

Several values of $\bar{e}_a$ and $\bar{e}_t$ are investigated, as shown in Table 4. As can be seen from Table 4, that, the modal frequencies, estimated using Eq. (65), are decreased with the increase in the axial eccentricity parameters $(\bar{e}_a)$ and transverse eccentricity parameters $(\bar{e}_t)$. These results are observed from cases (a) to (k). It is noted that, the results of the two cases (a) and (k) are verified using the FEM.

The second and third application in group (II) are for elevated and suspended partially filled tank, see Fig.4. The following beam model inputs are used:

Beam diameter =0.03m , $L$=1 m, $E$=$2.1 \times 10^{11}$Pa , $\rho$=7850 $\mathrm{kg/m^3}$. The tank dimensions are: $R$=0.15 m, $H$=0.3 m and $h$=0.2 m. the mass of the tank, fluid and beam are $m_t = 10.053$ kg , $m_f = 6.2832$ kg and $m_{beam} = 5.5488$ kg respectively. The tank radius of gyration is $\kappa = 0.12477$ m and the tank center of mass is located at distance $e_a = 0.15\ m$ from the end of the beam. $e_f = 0.1$ m in case of elevated tank and $e_f = 0.2$ m in case of suspended tank. The axial load is $P_f = m_f * 9.81 = 61.64$ N.

Table 5 shows the results due to the effect of partially filling the tank on the model natural frequency for both elevated and suspended tank situations. The model is solved

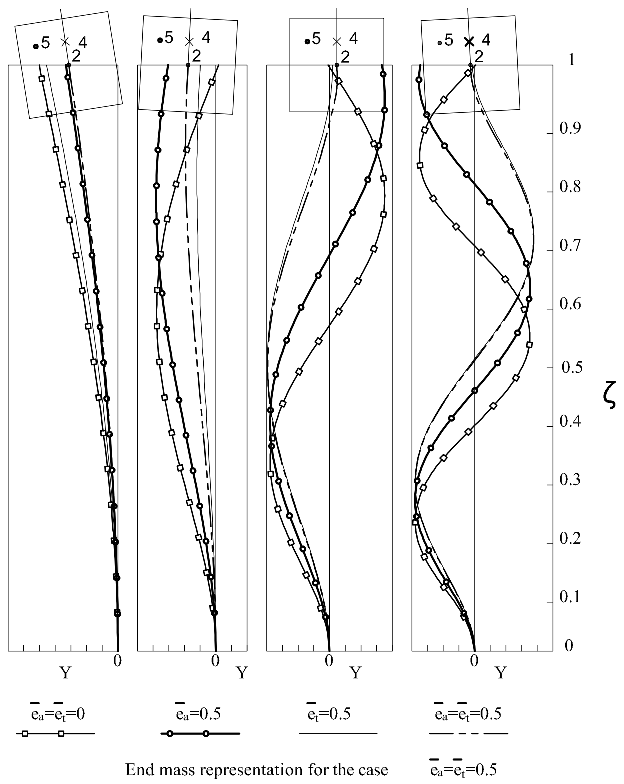


Fig. 6 The effect of the end mass eccentricity on the modal shapes for the cases of Table 4

using the present model and using ANSYS workbench FE modal analysis module. As can be seen from Table 5, that the results of the present model are in good agreement with those obtained using FEM.

The first four mode shapes for four selected cases are plotted in Fig.6. These cases are clamped from first end and with dimensionless end mass $\bar{m} = 5$ at the other end. These cases are selected with the following end conditions $(\bar{e}_a = 0, \bar{e}_t = 0)$ , $(\bar{e}_a = 0.5, \bar{e}_t = 0)$, $(\bar{e}_a = 0, \bar{e}_t = 0.5)$ and $(\bar{e}_a = 0.5, \bar{e}_t = 0.5)$. The natural frequencies for the first and fourth cases are listed in Table 4 case(a) and case(k) respectively. Fig. 6 shows the effect of the eccentricity on the beam mode shapes. The most significant observation is that the nodal points disappeared from the second mode except for the case where $(\bar{e}_a = 0, \bar{e}_t = 0)$ . These observations are validated using FEM in addition to the present analytical one.

## 6. Conclusions

A complex mathematical model of beam-column system is investigated and solved analytically based on the exact solution analysis. The model considers the case in which the double eccentric end mass is a rigid storage tank containing fluid. Both Timoshenko and Bernoulli-Euler beam bending theories are considered. Equation of motion, general solution and boundary conditions for the present system model are presented in dimensional and non-dimensional format. Symbolic and/or explicit formula of generalized frequency and mode shape equations are formulated. Most of the presented closed form frequency equations based on Timoshenko and Euler bending theory have not appeared in literature. Eight reduced models, representing significant simple applications, are tested. Selected cases are validated using FEM. The beam-column investigated in this work is of significant importance for many engineering applications, especially, in mechanical and structural systems.

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## Nomenclature

| | |
|---|---|
| $A$ | cross-section area of the beam. |
| $A_f$ | fluid cross-section area. |
| $B_a$ | Bending arm due to fluid. |
| $\bar{B}_e$ | non-dimensional fluid bending arm for elevated tank. |
| $\bar{B}_s$ | non-dimensional fluid bending arm for suspended tank. |
| $e_a$ | axial eccentricity. |
| $e_t$ | transverse eccentricity. |
| $e_f$ | Distance between fluid center of mass and point 2 |
| $E$ | Young's modulus of elasticity |
| $f_r$ | relative frequency, $(\lambda_i/\lambda_r)^2$ |
| $G$ | shear modulus of rigidity. |
| $h$ | Fluid level in tank |
| $H$ | Height of tank |
| $\bar{H}, \bar{h}$ | non-dimensional variables defined by $(H/L)$, $(h/L)$, respectively. |
| $I$ | moment of inertia of the beam cross section about the neutral axis. |
| $k'$ | shear deformation shape coefficient. |
| $\bar{\kappa}$ | non-dimensional radius of gyration. |
| $L$ | length of the beam (between 1 & 2). |
| $M_{fe}$ | end moment due to elevated tank. |
| $M_{fs}$ | end moment due to suspended tank. |
| $m$ | mass of the beam, $\rho AL$. |
| $m_f$ | Mass of the fluid inside the tank. |
| $m_t$ | mass of the tank. |
| $\bar{m}_f$ | $m_f/m$ |
| $\bar{m}_t$ | $m_t/m$ |
| $P_a$ | axial load |
| $P_f$ | axial load due to fluid inside the tank. |
| $p_a^2$ | axial load parameter $(P_a L^2/EI)$. |
| $p_f^2$ | Fluid axial load parameter $(P_f L^2/EI)$. |
| $r^2$ | rotary inertia parameter $(I/AL^2)$. |
| $s^2$ | shear deformation parameter $(Er^2/G\acute{k})$. |
| $Y$ | non-dimensional lateral deflection. |
| $x, y, \psi$ | system co-ordinates of the beam. |
| $Z_1, Z_2$ | end stiffness parameter defined as $k_1L^3/EI$, and $k_2L^3/EI$ respectively. |
| $\gamma_f$ | mass density of the fluid. |
| $\delta$ | distance between point of attachment 2 and spring location. |
| $\delta_1, \delta_2$ | parameter defined as in Eq. (13 a, b). |
| $\delta_3, \delta_4$ | parameters defined as in Eqs. (61 m, n). |
| $\delta_5, \delta_6$ | parameters defined as in Eqs. (61 o, p). |
| $\delta_7, \delta_8$ | parameters defined as in Eqs. (64). |
| $\zeta$ | non-dimensional beam length $x/L$. |
| $\lambda^4$ | frequency parameters, $\rho AL^4\omega^2/EI$. |
| $\nu$ | Poison's ratio. |
| $\rho$ | mass density of the beam material. |
| $\phi_1, \phi_2$ | end rotational spring stiffness. |
| $\Phi_1, \Phi_2$ | End rotational spring parameter $\phi_1 L/EI$ and $\phi_2 L/EI$ respectively. |
| $\psi$ | slope due to bending. |
| $L\Psi$ | non-dimensional slope due to bending. |
| $\omega$ | circular frequency. |

## Abbreviations

| | |
|---|---|
| $\hat{s}\ \hat{s}$h | sin(a) sinh (b). |
| $\hat{s}\ \hat{c}h$ | sin(a) cosh (b). |
| $\hat{c}\ \hat{s}$h | cos(a) sinh (b). |
| $\hat{c}\ \hat{c}$h | cos(a) cosh (b). |
| vs | very small value = 10 E-12. |
| vl | very large value = 10 E+12. |
| TBT | Timoshenko's beam theory. |
| EBT | Euler's beam bending theory. |

## Appendix A

AA.1 Useful relations

$$sin\, 2\theta \; = 2\, sin\, \theta\, cos\, \theta \tag{76}$$

$$2\sin A\sin B = \cos\,(A-B) - \cos\,(A+B) \tag{77}$$

$$\sin^2\theta\cos^2\theta = \frac{\sin^2 2\theta}{4} \tag{78}$$

$$\sin^2\theta\cos\theta = \sin\theta\sin 2\theta = \cos\theta - \cos 3\theta \tag{79}$$

$$\int_\pi^0 \frac{\sin^2 2\theta}{4}\; d\theta = \left|\frac{\theta}{2} - \frac{\sin 4\theta}{8}\right|_\pi^0 \tag{80}$$

$$\int_\pi^0 (\cos\theta - \cos 3\theta)\; d\theta = \left|\sin\theta - \frac{\sin 3\theta}{3}\right|_\pi^0 \tag{81}$$

$$\int_\pi^0 \sin^2\theta\, d\theta = \left|\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right|_\pi^0 \tag{82}$$