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Longitudinal vibration of double nanorod systems using doublet mechanics theory

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Abstract. This paper investigates the free and forced longitudinal vibration of a double nanorod system using doublet mechanics theory. The doublet mechanics theory is a multiscale theory spanning between lattice dynamics and continuum mechanics. Equations of motion and boundary conditions for the double nanorod system are obtained using Hamilton's principle. Clamped-clamped and clamped-free boundary conditions are considered. Frequencies and dynamic displacements are determined to demonstrate the effects of length scale parameter of considered material and geometry of the nanorods. It is shown that frequencies obtained by the doublet mechanics theory are bounded from above (van Hove singularity) and unlike classical elasticity theory doublet mechanics theory predicts finite number of modes depending on the length of the nanotube. The present doublet mechanics results have been compared to molecular dynamics, experimental and nonlocal theory results and good agreement is observed between the present and other mentioned results. The difference between wave frequencies of graphite is less than 10% between doublet mechanics and experimental results near to the end of the first Brillouin zone.

Keywords: double nanorod system; doublet mechanics; free vibration; forced vibration

1. Introduction

Carbon nanotubes (CNTs) (Iijima 1991) are used in many applications due to their excellent mechanical, electrical, chemical and thermal properties (Ebbesen 1997, Dirote 2004). These extraordinary properties of CNTs provide their usage in multiple nanostructures. Coupled nanorod systems have also become a crucial technology for the design of the nano electro-mechanical devices. There are many applications of nanotubes in the field of nanosensors, nanoresonators, nanoactuators and nano-opto-mechanical systems (NOMS). Thus, the determination of the frequencies of such devices may be useful for design procedures of these nano electro-mechanical systems.

Classical version of this problem has been considered in some of the previous studies. Erol and Gurgoze (2004) studied the vibration of the homogeneous double rod system by considering the classical rod model. Oniszcuk (2000) investigated the free vibrations of simply supported beams considering the elastic foundation by using the classical Euler-Bernoulli beam theory. Rosa and Lippiello (2007) examined the vibrations of double beams using the differential quadrature method. They modelled the beams system resting on springs at the ends so that rotation and translation effects are included in the analysis. One of the most important application fields of coupled nanorods are thin-film transistors which were investigated by Cao *et al.* (2007). They computed the capacitance of thin-film transistors consisting of a parallel array of spaced single walled CNTs with embedded in a gate dielectric using the finite element model. The proposed model for transistors is shown in Fig. 1 (Cao *et al.* 2007). Obtained results from that paper have become useful to the design of other electronic devices. Linear and nonlinear vibration of nanotubes have been studied by (Pradhan and Phadikar 2009, Besseghier et al. 2015, Arani *et al.* 2016) using nonlocal elasticity (NL) theory.

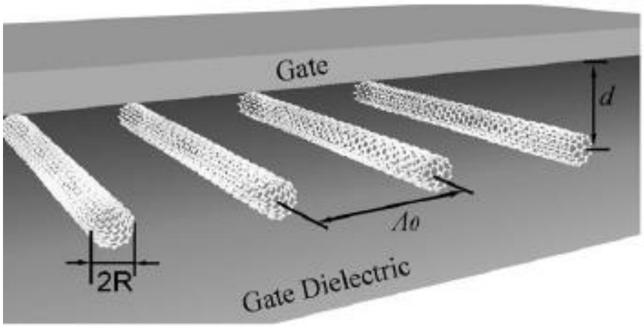

Fig. 1 Model of the thin-film transistors (R: nanotube radius, d: distance between each tube, $\wedge_0$: dielectric thickness) (Cao *et al*. 2007)

Recently, some size dependent continuum theories have been used in order to model multiple nano-structures. Longitudinal vibration of double nanorod systems has been investigated by Murmu and Adhikari (2010). They assumed that nanorods are linked to each other using vertical springs which are used to include the effect of elastic medium. Vibration frequencies of double nanorod systems have been obtained based on Eringen's nonlocal theory for clamped-clamped and clamped-free boundary conditions. Narendar and Gopalakrishnan (2011) studied the axial wave

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propagation in coupled nanorods using nonlocal theory. Wave characteristics of double nanorod systems have been investigated and cut-off frequency and escape frequency of nanorods have been obtained. Longitudinal vibration of nonlocal viscoelastic coupled nanorod systems has been studied by Karlicic *et al*. (2015). Results were compared with general numerical approaches for validation. Vibration analyses of double nanobeam systems under the initial pre-stress condition has been investigated by Murmu and Adhikari (2012). Natural frequencies of the double nanobeam system obtained for in phase type and out of phase type vibration modes. They used the nonlocal theory in their analysis and it was obtained that nonlocal effects change the flexural vibration characteristics of coupled nanobeams. Surface effects on the vibration and buckling of double nanobeam systems have been studied using Euler-Bernoulli beam theory in (Wang *et al*. 2011). Transverse vibration and axial buckling of double nanobeams have been examined for a rectangular and circular cross section with a surface layer. It was obtained that scale effects play a significant role on buckling and vibration in double nanobeam systems. Another study deals with the vibration of a double beam system by coupled mass-spring systems has been investigated by Pajand *et al*. (2018). The mass-spring system provides adjustable translation and rotation of the beams. Fourier transform method and finite element method were used in the analysis and the accuracy of both methods shown for vibration problem. Vibration of a double beam system has been studied by Mao and Nuttawit (2015) considering the elastic foundation. There are also some studies related to magnetic and thermal effects on statics and dynamics of double nanotube systems (Jun *et al*. 2010, Arani *et al*. 2014, Karličić *et al*. 2016, Pavlović *et al*. 2016) and statics, stability and dynamics of nanobeams, nanotubes and nanoplates (Oveissi *et al*. 2016a, Oveissi *et al*. 2016b, Foroutan *et al*. 2018, Mirkalantari *et al*. 2017, Saffari *et al*. 2017, Oveissi *et al*. 2015, Oveissi *et al*. 2017, Oveissi *et al*. 2018).

Classical elasticity theory is not adequate for modelling of nanostructures due to their length scale free nature. Thus, some size dependent continuum theories have been proposed like the nonlocal stress gradient theory, the strain gradient model, the modified couple stress model and the peridynamics (Eringen 1976, Eringen 1983, Cosserat and Cosserat 1909, Mindlin 1964, Kong *et al*. 2009) in the design of the nano scale structures. Although these size dependent continuum models give more accurate results than classical elasticity theory, they are phenomenological and definition of the length scale parameters is an open problem in these models. However, the definition of the length scale parameter is directly associated with the atomic structure of the materials.

Another size dependent theory known as Doublet Mechanics (DM) was developed by Granik (1978). This multiscale theory converges to lattice dynamics and continuum mechanics at the two limits. DM is not a phenomenological theory opposed to other size dependent models. Scale parameter in DM is obtained by considering the micro structure of the solid. In this model, micro strains and micro stresses are defined in each particle of the solid. Then, these micro deformations are transformed to macro-micro stress relations. Each atom is defined as a node in the solid in DM. Macro stress and strain relations can be obtained by expanding the micro stress and strain relations in Taylor series. Then, the length scale parameter of DM is included in the formulation and it can be controlled by taking different number of terms in the Taylor series expansion. The number of the terms in Taylor series expansion does not show the accuracy. It shows the order of the governing equations. In this paper, we use the first three terms in the Taylor series expansion during the deformation of the nodes and more terms can be considered in the analyses. However, more terms in the Taylor series expansion will make computation more complicated as an increased order of the equations of motions and boundary conditions. DM theory has been firstly employed to derive micro-equations for granular materials by Granik and Ferrari (1993) and Ferrari *et al*. (1997). Vajari and Imam (2016) and Vajari and Imam (2016) investigated the axial and torsional vibration of CNTs using DM. Obtained results predicted by DM were compared with classical elasticity results for validation in this paper. Recently, Gul *et al*. (2017) studied the longitudinal vibration and wave propagation of embedded nanorods considering the elastic medium via DM theory. They showed that scale dependent DM theory is in a good agreement with Lattice dynamics results. Furthermore, there are some static and dynamic analyses in nanotubes considering the DM theory in (Gul and Aydogdu 2018a, Gul and Aydogdu 2018b). According to reviewer's best knowledge the vibration of double nanorods has not been investigated using DM. It is believed that presented formulation may give alternative formulation for the other size dependent models such as nonlocal elasticity and strain gradient elasticity models.

In this paper, longitudinal vibration of double nanorod systems has been studied using DM theory. Scale effects on double nanorod systems are investigated. Governing equations and boundary conditions are obtained by using the Hamilton Principle. Clamped-clamped (C-C) and clamped-free (C-F) boundary conditions are considered. The effects of the geometry of the nanorods, internal length scale parameter and stiffness of springs on the vibration of double nanorod systems are examined. Natural frequencies obtained by using the DM model are compared to classical elasticity results for C-C and C-F boundary conditions. For validation, wave frequencies predicted by using the DM are compared with the experiment and other nonlocal studies in the literature. After a brief review of the DM theory in Section 2, equations of motion of a nanorod with boundary conditions are derived in Section 3. In Section 4, longitudinal free vibration of a double nanorod system is investigated for clamped-clamped and clamped-free boundary conditions. Longitudinal forced vibration of a double nanorod system has been analysed under the axial uniform distributed load and axial linear load in Section 5. Section 6 gives the natural frequencies predicted by DM and classical elasticity theories and results are compared with other nonlocal models for considered boundary conditions. Finally, some important results and concluding remark are presented in Section 7.

2. Analysis

2.1 Brief review of doublet mechanics

DM is a size dependent multi-scale mechanics theory which interconnects with the discrete system and continuum system. The main difference of DM from the other-scale dependent model is the definition of the internal length scale parameter. In DM, bonding length of atoms in the considered material is taken as a length scale parameter. Unlike DM, the other size dependent theories include a various phenomenological macro constants which cannot be easily determined by experiments. All of these theories are related to various tensors of stresses. However, it is important to emphasize that in case of a force applied at particle of a solid with microstructure is transferred only in stated directions. Unlike these theories of elasticity, in the DM theory, micro stresses and micro strains are being scalars or vectors, then these micro relations are transformed to macro-stress-strain relations. Doublet stress component in the present DM theory represents the microforces and microcouples of the directional particle interactions, adequately. This behavior provides a deeper comprehension into the mechanical behaviour of microstructure, thus it makes the DM theory more physical model. In DM theory, it is assumed that atoms are taken as nodes or points located at a finite distance. Any two points or nodes in a solid are called a doublet and distance between two nodes is known as a "length scale" of considered material. It is assumed that each node can translate and/or rotate with respect to a chosen reference node. During deformation of the material, three micro stresses are considered. These are elongation micro stress p_α, shear micro-stress t_α and torsional micro stress m_α. These micro stresses are shown in Fig.2. It is noted that these micro stresses are vector quantities. A symbolic deformation model of the points (nodes) for DM theory is denoted in Fig. 3. Each reference node α and their adjacent nodes are separated by their doublet separation distance $\eta_\alpha = |\xi_\alpha|$. After deformation of the solid, new positions of the nodes are shown as a'_α and b'_α (Fig. 3). The elongation micro strain is defined as (Ferrari *et al.* 1997):

$$\epsilon_\alpha = \frac{\vec{\tau}^0_\alpha . \Delta\vec{u}_\alpha}{\eta_\alpha} \tag{1}$$

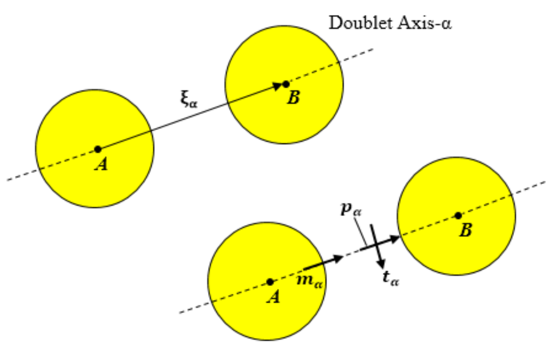


Fig. 2 Micro stresses in a doublet geometry (Ferrari *et al.* 1997)

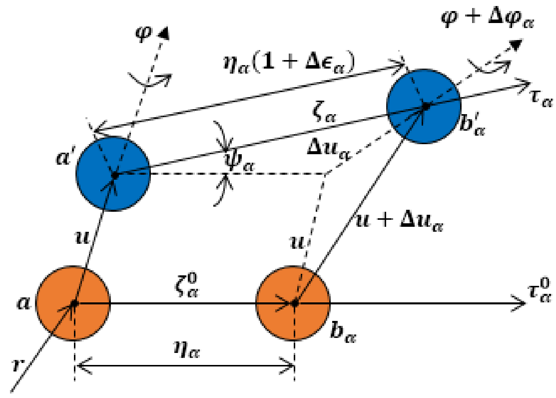


Fig. 3 Deformation of the doublet nodes (Ferrari *et al.* 1997)

where α=1,2,…z z is the number of doublets, $\vec{\tau}^0_\alpha$ denotes the unit vector in α direction, $\Delta\vec{u}_\alpha$ denotes the displacement vector and η_α denotes the doublet separation (internode) distance in the undeformed form. The increment of displacement vector is

$$\Delta u_\alpha = \vec{u}(\vec{r} + \vec{\xi}^0_\alpha, t) - \vec{u}(\vec{r}, t) \tag{2}$$

where $\vec{r}$ is the position vector of node α, and t is the time. Then, the micro strain ϵ_α can be expanded in a Taylor series, as follows:

$$\epsilon_\alpha = \sum_{\chi=1}^{M} \frac{(-\eta_\alpha)^{\chi-1}}{\chi!} \vec{\tau}^0_\alpha . \left(\vec{\tau}^0_\alpha . \vec{\nabla}\right)^\chi \vec{u} \tag{3}$$

Here, the Delta operator and the number of the terms in the Taylor series expansion are denoted by $\vec{\nabla}$ and M, respectively. It should be noted that M shows the discreteness of the elastic medium and it is not related to the accuracy of the approach. The optimum number of Taylor series expansion M may change of the atomic structure of the considered material. The elongation micro stress p_α can be expressed as (Ferrari *et al.* 1997):

$$p_\alpha = \sum_\beta B_{\alpha\beta}\, \epsilon_\beta \tag{4}$$

where p_α is the micro stress in the α doublet and ϵ_β is the axial micro strain associated with β doublet, $B_{\alpha\beta}$ is the tension micro modulus between nodes α and β. The relation between micro stresses and macro stresses is defined as (Ferrari *et al.* 1997):

$$\sigma = \sum_{\alpha=1}^{n} \vec{\tau}^0_\alpha \vec{\tau}^0_\alpha \sum_{\chi=1}^{M} \frac{(-\eta_\alpha)^{\chi-1}}{\chi!} (\vec{\tau}^0_\alpha \cdot \vec{\nabla})^{\chi-1} p_\alpha \tag{5}$$

DM formulation given until so far is appropriate for 3-D dimensional formulations. Considering the plane stress problem, the relation between micro stresses and macro stresses is (Ferrari *et al.* 1997):

$$\{\sigma\} = [H]\{p\} \tag{6}$$

where the [H] matrix contains the unit vectors of considered material and it is used to link micro stress to macro stress.

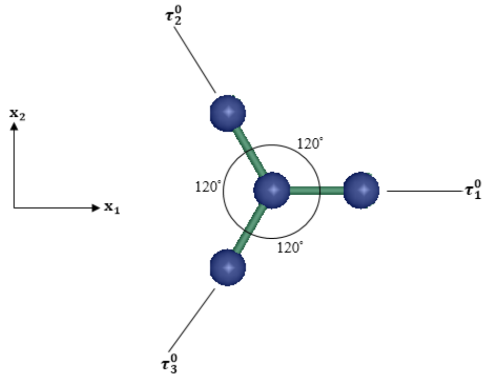


Fig. 4 Array of atoms in a zigzag nanotube

$\{\sigma\}$, *[H]* and *{p}* have been defined in the previous study (Gul and Aydogdu 2018b) for the plane stress problem.

2.2 Modelling of CNT using doublet mechanics theory

Now, DM theory is used for the dynamic analysis of double nanorod systems. We consider a zigzag nanotube model shown in Fig. 4. For a CNT a node of α has three adjacent nodes. The direction of these nodes can be presented in terms of direction cosines (Fig. 4):

$\tau_1(1,0,0)$, $\tau_2(-1/2,\sqrt{3}/2,0)$, $\tau_3(-1/2,-\sqrt{3}/2,0)$.

It is noted that the direction of unit vectors can be obtained for other nanotube models (armchair, chiral) by changing the chiral angle of the CNT. By using the macro-micro stress relations given in (Gul and Aydogdu 2018b), the macro stress-macro strain relation can be written as

$$\sigma_{ij}^{(M)} = \sum_{\alpha=1}^{n} B_0 \tau_{\alpha i}^0 \tau_{\alpha j}^0 \tau_{\alpha m}^0 \tau_{\alpha z}^0 \left(\varepsilon_{mz} + \frac{1}{12} \eta_\alpha^2 \tau_{\alpha t}^0 \tau_{\alpha s}^0 \frac{\partial^2 \varepsilon_{mz}}{\partial x_t \partial x_s} \right) \tag{7}$$

In plane stress condition, $-\frac{\varepsilon_{33}}{\varepsilon_{11}} = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = \nu = \frac{1}{3}$ and using only the axial strain, stress equation in DM theory can be written as (Ferrari *et al.* 1997):

$$\sigma_{xx}^{M} = B_0 \left(\varepsilon_{11} + \frac{1}{12} \eta^2 \frac{\partial^2 \varepsilon_{11}}{\partial x^2} \right) \tag{8}$$

The first term in Eq. (8) is classical Cauchy stress and the second term can be expressed as a doublet stress. Therefore, Eq. (8) can be re-expressed as

$$\sigma_{xx}^{M} = \kappa_0 \sigma_{xx}^{LE} + \kappa_1 \frac{\partial^2 \sigma_{xx}^{LE}}{\partial x^2} \tag{9}$$

where κ_0=1 and $\kappa_1 = \eta^2/12$ for the present zigzag CNT. If one chooses CNT with different chirality angles, these two parameters will take different values from present zigzag model. When κ_0=1 and κ_1=0 lead to the classical elasticity stress. It should be noted that number of stress terms given in the right hand side of equation (9) change with the number of terms in Taylor series expansion and node arrangement in the doublet configurations.

3. Governing equations and boundary conditions of a nanorod

The equation of motion and the boundary conditions of the nanorod in the framework of DM can be obtained by using the Hamilton principle:

$$\int_{t_0}^{t_1} \delta(U - T - W)dt = 0 \tag{10}$$

where δ is the variational symbol, U is the strain energy, W is the work done by external forces and T is the kinetic energy are determined as below:

$$U = \frac{1}{2}\int_0^L AE\left[(u')^2 - \frac{\eta^2}{12}(u'')^2\right]dx = 0 \tag{11}$$

$$\delta W = \int_0^L f\delta u dx + [P\delta u]_0^L + [R\delta u']_0^L \tag{12}$$

where *f*, *P* and *R* denote the distributed force, axial force and double force, respectively.

$$T = \frac{1}{2}\int_0^L \rho A \dot{u}^2\, dx \tag{13}$$

Applying the Hamilton principle, equation of motion of the nanorod in the framework of the DM theory is (Gul and Aydogdu 2018b):

$$AEs^2 u^{IV}(x,t) + AEu''(x,t) + f = m\ddot{u}(x,t) \tag{14}$$

where the internal length scale parameter *s* equals to $s^2 = \eta^2/12$ and A the cross-sectional area, E the elasticity modulus of the nanorod. In the DM theory, modeling of zigzag CNT with three terms increases the order of equation of motion of the classical elasticity rod equation from two to four. As it will be explained below the number of the boundary conditions will also be increased. The initial conditions satisfy the equation

$$\dot{u}(x,t_1)\delta u(x,t_1) - \dot{u}(x,t_0)\delta u(x,t_0) = 0 \tag{15}$$

and the boundary conditions are obtained as

$$\begin{aligned}\big[P(L,t) - AE[u'(L,t) + s^2u'''(L,t)]\big]\delta u(L,t) \\ - \big[P(0,t) \\ - AE[u'(0,t) \\ + s^2u'''(0,t)]\big]\delta u(0,t) = 0,\end{aligned} \tag{16a}$$

$$\begin{aligned}\big[R(L,t) - AE[-s^2u''(L,t)]\big]\delta u'(L,t) - \\ \big[R(0,t) - AE[-s^2u''(0,t)]\big]\delta u'(0,t) = 0.\end{aligned} \tag{16b}$$

Here displacement u or the axial force $P = AE(u' + s^2u''')$ (modified with an additional term containing scale effect) are the classical boundary conditions. The boundary strain u' or the boundary double force $R = AEs^2u''$ can be defined as the non-classical boundary conditions of the present DM theory. If the length scale parameter s^2 equals

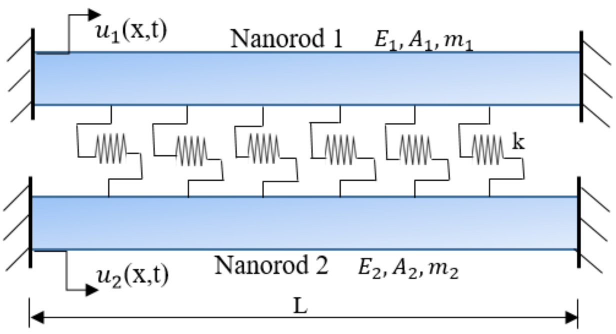


Fig. 5a A DNRS for C-C boundary conditions

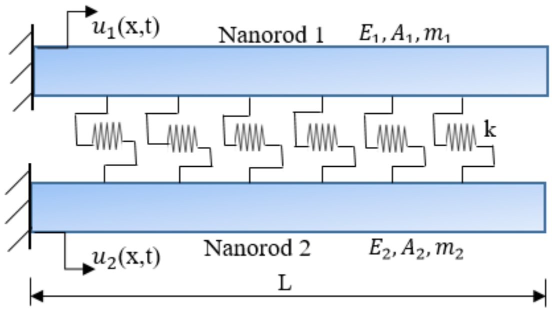


Fig. 5b A DNRS for C-F boundary conditions

to zero in Eqs. (14) and (16), it yields to classical elasticity equations and classical boundary conditions.

4. Free vibration of a double nanorod system

A double nanorod system (DNRS) is considered as shown in Fig. 5. Two nanorods are connected with distributed springs which are used to include the effect of elastic medium or van der Waals interaction between CNTs. The stiffness of the springs is assumed as Winkler constant in the model. Clamped-clamped (C-C) and clamped-free (C-F) boundary conditions are considered. Using the Eq. (14), governing equations of double nanorod systems can be written as

$$-E_1A_1u_1''(x,t) + k[u_1(x,t) - u_2(x,t)] + m_1\ddot{u}_1(x,t) - E_1A_1s^2u_1^{IV}(x,t) = 0, \tag{17a}$$

$$-E_2A_2u_2''(x,t) - k[u_1(x,t) - u_2(x,t)] + m_2\ddot{u}_2(x,t) - E_2A_2s^2u_2^{IV}(x,t) = 0. \tag{17b}$$

where $u_1(x,t)$ and $u_2(x,t)$ denote the longitudinal displacement of the first and second nanorods, respectively, m denotes the mass per unit length, k denotes stiffness of the springs between two nanorods, and primes ($''$) and dots ($\dot{}$) define the derivatives with respect to coordinate x and time t, respectively. It should be noted that DM effects were not considered in the elastic medium effects in order to simplify the present analysis. For simplicity, the following expressions are defined (Erol and Gürgöze 2004):

$$E_1A_1 = E_2A_2 = \bar{e} \equiv \text{constant}, \quad m_1 = m_2 = m \equiv \text{constant} \tag{18}$$

Thus, inserting Eq. (18) into Eqs. (17), gives

$$-\bar{e}u_1''(x,t) + k[u_1(x,t) - u_2(x,t)] + m\ddot{u}_1(x,t) - \bar{e}s^2u_1^{IV}(x,t) = 0, \tag{19a}$$

$$-\bar{e}u_2''(x,t) - k[u_1(x,t) - u_2(x,t)] + m\ddot{u}_2(x,t) - \bar{e}s^2u_2^{IV}(x,t) = 0. \tag{19b}$$

Eqs. (19) are the system of fourth order ordinary differential equations. Although they can be solved using standard solution procedures, in order to simplify analysis the following variable change method will be used (Erol and Gürgöze 2004). Let us define a function $u(x,t)$ in the following form:

$$u(x,t) = u_1(x,t) - u_2(x,t) \tag{20}$$

Here $u(x,t)$ is the relative displacement of the rods. Thus, Eq. (20) can be expressed in the following form:

$$u_1(x,t) = u(x,t) + u_2(x,t) \tag{21}$$

Considering the definitions in Eqs. (20) and (21) and inserting into the Eq. (19), the following expressions are obtained:

$$-\bar{e}u''(x,t) + 2ku(x,t) + m\ddot{u}(x,t) - \bar{e}s^2u^{IV}(x,t) = 0, \tag{22a}$$

$$-\bar{e}u_2''(x,t) + m_2\ddot{u}_2(x,t) - \bar{e}s^2u_2^{IV}(x,t) = ku(x,t). \tag{22b}$$

It should be noted that by setting $s^2 = 0$ in Eqs. (22), leads to the identical classical elasticity equations obtained in Ref. (Erol and Gürgöze 2004). Firstly, for relative displacement we will solve the $u(x,t)$ in Eqs. (22a). Then, Eq. (22b) can be solved. When u=0 two tubes vibrate with the same amplitude as a single nanorod. For harmonic vibration, the general solution of Eq. (22a) is assumed as

$$u(x,t) = U(x)e^{i\omega t} \tag{23}$$

where ω denotes the natural frequency and i^2=-1. By substituting Eq. (23) into Eq. (22a), following fourth-order differential equation can be obtained.

$$-\bar{e}s^2U^{IV}(x,t) - \bar{e}U''(x,t) + 2kU(x,t) - m\omega^2U(x,t) = 0 \tag{24}$$

Eq. (24) can be written in dimensionless form as follows:

$$g^2U^{IV}(\bar{x},t) + U''(\bar{x},t) + (\Omega^2 - 2K)U(\bar{x},t) = 0 \tag{25}$$

where the dimensionless terms are defined as

$$K = \frac{kL^2}{\bar{e}}, \qquad \Omega^2 = \frac{m\omega^2L^2}{\bar{e}}, \quad g^2 = \frac{\eta^2}{12L^2} = \frac{s^2}{L^2}, \quad \bar{x} = \frac{x}{L}. \tag{26}$$

Solution of Eq. (25) can be written as follows

$$U(\bar{x}) = c_1e^{\gamma\bar{x}} + c_2e^{-\gamma\bar{x}} + c_3e^{\theta\bar{x}} + c_4e^{-\theta\bar{x}} \tag{27}$$

where c_i (i=1,2,3,4) are the arbitrary constants and

$$\gamma = \frac{\sqrt{-\frac{\sqrt{1-4\Omega^2 g^2+8g^2K}+1}{g^2}}}{\sqrt{2}}, \tag{28a}$$

$$\theta = \frac{\sqrt{\frac{\sqrt{1-4\Omega^2 g^2+8g^2K}-1}{g^2}}}{\sqrt{2}}. \tag{28b}$$

By using the classical and non-classical boundary conditions defined in Eq. (16) natural frequencies of double nanorods can be obtained for C-C and C-F boundary conditions. Classical and non-classical boundary conditions are defined for C-C boundary conditions in dimensionless form:

C-C: $U_1(0,t)=0,\ U_1(1,t)=0$ (classical),
$U_1''(0,t)=0,\ U_1''(1,t)=0$ (non-classical) at $\bar{x}$=0,1 *for nanorod*-1.
$U_2(0,t)=0,\ U_2(1,t)=0$ (classical),
$U_2''(0,t)=0,\ U_2''(1,t)=0$ (non-classical) at $\bar{x}$=0,1 *for nanorod*-2. (29)

Using Eq. (20) we obtain

$$U(0,t)=U_1(0,t)-U_2(0,t),\quad U(1,t)=U_1(1,t)-U_2(1,t), \tag{30a}$$

$$U''(0,t)=U_1''(0,t)-U_2''(0,t),\quad U''(1,t)=U_1''(1,t)-U_2''(1,t). \tag{30b}$$

And using Eqs. (30a, 30b), the following expression can be written

$$U(0,t)=0,\quad U''(0,t)=0,\quad U(1,t)=0,\quad U''(1,t)=0 \tag{31}$$

Thus, using Eq. (27) and Eq. (31), general solution for C-C double nanorod systems gives a system of homogeneous equations with four equations and four unknowns. For a nontrivial solution, the determinant of the coefficient matrix should vanish.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \gamma^2 & \gamma^2 & \theta^2 & \theta^2 \\ e^{\gamma} & e^{-\gamma} & e^{\theta} & e^{-\theta} \\ -\frac{e^{\gamma}+e^{\gamma}y}{2g^2} & -\frac{e^{-\gamma}+e^{-\gamma}y}{2g^2} & \frac{e^{\theta}y-e^{\theta}}{2g^2} & \frac{e^{-\theta}y-e^{-\theta}}{2g^2} \end{vmatrix} = 0 \tag{32}$$

where

$$y=\sqrt{1-4g^2\Omega^2+8g^2K} \tag{33}$$

The determinant (32) can be obtained as

$$ye^{-2\theta}(2\sinh(\theta)\,e^{\gamma+2\theta}-2\sinh(\theta)\,e^{2\theta-\gamma})(\gamma^2-\theta^2)=0 \tag{34}$$

This relation is obtained for $U_1-U_2\neq 0$ case. We also consider the vibration of second nanorod with using the $U_1-U_2=0$ case. The same procedure can be used for solving Eq. (22b) in C-C double nanorod system. Using Eqs. (22b-31) with required modifications, natural frequencies are obtained. Similar to previous case homogeneous equations with four equations and four unknowns can be found. The natural frequencies obtained for $U_1-U_2\neq 0$ contains the exact frequency results for $U_1-U_2=0$ case. It should be noted that natural frequencies obtained for $U_1-U_2=0$ case are independent from stiffness of the springs between the two nanorods.

For C-F boundary condition, similar procedures can be used. Classical and non-classical boundary conditions are defined for C-F boundary conditions in dimensionless form as:

C-F: $U_1(0,t)=0,\ P_1(1,t)=(U_1'(1,t)+g^2U_1'''(1,t))=0$ (classical),
$U_1''(0,t)=0,\ U_1''(1,t)=0$ (non-classical) at $\bar{x}$=0,1 *for nanorod*-1.
$U_2(0,t)=0,\ P_2(1,t)=(U_2'(1,t)+g^2U_2'''(1,t))=0$ (classical),
$U_2''(0,t)=0,\ U_2''(1,t)=0$ (non-classical) at $\bar{x}$=0,1 *for nanorod*-2. (35)

Using Eq. (20) we obtain

$$U(0,t)=0,\quad U''(0,t)=0,\quad [U'(1,t)+g^2U'''(1,t)]=0,\quad U''(1,t)=0. \tag{36}$$

Substituting Eq. (36) into Eq. (27), general solution of Eq. (24) for the C-F double nanorod systems yields to a system of homogeneous equations with four equations and four unknowns. For a nontrivial solution, the determinant of the coefficient matrix of this system should vanish.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \gamma^2 & \gamma^2 & \theta^2 & \theta^2 \\ \frac{e^{\gamma}\gamma-e^{\gamma}\gamma y}{2} & \frac{e^{-\gamma}\gamma y-e^{-\gamma}\gamma}{2} & \frac{e^{\theta}\theta-e^{\theta}\theta y}{2} & \frac{-e^{-\theta}\theta-e^{-\theta}\theta y}{2} \\ -\frac{e^{\gamma}+e^{\gamma}y}{2g^2} & -\frac{e^{-\gamma}+e^{-\gamma}y}{2g^2} & \frac{e^{\theta}y-e^{\theta}}{2g^2} & \frac{e^{-\theta}y-e^{-\theta}}{2g^2} \end{vmatrix} = 0 \tag{37}$$

The determinant (Eq. 37) leads to

$$A_1[A_2-A_3+A_4+A_5-A_6]=0 \tag{38}$$

where coefficients are

$$\begin{aligned} A_1 &= e^{-2\theta}(\gamma^2-\theta^2), \\ A_2 &= e^{\gamma+3\theta}(\theta-\gamma+2\gamma y)-e^{3\theta-\gamma}(\gamma+\theta-2\gamma y), \\ A_3 &= e^{\theta-\gamma}(\theta-\gamma+2\gamma y+2\theta y), \\ A_4 &= e^{\theta+\gamma}(\theta+\gamma-2\gamma y+2\theta y), \\ A_5 &= 2y^2\sinh(\theta)\,e^{2\theta-\gamma}(\theta-\gamma), \\ A_6 &= 2y^2\sinh(\theta)\,e^{2\theta+\gamma}(\theta+\gamma). \end{aligned} \tag{39}$$

These solutions are obtained for $U_1 - U_2 \neq 0$ case. We also consider the vibration of second nanorod with using the $U_1 - U_2 = 0$ case for the second nanorod. Similar to C-C case, homogeneous equations with four equations and four unknowns can be obtained considering the Eqs. (22b-28, 35-36). Again, the natural frequencies obtained for $U_1 - U_2 \neq 0$ contain exactly same results for $U = U_1 - U_2 = 0$ case. So, there is no need to show the solution of the $U_1 - U_2 = 0$ case in the present study.

5. Forced vibration of a double nanorod system

This section considers the forced vibration of the DNRS under the uniform and linear axial loads for the C-C boundary conditions. The harmonic force applied on the DNRS can be assumed as follows:

$$f_i(\bar{x},t) = F_i(\bar{x}) sin\omega t \quad i = 1,2. \tag{40}$$

For uniform distributed load ($F_i(\bar{x}) = F_i$), inserting Eqs. (23) and (40) into Eq. (22a) by considering the axial force ($F_i(\bar{x}) = F_i$) leads to

$$-g^2 U^{IV}(\bar{x},t) - U''(\bar{x},t) + (2K - \Omega^2)U(\bar{x},t) = F_1 - F_2 \tag{41}$$

where

$$F_1 = \frac{f_1 L^2}{\bar{e}}, \qquad F_2 = \frac{f_2 L^2}{\bar{e}}, \qquad \bar{\bar{u}}(\bar{x}) = \frac{U(\bar{x})}{L}. \tag{42}$$

The general solution of Eq. (41) can be obtained as

$$\bar{\bar{u}}(\bar{x}) = c_1 e^{\gamma\bar{x}} + c_2 e^{-\gamma\bar{x}} + c_3 e^{\theta\bar{x}} + c_4 e^{-\theta\bar{x}} + \frac{F_2 - F_1}{\Omega^2 - 2K} \tag{43}$$

Here, dimensionless dynamic displacements are defined as

$$\bar{\bar{u}}(\bar{x}) = U(x)\frac{\bar{e}}{(F_1 - F_2)L^2} \tag{44}$$

Applying the boundary conditions (31) in Eq. (43), undetermined coefficients (c_1, c_2, c_3, c_4) can be computed as

$$\begin{aligned} c_1 &= \frac{\frac{1}{\Omega^2}(\theta^2 e^{-\gamma} - \theta^2)}{\gamma^2 e^{\gamma} - \theta^2 e^{\gamma} - \gamma^2 e^{-\gamma} + \theta^2 e^{-\gamma}}, \\ c_2 &= \frac{\frac{1}{\Omega^2}(\theta^2 - \theta^2 e^{\gamma})}{\gamma^2 e^{\gamma} - \theta^2 e^{\gamma} - \gamma^2 e^{-\gamma} + \theta^2 e^{-\gamma}}, \\ c_3 &= \frac{\frac{1}{\Omega^2}(\gamma^2 - \gamma^2 e^{-\theta})}{\gamma^2 e^{\theta} - \theta^2 e^{\theta} - \gamma^2 e^{-\theta} + \theta^2 e^{-\theta}}, \\ c_4 &= \frac{\frac{1}{\Omega^2}(\gamma^2 e^{-\theta} - \gamma^2)}{\gamma^2 e^{\theta} - \theta^2 e^{\theta} - \gamma^2 e^{-\theta} + \theta^2 e^{-\theta}}. \end{aligned} \tag{45}$$

Thus, dimensionless dynamic displacement can be obtained for the C-C boundary condition:

$$\begin{aligned} \bar{\bar{u}}(\bar{x}) &= \frac{\frac{1}{\Omega^2}(\theta^2 e^{-\gamma} - \theta^2)}{\gamma^2 e^{\gamma} - \theta^2 e^{\gamma} - \gamma^2 e^{-\gamma} + \theta^2 e^{-\gamma}} e^{\gamma\bar{x}} + \frac{\frac{1}{\Omega^2}(\theta^2 - \theta^2 e^{\gamma})}{\gamma^2 e^{\gamma} - \theta^2 e^{\gamma} - \gamma^2 e^{-\gamma} + \theta^2 e^{-\gamma}} e^{-\gamma\bar{x}} \\ &+ \frac{\frac{1}{\Omega^2}(\gamma^2 - \gamma^2 e^{-\theta})}{\gamma^2 e^{\theta} - \theta^2 e^{\theta} - \gamma^2 e^{-\theta} + \theta^2 e^{-\theta}} e^{\theta\bar{x}} + \frac{\frac{1}{\Omega^2}(\gamma^2 e^{-\theta} - \gamma^2)}{\gamma^2 e^{\theta} - \theta^2 e^{\theta} - \gamma^2 e^{-\theta} + \theta^2 e^{-\theta}} e^{-\theta\bar{x}} + \frac{F_2 - F_1}{\Omega^2 - 2K} \end{aligned} \tag{46}$$

For the linear axial load case ($F_i(\bar{x}) = F_i \bar{x}$), general solution of Eq. (41) is obtained by

$$\bar{\bar{u}}(\bar{x}) = c_1 e^{\gamma\bar{x}} + c_2 e^{-\gamma\bar{x}} + c_3 e^{\theta\bar{x}} + c_4 e^{-\theta\bar{x}} + \frac{F_2 x - F_1 x}{\Omega^2 - 2K} \tag{47}$$

Substituting the boundary conditions (31) in Eq. (47), undetermined coefficients (c_1, c_2, c_3, c_4) can be found.

$$\begin{aligned} c_1 &= \frac{\frac{1}{\Omega^2}\theta^2}{\gamma^2 e^{\gamma} - \theta^2 e^{\gamma} - \gamma^2 e^{-\gamma} + \theta^2 e^{-\gamma}}, \\ c_2 &= \frac{\frac{1}{\Omega^2}\theta^2}{\gamma^2 e^{\gamma} - \theta^2 e^{\gamma} - \gamma^2 e^{-\gamma} + \theta^2 e^{-\gamma}}, \\ c_3 &= \frac{\frac{1}{\Omega^2}\gamma^2}{\gamma^2 e^{\theta} - \theta^2 e^{\theta} - \gamma^2 e^{-\theta} + \theta^2 e^{-\theta}}, \\ c_4 &= \frac{\frac{1}{\Omega^2}\gamma^2}{\gamma^2 e^{\theta} - \theta^2 e^{\theta} - \gamma^2 e^{-\theta} + \theta^2 e^{-\theta}}. \end{aligned} \tag{48}$$

Eventually, dimensionless dynamic displacement under the linear axial load is written in the following form:

$$\begin{aligned} \bar{\bar{u}}(\bar{x}) &= \frac{\frac{1}{\Omega^2}\theta^2}{\gamma^2 e^{\gamma} - \theta^2 e^{\gamma} - \gamma^2 e^{-\gamma} + \theta^2 e^{-\gamma}} e^{\gamma\bar{x}} + \frac{\frac{1}{\Omega^2}\theta^2}{\gamma^2 e^{\gamma} - \theta^2 e^{\gamma} - \gamma^2 e^{-\gamma} + \theta^2 e^{-\gamma}} e^{-\gamma\bar{x}} \\ &+ \frac{\frac{1}{\Omega^2}\gamma^2}{\gamma^2 e^{\theta} - \theta^2 e^{\theta} - \gamma^2 e^{-\theta} + \theta^2 e^{-\theta}} e^{\theta\bar{x}} + \frac{\frac{1}{\Omega^2}\gamma^2}{\gamma^2 e^{\theta} - \theta^2 e^{\theta} - \gamma^2 e^{-\theta} + \theta^2 e^{-\theta}} e^{-\theta\bar{x}} + \frac{F_2 - F_1}{\Omega^2 - 2K} x \end{aligned} \tag{49}$$

It should be noted that if the length scale parameter $s^2 = \eta^2/12$ set to zero dynamic displacements (in Eqs. (46) and (49)) can be obtained for the LE theory.

Table 1 Comparison of axial wave frequency (THz) of graphite for LE, DM, and Experiment (Aizawa *et al.* 1991)

Reduced wavenumber (k/k_{max})	LE	DM	Experiment (Aizawa *et al.* 1991)
0.2	10.0317	9.3483	7.2500
0.4	20.0634	13.8076	12.6940
0.6	30.0950	18.4717	18.6180

Table 2 Comparison of first four natural frequencies (GHz) of DNRS for C-F boundary conditions (k=8 N/nm, $\bar{e}=1$ nN, L=1 nm, m=10^{-9} kg/m)

n	NL (e_0a=0.5 nm)	NL (e_0a=1 nm)	NL (e_0a=1.5 nm)	NL (e_0a=2 nm)	Classical	DM
1	1.2353	0.8436	0.6137	0.4764	1.5708	1.5680
2	4.1864	4.0880	4.0468	4.0283	4.2974	4.2970
3	1.8411	0.9782	0.6601	0.4972	4.7124	4.6268
4	4.4033	4.1179	4.0541	4.0308	6.1812	6.1194

*(NL theory (Murmu and Adhikar 2010), Classical theory (Erol and Gürgöze 2004))

6. Numerical results and discussion

6.1 Validation of present DM theory

In order to validate the present DM formulation, firstly, axial wave propagation in graphite is compared with experimental results, DM and LE theories. Wave frequencies are compared with experimental results (Aizawa *et al.* 1991) for various wave frequencies with reduced wave number (k/k_{max}). Experimental results of phonon dispersion for graphite have been investigated by Aizawa *et al.* (1991). Density, elasticity modulus and bond length of graphite assumed as $\rho = 2266\frac{kg}{m^3}$, E=345GPa and $\eta = 0.246\,nm$, respectively. It is seen that DM results are in good agreement with experimental results. The DM results and experimental results give close values at the end of the first Brillouin zone (k/k_{max}=0.5). However, LE gives very different results with experiment compared to DM model for the given reduced wave number (Table 1). The percentage difference between experiment and DM theory is between 0.7% and 29%, whereas it is between 38% and 61.64% for the local elasticity. It is interesting to note that the percentage difference is decreasing with increasing the wave number for the DM theory, whereas the reverse is true for the local elasticity (LE) theory.

A second comparison is given for DNRS in Table 2. Here, $\bar{e}=1$ nN is taken into account in the calculations (Murmu and Adhikari 2010). As it can be observed from Table 2, the first four natural frequencies of DNRS predicted by DM are compared with classical (Erol and Gürgöze 2004) and nonlocal (Murmu and Adhikari 2010) theories for the C-F boundary condition. DM results give lower values than classical elasticity theory and the difference between DM and nonlocal models decreases with decreasing the nonlocal parameter (e_0a). It is noted that e_0 is the constant of the length scale parameter in the nonlocal theory. The stress gradient nonlocal results change in a wide band in terms of nonlocal parameter (e_0a).

Table 3 Comparison of the first natural frequency (THz) of DNRS (5,5) with nonlocal theory (Karličić *et al.* 2015) and MD (Cao *et al.* 2006) simulation for C-F boundary conditions

n	(e_0a=0 nm)	(e_0a=1 nm)	(e_0a=1.5 nm)	(e_0a=2 nm)	MD	DM
1	1.2353	0.8436	0.6137	0.4764	1.5708	1.5680

Last validation of the present results is given in Table 3. The first mode of vibration frequency of DNRS obtained by DM is compared with Molecular Dynamics (MD) simulation (Cao *et al.* 2006), classical elasticity and nonlocal models (Karličić 2015) in Table 3 for C-F single-walled zigzag (5,5) nanotube. The natural frequencies are presented in THz (f=ω/2π) and material properties and geometrical properties are taken into account as ρ=9517 kg/m^3, E=6.85 TPa, L=12.2 nm (Cao *et al.* 2006). It is clearly seen that DM results are in a good agreement with MD and other given models (Table 3). The value of length scale parameter e_0 is not constant opposed to bond length of C-C atoms η in CNTs. e_0 should be optimized to get the better results in Table 3. Moreover, satisfying results can be obtained by using DM which does not need to any modification of length scale parameter in the considered material. DM directly uses the bonding length of atoms as a length scale parameter.

After validation of the present formulation, longitudinal free and forced vibrations of a DNRS are presented for C-C and C-F boundary conditions in Figs. 6-17. In Figs. 6-9, dimensionless frequency parameter (DFP) results have been presented for different mode numbers. The results are given for L=1 nm and L=2 nm. The variation of DFP predicted by the LE increases linearly with increasing the mode number without any limit.

However, DM has a van Hove singularity (Hove 1953, Wu and Layman 2004) when $\frac{d\omega}{dk}=0$, (group velocity is zero). Therefore the predicted frequencies for negative group velocity are not physical, so they should be omitted. Wave propagation in nanotubes has been studied in using DM theory (Gul *et al.* 2017, Gul and Aydogdu 2017). It can be seen from these papers that, the van Hove singularity occurs at k=1.73/η (boundary of the Brillouin zone is 2.20). Wavelength is restricted by λ>2.55η.

Group velocity predicted by DM is zero when n>5 for L=1 nm and the results are not physically meaningful when n>5. Maximum meaningful value of mode number is 5 for L=1 nm and 11 for L=2 nm in DM model. This restriction cannot be observed in the classical elasticity models. Similar bounded frequencies can be seen in some previous studies (Vajari and Imam 2016, Gul *et al.* 2017). It is interesting to note that this behaviour has not been mentioned in these papers. Dimensionless frequencies obtained for C-C and C-F boundary conditions for single walled nanotubes are (Vajari and Imam 2016, Gul *et al.* 2017):

$$\Omega = \sqrt{\frac{1-[n^2\pi^2 2g^2-1]^2}{4g^2}} \quad for\ \mathrm{C-C} \tag{50a}$$

and

$$\Omega = D_1 D_2 [D_3 - D_4 + D_5 - D_6] \quad for\ \mathrm{C-F} \tag{50b}$$

where

$$\varphi = \sqrt{\frac{1+\sqrt{1-4g^2(\Omega^2)}}{2g^2}},$$

$$\varsigma = \sqrt{\frac{1-\sqrt{1-4g^2(\Omega^2)}}{2g^2}},$$

$$D_1 = \varsigma^2 - \varphi^2, \qquad D_2 = \varsigma\varphi,$$

$$D_3 = \sin(\varphi)\left[2sin\left(\frac{\varsigma}{2}\right)^2 - 1\right] g^2\varsigma^2\varphi, \tag{51}$$

$$D_4 = \sin(\varsigma)\left[2sin\left(\frac{\varphi}{2}\right)^2 - 1\right] g^2\varsigma\varphi^2,$$

$$D_5 = \varsigma)\left[2sin\left(\frac{\varphi}{2}\right)^2 - 1\right] \varsigma,$$

$$D_6 = \sin(\varphi)\left[2sin\left(\frac{\varsigma}{2}\right)^2 - 1\right] \varphi.$$

It is seen that these frequencies are limited for wave number whereas infinite number of mode numbers are obtained for the classical elasticity theory. Bounded frequencies have been also observed for the stress gradient nonlocal theory (Aydogdu 2009), but mode numbers of frequencies are not bounded for the nonlocal elasticity. The difference between DM results and LE model increases with increasing mode number. The difference between LE and DM theories decrease with increasing the nanorod length. Frequency results are closer each other for C-F boundary condition compared to C-C DNRS for higher modes of vibration.

The variation of DFP with dimensionless stiffness parameter has been demonstrated in Figs. (10-11). It is clearly seen that dimensionless stiffness parameter increases the DFP for both theories. DM results are lower than classical results especially for high modes of vibration and for L=1 nm. Slope of frequency stiffness parameter curve is higher for lower modes i.e. lower modes are more affected from stiffness parameter than higher modes. By increasing the rod length (L=2 nm) both theories give almost same value for the first two modes (Figs. 10 and 11).

The variation of the DFP with beam length is given for the first three modes for DM and LE theories (Fig.12-13). There is a very small difference between DM and LE theories for the first mode. The difference between the two theories is more obvious for higher modes of vibration and for small nanorod length. DFP predicted by DM is less than LE theory and it is more apparent for C-C boundary condition compared to C-F (Fig. 12). The effect of the elastic medium on vibration is also shown in DNRS in Figs. 12-13 for different nanorod length. It is seen that DFP increases with increasing the elastic medium stiffness of the springs for all modes (Figs. 12-13). LE model predicts higher frequencies than DM especially for short length (L=1-3 nm). Scale effects become insignificant and DM results converge to LE results by increasing the nanorod length.

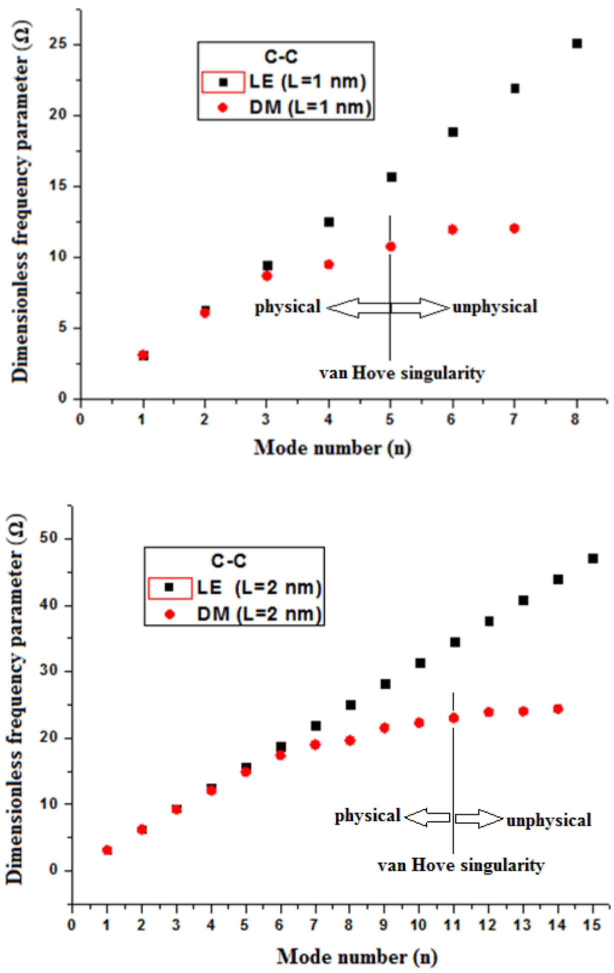
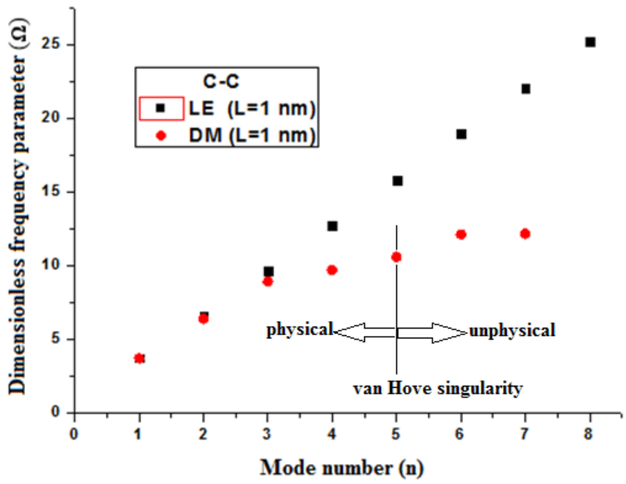


Fig. 6 Variation of dimensionless frequency parameter with mode number for C-C DNRS (K=0)



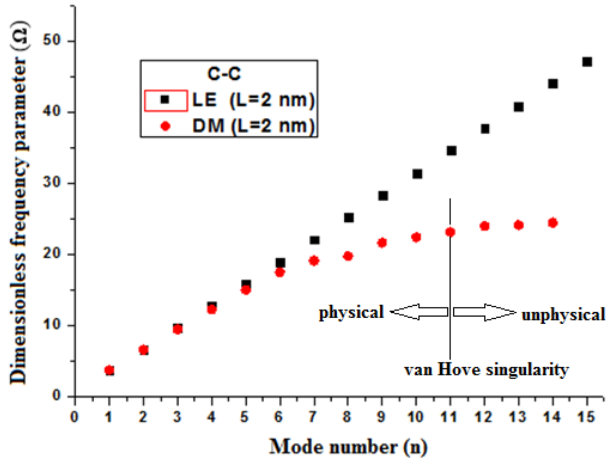


Fig. 7 Variation of dimensionless frequency parameter with mode number for C-C DNRS (*K*=2)

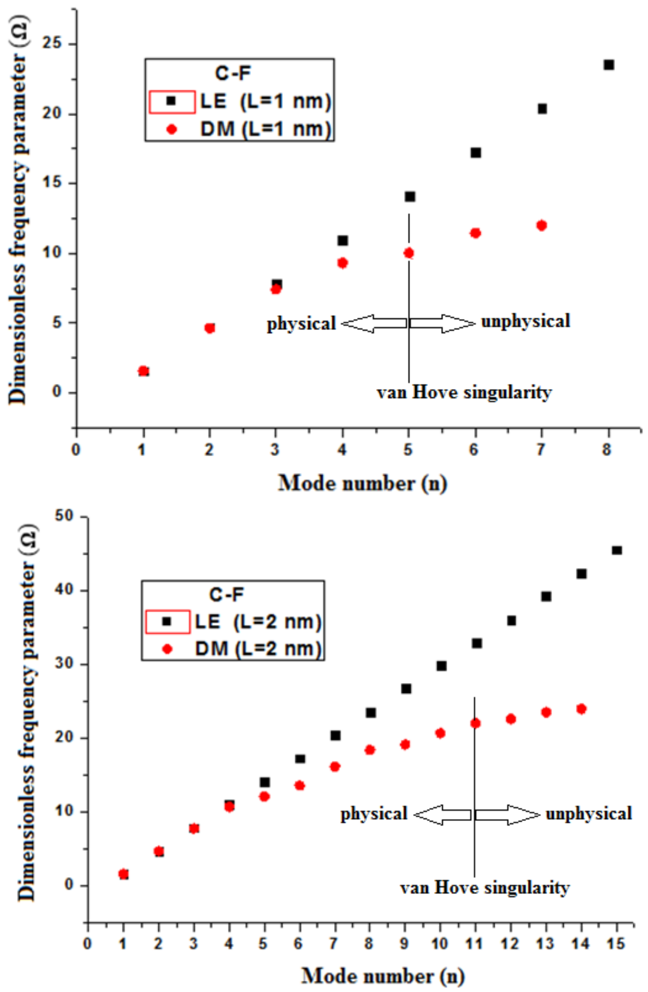


Fig. 8 Variation of dimensionless frequency parameter with mode number for C-F DNRS (*K*=0)

Figs. 14-15 demonstrate the variation of dimensionless dynamic displacement of the DNRS with DFP for different rod lengths. The results are obtained for the uniform axial load for the C-C boundary condition. The dynamic displacement increases with increasing DFP for both theories. The difference between DM and LE model is more apparent for shorter nanorod length. This is due to DM gradient effects which are more obvious for shorter nanorods and it vanishes for longer nanorods. It is seen that dynamic displacements take smaller values for axial linear load compared to uniform load case (Fig. 14). Also, scale effects decrease with increasing the rod length especially for lower modes.

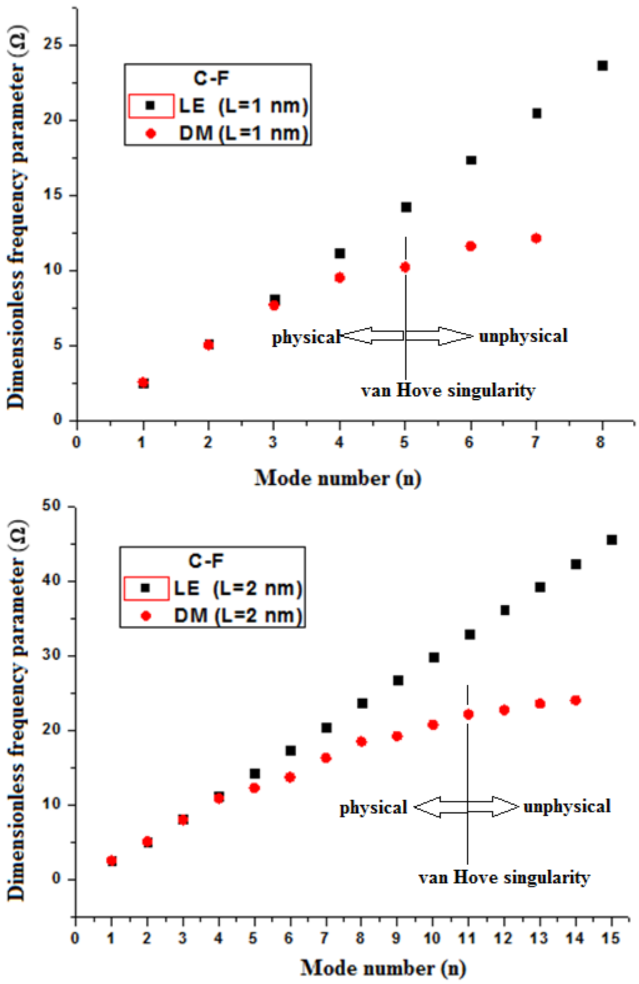
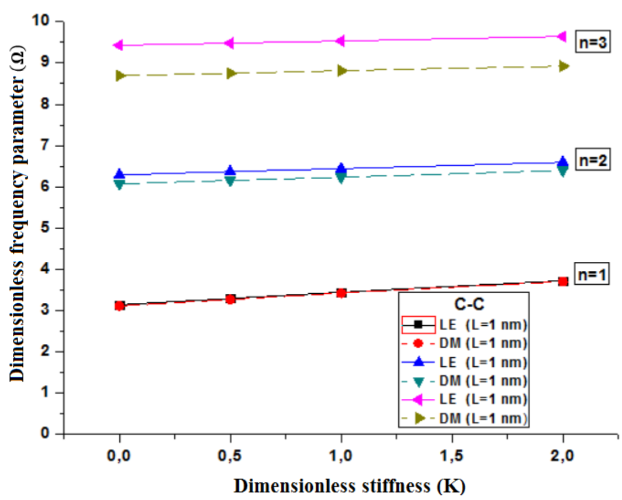


Fig. 9 Variation of dimensionless frequency parameter with mode number for C-F DNRS (*K*=2)



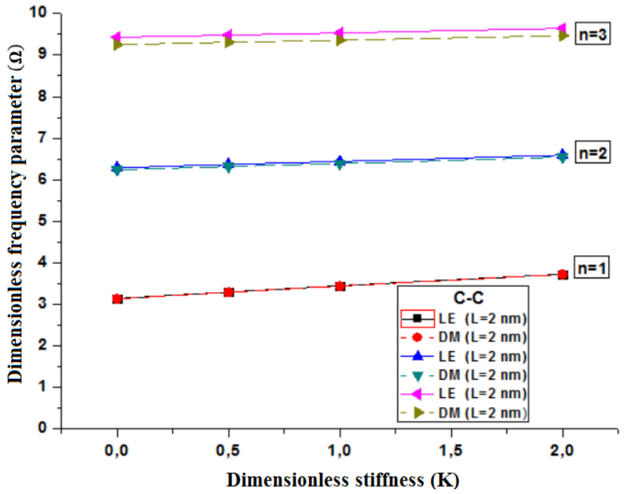


Fig. 10 Variation of dimensionless frequency parameter with dimensionless stiffness for the first three modes for C-C DNRS

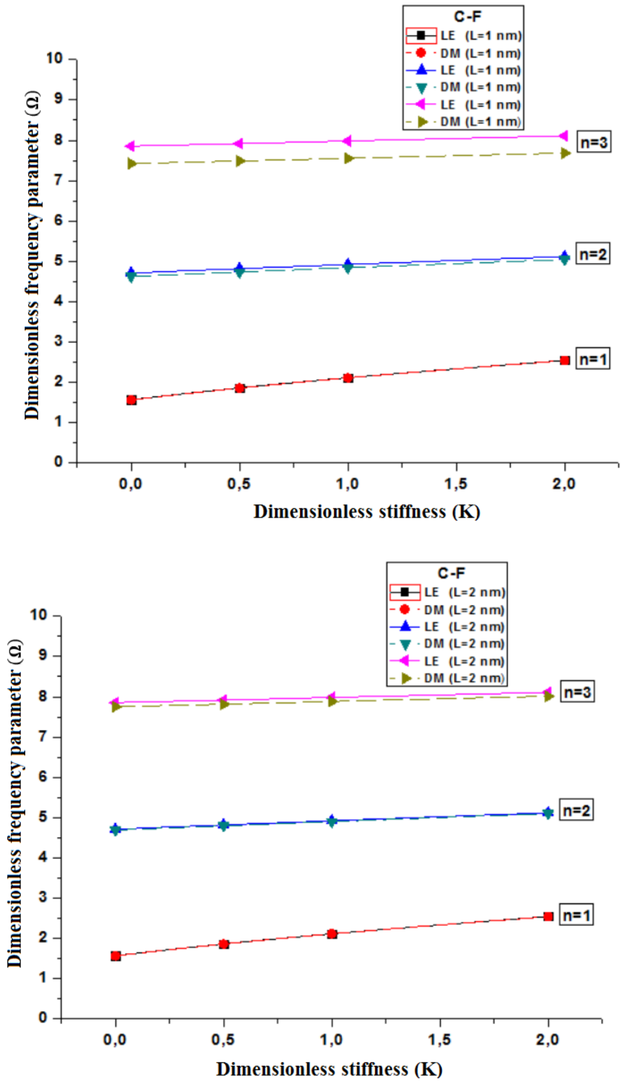


Fig. 11 Variation of dimensionless frequency parameter with dimensionless stiffness for the first three modes for C-F DNRS

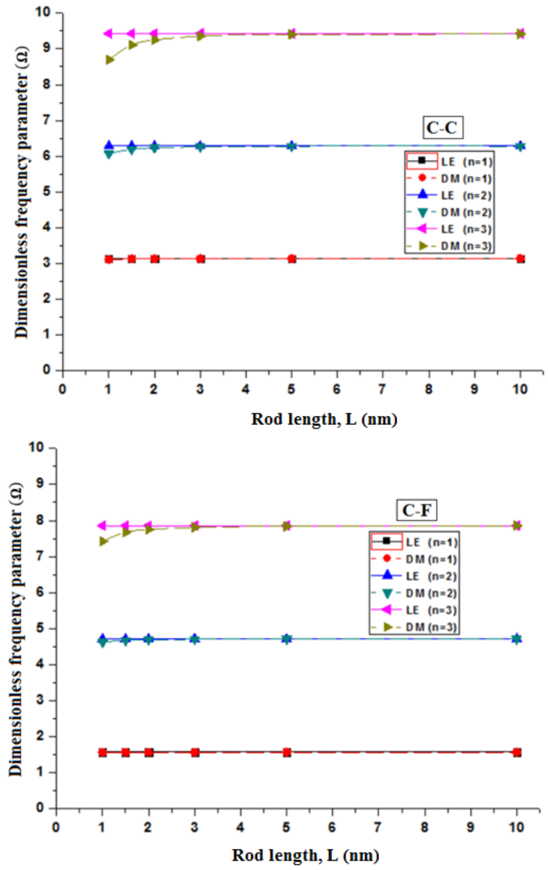


Fig. 12 Variation of dimensionless frequency parameter with rod length (K=0)

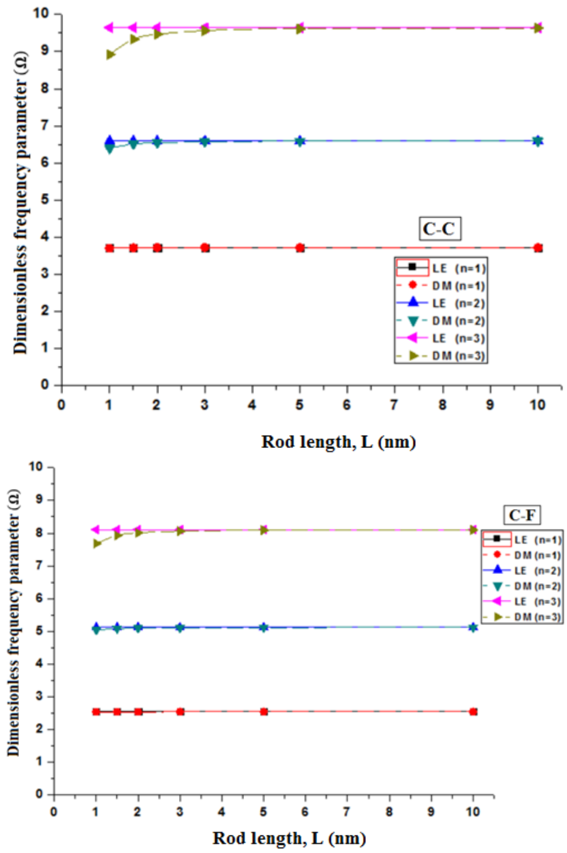
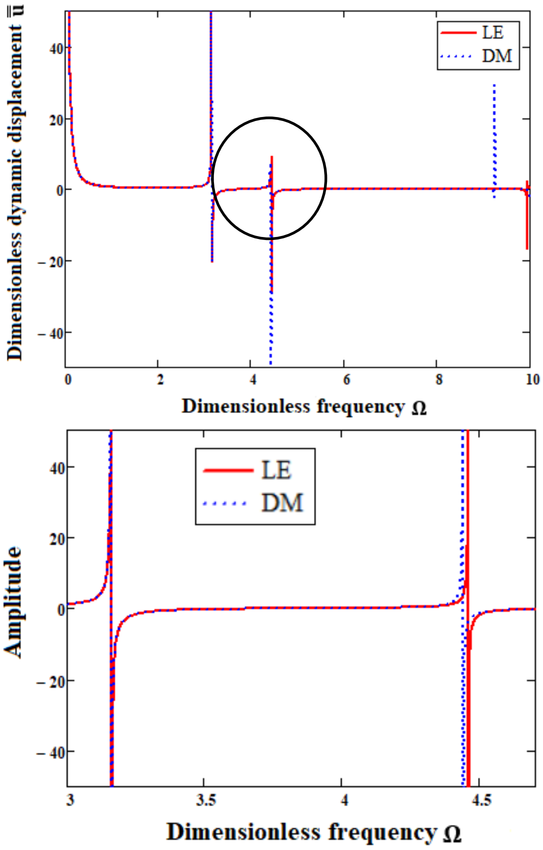
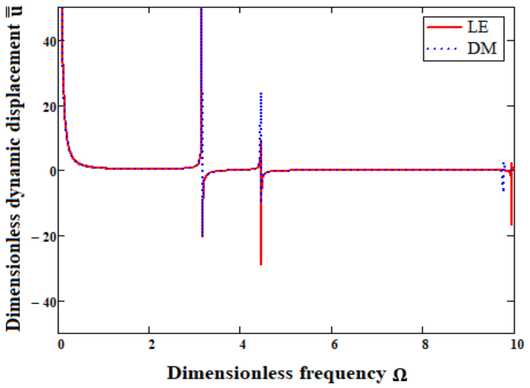


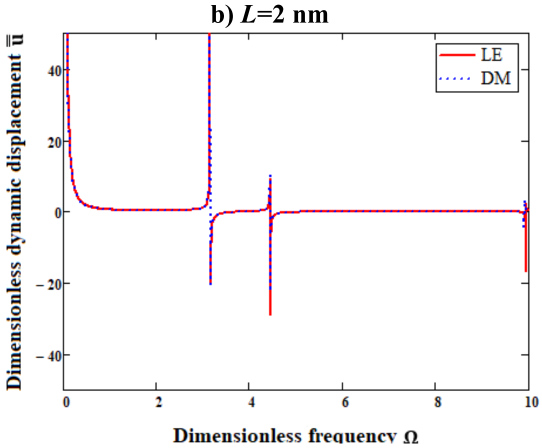
Fig. 13 Variation of dimensionless frequency parameter with rod length (K=2)



a) *L*=1 nm

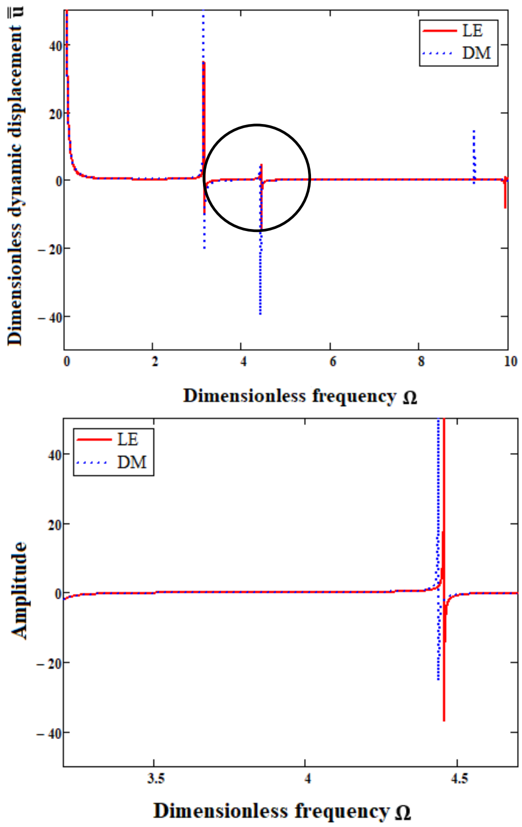


b) *L*=2 nm

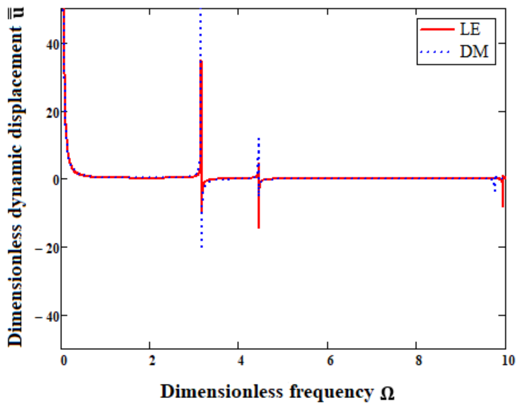


c) *L*= 5 nm

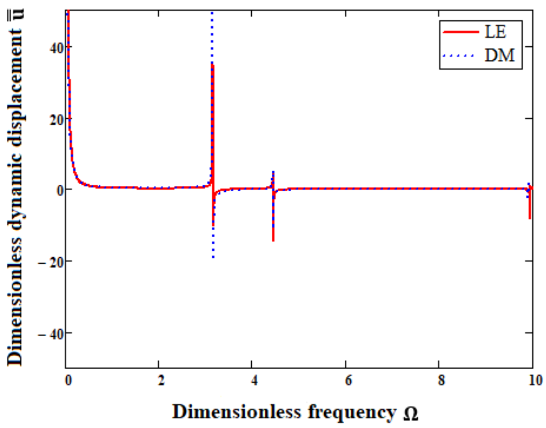
Fig. 14 Variation of dimensionless dynamic displacement with DFP under uniform axial load at C-C boundary condition (*K*=5)



a) *L*=1 nm

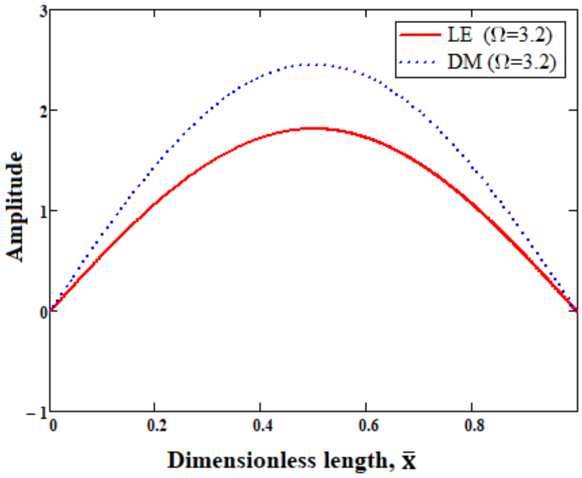


b) *L*=2 nm

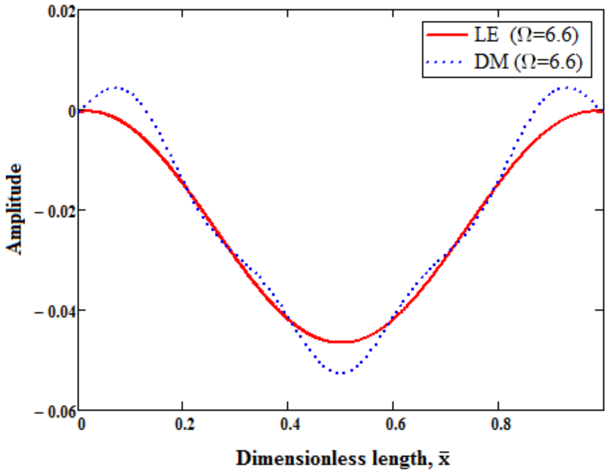


c) *L*=5 nm

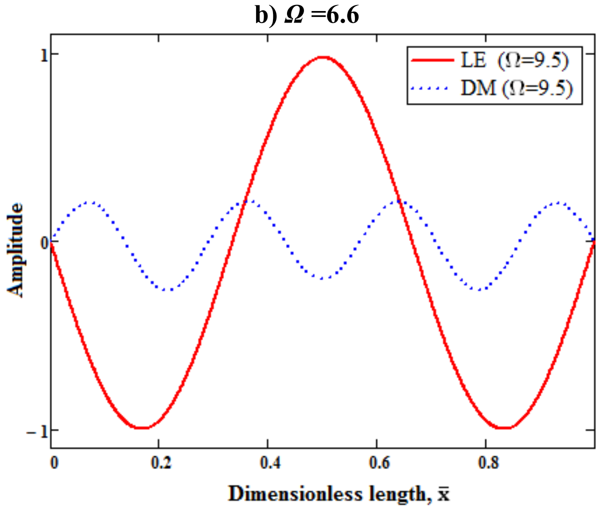
Fig. 15 Variation of dimensionless dynamic displacement with DFP under axial linear load at C-C boundary condition (*K*=5)



a) *Ω*=3.2



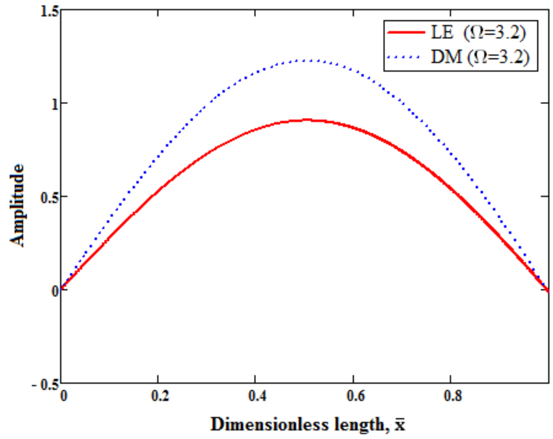
b) *Ω* =6.6



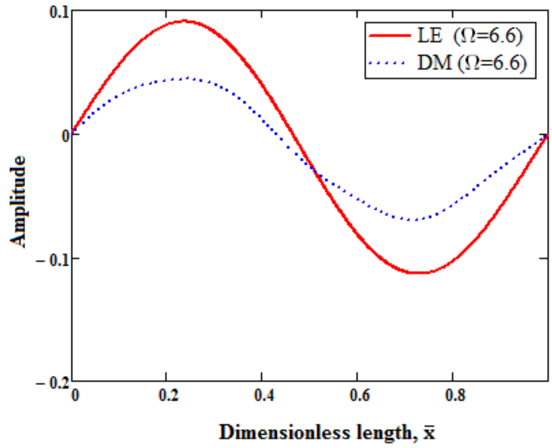
c) *Ω* =9.5

Fig. 16 Relative mode shapes of DNRS under uniform axial load at C-C boundary condition (*L*=1 nm, *K*=0.5)

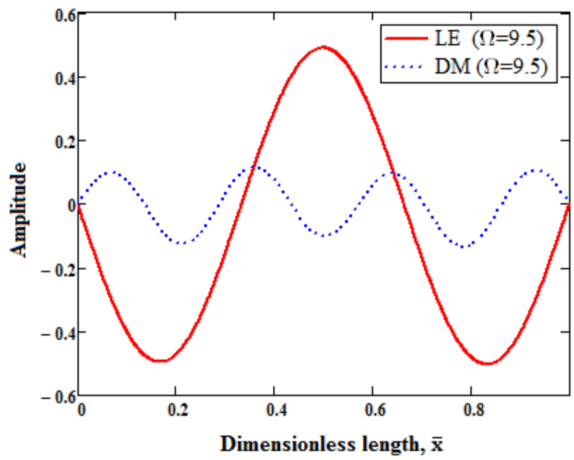
Relative mode shapes are demonstrated for different DFP for C-C boundary condition in Figs. (16-17). The frequencies of external loads are chosen as *Ω*=3.2, 6.6 and 9.5. These frequencies have been chosen near to the free vibration frequencies in the LE and DM theory. Different relative mode shapes are obtained in DNRS with LE and DM models. Especially for uniform load, mode shapes are different when *Ω*=6.6 and *Ω*=9.5 for two theories. This difference is more pronounced when the rod length is *L*=1 nm (here the results are presented only for short nanotubes). It means DM gradient effects can change the mode shapes. However, relative mode shapes are similar under the linear load case compared to uniform load case in DNRS. It should be noted that the number of nodes at the mode shapes are different for different theories and this point will be very important when designing nano-scale devices.



a) *Ω*=3.2



b) *Ω*=6.6



c) *Ω* =9.5

Fig. 17 Relative mode shapes of DNRS under linear axial load at C-C boundary condition (*L*=1 nm, *K*=0.5)

7. Conclusions

This paper is concerned with an exact analytical solution for DNRS using a length scale theory. Free and forced vibration analysis of DNRS are investigated for C-C and C-F boundary conditions. The study demonstrates that length scale plays a significant role in the calculations and the DM model should be used at the nano length scale. It is obtained that van-Hove singularity limits the frequencies from the upper for the DM theory. It is concluded from the present study that DM predicts a softening in the structures and experimental results agree well with DM for wave propagation of graphite. Moreover, DM theory is in a good agreement with MD results for DNRS unlike the NL theories given in the literature (Karličić *et al*. 2015). Frequencies of DNRS increase with increasing the elastic medium stiffness of the springs and decreases with increasing the rod length for DM and LE models. The difference between the two theories is more apparent for higher modes of vibration and for C-C boundary conditions compared to C-F. Also, different relative mode shapes of DNRS are obtained under the uniform and linear load cases for LE and DM models at C-C boundary condition. It is finally worth mentioning that the present study may be extended to the more complex n-nanorod systems and may be helpful for designing the future nano-devices.

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CC

Nomenclature

A	cross-section area of a nanorod
$\vec{a}_1, \vec{a}_2$	unit vectors
a'_α, b'_α	positions of the nodes after deformation
$[B]$	micro-moduli matrix
$B_{\alpha\beta}$	tension micro modulus between nodes α and β
B_{ij}	terms of micro moduli matrix (i,j=1,2,3)
c_i	undetermined coefficients (i=1,2,3,4)
e_0a	nonlocal parameter
E	Elasticity modulus the nanorods
f	distributed force
$[H]$	transformation matrix
k	stiffness of the springs between two nanorods
K	dimensionless stiffness of the springs between
L	length of the nanorod
m	mass per unit length
M	number of terms in the Taylor series expansion
m_α	torsional micro stress at the αth-node
n	mode number
p_α	axial micro-stress at the αth-node
P	axial force
$\vec{r}$	position vector of node α
R	double force
s^2	length scale parameter of the doublet mechanics theory
t_α	shear micro-stress at the αth-node
t	time
T	kinetic energy
$\vec{u}$	displacement vector
U	strain energy
$u(x,t)$	relative displacement of the nanorods
W	work done by external forces
x,y and z	Cartesian coordinates

Greek and mixed letters

∇	Delta operator
$\Delta\vec{u}_\alpha$	increment displacement vector
ε	macro strain
ϵ_α	axial micro strain
ϵ_β	axial micro strain at the β doublet
ε_{mn}	macro strain components
$\eta_\alpha, \lvert\xi_\alpha\rvert$	doublet separation distance
g^2	dimensionless scale parameter
δ	variational symbol
λ	wavelength
ω	natural frequency
Ω	dimensionless frequency parameter
ρ	density of the nanorod
σ	macro stress
$\{\sigma\}$	macro stress tensor
σ_{xx}^{LE}	Cauchy stress
$\vec{\tau}_\alpha^0$	unit vector at the αth-direction
$\tau_{\alpha m,n,s,t}^0$	unit vector components

Abbreviations

C-C	clamped-clamped
C-F	clamped-free
CNT	carbon nanotube
DFP	dimensionless frequency parameter
DM	doublet mechanics
DNRS	double nanorod system
LE	local elasticity
MD	molecular dynamics
NL	nonlocal elasticity theory
NOMS	nano-opto-mechanical systems